PHILOSOPHICAL

TRANSACTIONS

OF THE

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MDCCCLXXVI

CONTENTS.

PART II.

LIST OF ILLUSTRATIONS

Adjudication of the Medals of the Royal Society for the year 1875 by the President and Council.

The Copley Medal to Professor August Wilhelm Hofmann, F.R.S., for his numerous contributions to the science of Chemistry, and especially for his Researches on the Derivatives of Ammonia

A Royal Medal to Mr. William Crookes, F.R.S., for his various chemical and physical researches, more especially for his discovery of Thallium, his investigation of its compounds, and determination of its atomic weight; and for his discovery of the repulsion referable to radiation.

A Royal Medal to Dr. Thomas Oldham, F.R.S., for his long and important services in the Science of Geology, first as Professor of Geology, Trinity College, Dublin, and Director of the Geological Survey of Ireland, and chiefly for the great work which he has long conducted as Superintendent of the Geological Survey of India, in which so much progress has been made that, in a few years, it will be possible to produce a Geological Map of India comparable to the Geological Map of England executed by the late Mr Greenough, also for the series of volumes of Geological Reports and Memoirs, including the 'Palæontologica Indica,' published under his direction

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The Bakerian Lecture was delivered by Professor W. G. Adams, F.R.S.: it was entitled "On the Forms of Equipotential Curves and Surfaces and Lines of Electric Force"

The Croonian Lecture was delivered by Professor David Ferrier, M.D.: it was entitled "Experiments on the Brain of Monkeys (Second Series)"

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CONTENTS

OF VOL. 165

PART I.

PART II.

LIST OF ILLUSTRATIONS.

XI. *On the Tides of the Arctic Seas.*

By the Rev. SAMUEL HAUGHTON, *M.D. Dubl., D.C.L. Oxon., F.R.S.*,
Fellow of Trinity College, Dublin.

Part IV. *On the Tides of Northumberland Sound, at the Northern Outlet of Wellington Channel.*

Received July 11,—Read November 19, 1874

THESE Tidal Observations were made on board H.M.S. 'Assistance,' Captain Sir EDWARD BELCHER, R.N., K.C.B., from 24th May to 6th July 1853, the exact position of the ship being 76° 52′ N. lat. and 97° 00′ W. long. Sir LEOPOLD M'CLINTOCK kindly procured for me, from Sir EDWARD BELCHER, a copy of the Observations, and in forwarding them to me writes thus:—"Sir EDWARD BELCHER wishes me to tell you how his Tidal Observations in 1853 were made. He says they did not depend upon the guess of any one, but resulted from machinery connected with the bottom, which moved a ratchet-wheel, each cog or inch of gauge ringing a bell, and the rise and fall was not that of the ship, but of the whole floe in which she was fixed. This machinery is described in his narrative, 'The last of the Arctic Voyages,' vol. i. p. 141. He further states that this rise was repeatedly verified by Theodolite Observations."

The following Table contains the Time and Height of High Water and Low Water, extracted from the original observations (which are forwarded with this paper), also the Diurnal Tide at High Water and Low Water, calculated from the heights by means of the formula

$$\text{Diurnal Tide} = \frac{h_1 - 4h_2 + 6h_3 - 4h_4 + h_5}{16}, \quad . \quad . \quad . \quad . \quad (1)^*$$

which gives the fourth difference of the successive heights.

* This expression for the Diurnal Tide is used and explained by Mr. AIRY in his paper "On the Tides of the Coasts of Ireland" (Phil. Trans. 1845), and by the author in his paper "On the Diurnal Tides of the Coasts of Ireland" (Trans. Royal Irish Academy, 1855).

A. *Diurnal Tide* (Heights).

Northumberland Sound.

Time	High Water Height.	Low Water Height.	Diurnal Tide at High Water	Diurnal Tide at Low Water
1853 h m	ft. in	ft. in	ft	ft.
May 27. 4 0 P.M	16 1			
27. 10 0 "	. .	14 4½		
28. 4 40 A.M. .. .	16 4½	.....	0·187	
28. 12 5 P.M.	.. .	15 4	. . .	0 481
28 4 0 "	15 11	. .	0·193	
28. 11 0 "	..	14 4½		0·443
29. 4 30 A.M.	16 3	. ..	0·201	
29 1 0 P.M. . . .	. .	15 3	.	0·381
29. 6 0 " .	15 10	. .	0·219	
29 Midnight	.	14 8		0·339
30. 6 40 A.M.	16 4	. ..	0·224	
30. 1 40 P.M.	. .	15 5	. .	0 292
30. 7 0 "	15 11	....	0·208	
31. 12 50 A.M.	.... .	14 11	. .	0·230
31. 8 0 "	16 4	.	0·198	
31. 3 30 P.M.	.... .	15 3		0·187
31. 9 0 "	16 0	...	0·214	
June 1. 2 40 A.M.	. .	14 10	. .	0·146
1. 8 32 "	16 6	.	0·234	
1. 3 30 P.M	.	15 1	...	0·062
1. 9 40 "	16 0		0·229	
2. 3 30 A.M.	......	15 0¾	......	0·088
2. 9 40 "	16 5	. . .	0·209	
2. 4 20 P.M. . ..	. ....	14 10	....	0·083
2. 10 30 "	16 0	. ....	0·172	
3. 4 35 A.M.	. . .	14 11	..	0·121
3. 10 30 "	16 3¾	......	0·182	
3. 5 0 P.M.	.... .	14 7	. ....	0·177
3. 11 0 " . ..	16 2	.	0 062	
4. 5 0 A.M.	.. ..	15 0	......	0·224
4. 10 48 "	16 3	.....	0·010	
4. 5 30 P.M	..	14 6	...	0·255
5. 12 15 A.M.	16 3	....	0·015	
5. 5 30 "	..	15 0	. .	0·313
5. 11 45 "	16 2	......	0·031	
5. 6 0 P.M.	. ...	14 3½	......	0·370
6 12 50 A.M.	16 3½	.. .	0·010	
6. 6 30 "	....	15 2	......	0·380
6. 12 15 P.M.	16 5½	....	0·021	
6. 6 20 "	.. .	14 6½	......	0·391
7. 1 10 A.M.	16 8	.....	0·088	
7. 7 22 "	. ...	15 5	.. ..	0·438
7. 12 25 P.M . . .	16 5	.	0·117	
7. 6 21 " . .		14 6	.....	0·474
8 1 0 A.M. . .	16 8	. .	0·117	
8. 7 20 "	......	15 6½	....	0·475
8. 12 40 P.M	16 6	. ..	0·128	
8. 8 0 " . .	. .	14 8½	. . .	0·474
9 2 0 A.M.	16 10½	... .	0·110	
9. 8 22 "	. .	15 9	.. ...	0·477
9. 1 15 P.M.	16 8½	.....	0 107	
9. 7 43 "	..	14 10	......	0·479
10. 2 38 A.M.	16 10½	. ..	0 112	
10. 8 50 "	......	15 9	......	0·484
10. 2 0 P.M.	16 6½	.. ...	0·125	

Northumberland Sound (continued).

Time.	High Water Height.	Low Water Height.	Diurnal Tide at High Water	Diurnal Tide at Low Water
1853 h m	ft. in	ft. in	ft	ft.
June 10. 8 30 P.M.		14 8		0·428
11. 3 40 A.M.	16 8		0·151	
11. 9 35 „		15 6¼		0·256
11. 2 30 P.M.	16 3		0·214	
11. 10 7 „		15 5		0·234
12. 4 0 A.M.	16 9		0·240	
12. 10 30 „		15 8		0·251
12. 3 15 P.M.	16 3		0·203	
12. 10 0 „		14 9¾		0·339
13. 4 53 A.M.	16 8		0·068	
13. 11 15 „		15 6		0·339
13. 10 30 P.M.		14 10		0 328
14. 5 45 A.M.	16 8		0·151	
14. 12 15 P.M.		15 5¼		0·286
14. 6 0 „	15 11¾		0·307	
14. Midnight		14 11		0·219
15. 6 45 A.M.	16 8		0·354	
15. 1 10 P.M.		15 3½		0·143
15. 8 0 „	15 11½		0 360	
16. 12 50 A.M.		15 2		0 005
16. 8 0 „	16 9		0 339	
16. 2 15 P.M.		15 2¾		0·005
16. 8 10 „	16 2		0 307	
17. 2 10 A.M.		15 3		0·078
17 8 30 „	16 9½		0·271	
17. 3 0 P.M.		14 11½		0·151
17. 9 30 „	16 4		0·224	
18. 3 12 A.M.		15 4		0 187
18. 9 15 „	16 10		0·088	
18. 4 35 P.M.		14 11		0·214
18. 10 45 „	16 9		0·099	
19. 4 50 A.M.		15 3		0·250
19. 10 12 „	17 0		0·078	
19 4 55 P.M.		14 8		0·344
19. Midnight	16 10½		0·041	
20. 5 30 A.M.		15 6		0·427
20. 11 20 „	17 0½			
20. 6 0 P.M.		14 7½		
21. Noon	——			
21. 7 0 P.M.		——		
22. 1 0 „	——			
22. 7 20 „		13 2		
23. 2 0 A.M.	15 7			
23. 8 0 P.M.		14 1		
24. 2 0 „	16 8			
24. 8 30 „		——		
25. 3 0 P.M.	16 5			
25. 9 55 „		14 8		
26. 4 30 A.M.	17 0		0·333	
26. 11 30 „		15 6		0·407
26. 4 0 P.M.	16 3¼		0·307	
26. 10 40 „		14 8½		0·401
27. 5 45 A.M.	16 10		0·266	
27. 12 16 P.M.		15 7½		0·339
27. 6 0 „	16 5		0·266	
27. 11 0 „		15 2		0·250

Northumberland Sound (continued).

Time		High Water Height.	Low Water Height.	Diurnal Tide at High Water	Diurnal Tide at Low Water
1853	h m	ft. in.	ft. in.	ft.	ft.
June 28.	6 40 A.M.	17 0½	....	0·302	
28.	1 38 „		15 8	. .	0·052
28.	6 20 P.M.	16 5½		0·453	
28.	Midnight		15 6	. .	0·240
29.	7 20 A.M.	17 3		0·635	
29.	2 10 P.M.		13 11	. .	0·297
29.	7 40 „	14 9		0·656	
30	1 30 A.M.		13 11½	....	0·125
30.	8 30 „	15 4		0·364	
30	3 45 P.M.		13 11	.	0·057
30.	9 0 „	14 10¾		0·193	
July 1.	12 40 A.M.		14 1	. . .	0·107
1.	9 5 „	15 2½		0·135	
1.	4 35 P.M.		13 10		0·172
1.	9 40 „	15 0		0·088	
2.	4 20 A.M.		14 3	.	0·229
2	9 38 „	15 2		0·115	
2.	4 10 P.M.		13 8		0·281
2	10 45 „	15 1		0·000	
3.	4 45 A.M.		14 2		0·339
3.	10 30 „	14 11½		0·026	
3.	5 4 P.M.		13 4		0·151
3	11 13 „	15 0		0·026	
4.	5 0 A.M.		14 2	...	0·448
4.	10 45 „	14 11¼		0·021	
4.	5 30 P.M.		13 2½	. . .	0·490
5.	12 15 A.M.	15 1		0·015	
5	6 0 „		14 2	.	0·500
5	12 45 P.M.	15 1		0·031	
5.	6 15 „		13 1		0·511
6.	1 0 A.M.	15 1½		0·073	
6.	8 0 „		14 0	. ...	0·521
6.	1 0 P.M.	14 10		0·028	
6	6 40 „		12 9½	. . .	0·521
7.	2 0 A.M.	15 0			
7	8 0 „		13 8		

The general expression for the Diurnal Tide is the following:—

$$D=S\sin 2\bar{\sigma}\cos(s-i_s)+M\sin 2\bar{\mu}\cos(m-i_m), \quad \ldots\ldots \quad (2)$$

where

D=height of tide,

$\bar{\sigma}$, $\bar{\mu}$=Solar and Lunar Declinations, corrected for Age of Tide,

s, m=Solar and Lunar Hour-angles,

i_s, i_m=Diurnal Solitidal and Lunitidal Intervals,

S, M=Solar and Lunar Coefficients, uncorrected for Parallax, &c.

It would be impossible to obtain any result as to the Diurnal Tides from so short a series of observations, only for a lucky chance which simplifies the calculation at this station, and enables us to obtain the Solar Diurnal Tide, although it is not easy to

determine the Lunar Diurnal Tide *. It so happened, during the observations, that the time of vanishing of the whole Diurnal Tide at Low Water corresponded very closely with the time of vanishing of the Moon's declination.

So that we have, at the same times,

$$\mu=0, \quad D=0,$$

which reduces the general expression at these times to

$$S \sin 2\sigma \cos (s-i_s)=0. \quad \ldots \ldots \quad (3)$$

The times corresponding to

$$\mu=0, \quad D=0$$

were

		h m
1st June	. . .	3 0 A.M.
15th „	. . . . . .	4 30 P.M.
28th „	. . . .	9 30 A.M.

If we now take the hours of Low Water of the Tides occurring nearest to the time of the Moon's declination vanishing, we find.—

			h m
1st June	. . .	$s=$	2 40 A.M.
15th „	. . . .		1 10 P.M.
28th „	. . .		1 38 A.M.
Mean value of s	.		1 49

Now from equation (3) we have

$$s-i_s=6^h \text{ or } 18^h,$$

hence

$$1^h\ 49^m-i_s=6^h \text{ or } 18^h,$$

and, finally,

$$i_s=-4^h\ 11^m$$

$$\text{or}\dagger \quad +7^h\ 49^m.$$

The Diurnal Tide at High Water, when $\mu=0$, is represented by

$$D=S \sin 2\sigma \cos (s-i_s),$$

and had the following values —

1st June	. . . .	D = +0·234 ft
16th „	. . . .	−0 360 „
28th „	. . . . .	+0·302 „
Mean value of D	. .	0 299 „

* See Note A, p. 327.

† An examination of the signs of the numerical values of the tide shows that the negative value of i_s must be chosen.

The hours of High Water corresponding to these values, and nearest to the time of the Moon's declination vanishing, were :—

	h m
1st June	8 32
15th „	8 00
28th „	6 40
Mean	7 44

Hence, using the mean value of 2σ during the observations (45° 50′), we obtain

$$\pm 0\ 299 = \mathrm{S} \sin 45^\circ\ 50' \cos(s - i_s);$$

but

$$\begin{aligned} & \quad\ \ \mathrm{h} \quad \mathrm{m} \\ s &= \ \ 7 \quad 44 \\ i_s &= -\ 4 \quad 11 \\ s - i_s &= \ 11 \quad 55 \end{aligned}$$

which corresponds to 180°, or the cosine equal to unity. Hence we have

$$0\ 299 = \mathrm{S} \sin 45^\circ\ 50',$$

or, finally,

$$\mathrm{S} = \frac{0{\cdot}299 \text{ ft.}}{\sin 45^\circ 50'}$$

$$= 0{\cdot}417 \text{ ft.} = 5{\cdot}00 \text{ inches.}$$

B *Semidiurnal Tide* (Heights).

If the preceding Table be plotted to scale, it is easy to separate the Semidiurnal Tide from the Diurnal Tide just discussed, but it is not possible, from observations made at the Solstice only, to separate the Solar and Lunar Tide and determine their coefficients.

The general expression for the Semidiurnal Tide is

$$\mathrm{T} = \mathrm{S}' \cos 2(s - i_s) + \mathrm{M}' \cos 2(m - i_m), \quad . \quad . \quad . \quad . \quad . \quad (3^*)$$

where

S', M' = Solar and Lunar Coefficients, not corrected for declination or parallax,

s, m = Hour-angles of Sun and Moon,

i_s, i_m = true Solitidal and Lunitidal Intervals.

This expression may be thrown into the form

$$\mathrm{T} = \mathrm{A} \cos 2(m - \mathrm{B}), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where

$$\left.\begin{aligned} \mathrm{A} &= \sqrt{\mathrm{M}'^2 + \mathrm{S}'^2 + 2\mathrm{M}'\mathrm{S}' \cos 2(\overline{m - s} - \overline{i_m - i_s})}, \\ \tan 2\mathrm{B} &= \frac{\mathrm{M}' \sin 2i_m + \mathrm{S}' \sin 2(\overline{m - s} + i_s)}{\mathrm{M}' \cos 2i_m + \mathrm{S}' \cos 2(\overline{m - s} + i_s)}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

A is the apparent height of the tide, and

B is the apparent Lunitidal Interval at High Water and Low Water.

At Spring-Tides we have

$$\left.\begin{array}{r} m-s=i_m-i_s, \\ A=M'+S', \end{array}\right\} \qquad (6)$$

at Neap-Tides we have

$$\left.\begin{array}{r} \overline{m-s}-\overline{i_m-i_s}=90^\circ, \\ A=M'-S'. \end{array}\right\} \qquad (7)$$

The maximum Spring-Tides occurred —

	Sun's Hour-Angle h m	Moon's Hour-Angle h m
6th June . .	12 50 A.M.	+1 18
	12 15 P.M.	+0 21
19th June . . .	Midnight.	+1 6
20th „ . . .	11 20 A.M.	+0 2
Mean .	0 6	+0 42

The minimum Neap-Tides occurred —

	Sun's Hour-Angle. h m	Moon's Hour-Angle h m
30th May	6 40 A.M.	−0 15
	7 0 P.M.	−0 19
12th June . . .	4 0 A.M.	−0 30
	3 15 P.M.	−1 40
Mean .	+5 14	−0 26

From the Spring-Tides we find, by equation (6),

$$i_m-i_s=36^m;$$

and from the Neap-Tides, by equation (7), we obtain

$$i_m-i_s=40^m.$$

The mean of these values gives us

$$i_m-i_s=38^m. \qquad (8)$$

We have no means of determining i_m and i_s separately.

The maximum and minimum ranges of the Tide, corrected for Diurnal Inequality only, were:—

Springs.	Range.
6th June	19·3 inches.
20th June	23·7 „
Mean . .	21·5 „

Neaps		Range.
30th May	. . . . . .	12 0 inches.
12th June	. . . . . .	11·3 „
	Mean . .	11·65 „

Substituting these values in (6) and (7), we find

$$2(M'+S') = 21{\cdot}5,$$

$$2(M'-S') = 11{\cdot}65,$$

and, finally,

$$\frac{S'}{M'} = 0\ 297 \qquad \ldots\ldots\ldots\ldots \qquad (9)$$

It will be observed that the Diurnal Solar Tide Range, already determined (9·40 in), bears a very large proportion to the Semidiurnal Tide Ranges.

C. *Diurnal Tide* (Times).

The following Table contains the hour in local time of High Water and Low Water, and also the Lunitidal Intervals at High Water and Low Water elapsed from the Moon's passage of the meridian of the place The Diurnal Tide in time might be calculated from the Lunitidal Intervals by first or second differences, as in the case of heights, but it is not worth the trouble to make the calculations, as the results can be more readily obtained by plotting the Lunitidal Intervals carefully to scale

When this is done the diagram shows a fairly regular Diurnal Tide, with vanishing epochs and range well marked.

The maximum accelerations and retardations of the time of High or Low Water occasioned by the Diurnal Inequality amounted, generally, to from 35 minutes to 40 minutes, and on 1st July, at Low Water, reached 65 minutes.

D *Semidiurnal Tide* (Times)

Northumberland Sound —Lunitidal Intervals.

	High Water	Low Water		High Water	Low Water
1853 h m	h m	h m	1853 h m	h m	h m
May 27 4 0 P.M	−0 46		June 1 2 40 A.M.	.	6 38
27 10 0 „		5 14	1. 8 32 „	+0 12	
28 4 40 A M.	−0 37		1. 3 30 P.M	..	7 10
28 12 5 P.M		6 48	1. 9 40 „	+0 56	
28. 4 0 „	−1 41		2 3 30 A.M.	....	6 46
28. 11 0 „		5 19	2. 9 40 „	+0 39	
29 4 30 A M	−1 38		2. 4 20 P.M.	...	7 31
29. 1 0 P M.	.	6 52	2 10 30 „	+1 5	
29. 6 0 „	−0 32		3. 4 35 A.M.	.. .	7 10
29. Midnight.		5 28	3. 10 30 „	+0 48	
30 6 40 A M.	−0 15		3. 5 0 P M.	.	7 18
30. 1 40 P.M.		6 45	3 11 0 „	+0 54	
30 7 0 „	−0 19		4. 5 0 A.M.	...	6 54
31. 12 50 A.M.	.	5 31	4 10 48 „	+0 24	
31. 8 0 „	+0 22		4. 5 30 P M.	. ....	7 6
31. 3 30 P.M.		7 52	5. 12 15 A.M.	+1 27	
31. 9 0 „	+0 58		5. 5 30 „	...	6 42

Northumberland Sound.—Lunitidal Intervals (continued).

1853 h m	High Water h m	Low Water h m
June 5. 11 45 P.M.	+0 37	
5. 6 0 „		6 52
6 12 50 A.M.	+1 18	
6. 6 30 „		6 58
6. 12 15 P.M.	+0 21	
6. 6 20 „		6 26
7. 1 10 A.M.	+0 52	
7. 7 22 „		7 4
7. 12 25 P.M.	−0 18	
7. 6 21 „		5 38
8. 1 0 A.M.	−0 7	
8. 7 20 „		6 13
8. 12 40 P.M.	−0 53	
8. 8 0 „		6 27
9. 2 0 A.M.	+0 3	
9. 8 22 „		6 25
9. 1 15 P.M.	−1 9	
9. 7 43 „		5 19
10. 2 38 A.M	−0 10	
10. 8 50 „		6 2
10. 2 0 P.M.	−1 15	
10. 8 30 „		5 15
11. 3 40 A.M.	+0 1	
11. 9 35 „		5 56
11. 2 30 P.M.	−1 36	
11. 10 7 „		6 1
12. 4 0 A.M.	−0 30	
12 10 30 „		6 0
12. 3 15 P.M.	−1 40	
12. 10 0 „		5 5
13. 4 53 A M.	−0 26	
13. 11 15 „		5 56
13. ——	——	
13 10 30 P.M.		4 47
14. 5 45 A.M.	−0 22	
14. 12 15 P.M.		6 8
14. 6 0 „	−0 30	
14. Midnight		5 30
15. 6 45 A.M.	−0 9	
15 1 10 P M.		6 16
15. 8 0 „	+0 42	
16. 12 50 A.M.		5 32
16. 8 0 „	+0 18	
16. 2 15 P.M.		6 33
16. 8 10 „	+0 3	
17 2 10 A.M.		6 3
17. 8 30 „	−0 1	
17. 3 0 P.M.		6 29
17. 9 30 „	+0 31	
18. 3 12 A.M.		6 13
18. 9 15 „	−0 8	
18. 4 35 P.M.		7 12
18. 10 45 „	+0 51	
19. 4 50 A.M.		6 56
19 10 12 „	−0 6	
19. 4 55 P.M.		6 36
19 Midnight	+1 6	
20. 5 30 A.M.		6 36
20. 11 20 „	+0 2	
20. 6 0 P.M.		6 42
20. ——	——	
21. ——		——
21. ——	——	

1853 h m	High Water h m	Low Water h m
June 21 ——		——
21 ——	——	
22. ——		——
22 ——	——	
22 7 20 P M.		5 55
23. 2 0 A.M.	−0 5	
23. ——		——
23. ——	——	
23 8 0 P.M		5 31
24 ——	——	
24. ——		——
24. 2 0 P.M.	−1 28	
24 8 30 „		5 2
25. ——	——	
25. ——		——
25. 3 0 P.M.	−1 23	
25. 9 55 „		5 32
26. 4 30 A.M.	−0 18	
26. 11 30 „		6 42
26. 4 0 P.M.	−1 12	
26. 10 40 „		5 28
27 5 45 A M.	+0 11	
27. 12 16 P M.		6 42
27. 6 0 „	+0 2	
27. 11 0 „		5 2
28. 6 40 A.M.	+0 23	
28. 1 38 P.M		7 21
28. 6 20 „	−0 21	
28 Midnight		5 19
29 7 20 A M.	+0 21	
29. 2 10 P.M.		7 11
29. 7 40 „	+0 17	
30 1 30 A M		6 7
30. 8 30 „	+0 50	
30. 3 45 P M.		8 5
30 9 0 „	+0 56	
July 1. 12 40 A.M.		4 36
1. 9 0 „	+0 38	
1. 4 35 P M		8 13
1. 9 40 „	+0 54	
2. 4 20 A M.		7 34
2 9 38 „	+0 33	
2. 4 10 P.M.		7 5
2 10 45 „	+1 16	
3. 4 45 A.M.		6 0
3. 10 30 „	+0 39	
3. 5 4 P.M		7 13
3 11 13 „	+0 58	
4. 5 0 A.M.		6 45
4. 10 45 „	+0 6	
4. 5 30 P.M.		6 51
5. 12 15 A.M.	+1 12	
5. 6 0 „		6 57
5. 12 45 P.M.	+1 17	
5. 6 15 „		6 47
6. 1 0 A.M.	+1 8	
6. 8 0 „		8 8
6 1 0 P.M.	+0 40	
6. 6 40 „		6 20
7. 2 0 A.M.	+1 16	
7. 8 0 „		7 16
Mean	+0 7·05	6 35·35

Having corrected the curve of Lunitidal Intervals for the Diurnal Inequality, the remainder is the acceleration or retardation on the time of the Semidiurnal Tide.

We have, by equation (5),

$$\tan 2B = \frac{M' \sin 2\iota_m + S' \sin 2\,(m-s+\iota_s)}{M' \cos 2\iota_m + S' \cos 2\,(m-s+\iota_s)}.$$

Differentiating this expression so as to obtain for B a maximum value, we find, as the equation of condition,

$$0 = M' \cos 2\,(\overline{m-s} - \overline{i_m - i_s}) + S'. \qquad (10)$$

Substituting in (5) we obtain

$$\tan 2B = \frac{\sqrt{M'^2 - S'^2}\sin 2\iota_m + S' \cos 2\iota_m}{\sqrt{M'^2 - S'^2}\cos 2\iota_m - S' \sin 2\iota_m}; \qquad (11)$$

and assuming

$$\frac{S'}{M'} = \sin 2\theta,$$

we find, after reduction,

$$\tan 2B = \tan 2\,(i_m + \theta);$$

and, finally,

$$2\,(B - i_m) = 2\theta$$

and

$$\frac{S'}{M'} = \sin 2\,(B - i_m). \qquad (12)$$

On examining the Lunitidal Curve, corrected for Diurnal Inequality, we find the following ranges from Springs to Neaps.—

High Water.	Low Water
h m	h m
+1 0	7 18
−1 6	5 30
2 6	1 48

or mean maximum range

$$2B = 1^h\ 57^m.$$

Although we have not found the value of ι_m, we may take as an approximation to it the Moon's mean Hour-Angle at High Water, already given in the Table,

$$\iota_m = +0^h\ 7^m.$$

Hence we have

$$2B - 2i_m = 1^h\ 57^m - 0^h\ 14^m,$$

or

$$\frac{S'}{M'} = \sin(1^h\ 43^m) = \sin(24^\circ\ 55') = 0{\cdot}421. \qquad (13)$$

Collecting together the partial results obtained at this most interesting Tidal Station, we obtain:—

Diurnal Tide.

1. Solitidal Interval,

$$i_s = -4^h\ 11^m.$$

2. Solar Coefficient, corrected for Declination,

$$S = 5{\cdot}0 \text{ inches.}$$

3. Lunitidal Interval,

$$i_m = -8^h\ 8^m$$

4. Solar Coefficient, corrected for Declination,

$$M = 4{\cdot}0 \text{ inches.}$$

Semidiurnal Tide.

1. Mean Lunitidal Interval,

High Water	Low Water
h m	h m
+0 7·05	6 35·35

2. Difference between Lunitidal and Solitidal Intervals,

$$i_m - i_s = 38^m$$

3 Approximate ratios of uncorrected Solar and Lunar Coefficients,

$$\frac{S'}{M'} = 0{\cdot}297 \text{ (Heights)}$$

$$= 0{\cdot}421 \text{ (Intervals).}$$

NOTE A.—Added July 1, 1875.

At the time of writing this paper I abandoned the attempt to determine the Lunar Diurnal Tide, in consequence of the breakdown of the observations which occurred in the neighbourhood of the 23rd June, which corresponds with one of the maxima of the Lunar Declination. This Tide may, however, be found from the tides of 8th and 9th June and 5th and 6th July, which also correspond to maxima of the Declination The Lunar Diurnal Tide is the difference between the Total Diurnal Tide and the Solar Diurnal Tide, which is determined in the paper

In the following Tables, the Solar Diurnal Tide is calculated from the formula

$$\text{Solar Tide} = 3{\cdot}58 \cos(s + 4^h\ 11^m),$$

founded on the constants

$$S = 5{\cdot}00 \text{ inches,}$$

$$i_s = -4^h\ 11^m.$$

North Declination of Moon a Maximum.

High Water.

	Diurnal Tide	Solar	Lunar
h m	inches	inches	inches.
June 7. 12 45 P.M.	−1·40	+1·20	−2·60
8. 1 0 A.M. .. .	+1·40	−0 76	+2·16
8. 12 40 P.M ..	−1·53	+1·06	−2·59
9. 2 0 A.M	+1·32	+0·17	+1 15
9 1 15 P.M. ...	−1·28	+0 53	−1·81
10. 2 38 A.M. ...	+1·34	+0·78	+0·56
		Mean	±1 812

Low Water.

June 7. 6 21 P.M.	−5 69	−3·28	−2·41
8. 7 20 A.M. ...	+5 70	+3·51	+2·19
8 8 0 P.M ..	−5·69	−3·58	−2·11
9. 8 22 A.M.	+5 72	+3 52	+2·20
9. 7 43 P.M. .	−5·75	−3·55	−2 20
10. 8 05 A.M	+5·81	+3·58	+2 23
		Mean ..	±2·223

High Water.

July 4 10 45 A.M ..	−0·25	+2·25	−2·50
5 12 15 A.M .	+0·18	−1·42	+1 60
5. 12 45 P.M	−0·37	+0·54	−0·91
6. 1 0 A.M.	+0·87	−0·76	+1·63
6. 1 0 P.M .	−0·33	+0 76	−1·09
		Mean .	±1·546

Low Water.

July 4. 5 30 P.M.	−5·88	−2 91	−2·97
5. 6 0 A.M.	+6·00	+3·18	+2·82
5. 6 15 P.M.	−6 13	−3·25	−2·88
6. 8 00 A.M.	+6 25	+3·58	+2·67
6. 6 40 P.M .	−6 25	−3·38	−2·87
		Mean .	±2·842

If m denote the Moon's Hour-Angle at High Water, we have at High Water,

$$\text{Lunar Tide} = M \sin \overline{2\mu} \cos \overline{m - \iota_m}, \qquad \text{(a)}$$

and at Low Water,

$$\text{Lunar Tide} = -M \sin \overline{2\mu} \sin \overline{m - \iota_m}. \qquad \text{(b)}$$

Hence we find (observing that the Lunar Tide has the same sign at High Water and Low Water)

$$\text{June 7, 8, 9} \quad . \quad . \quad . \quad . \quad . \quad \overline{2\mu} = 49° \ 30' \ \text{N}.$$

$$\tan \overline{m - \iota_m} = -\frac{2223}{1812},$$

$$m - \iota_m = -50° \ 49' \text{ or } +129° \ 11'$$

$$= -3^h \ 23^m \text{ or } +8^h \ 37^m.$$

The value of m, as found from all the observations, is given in the paper,

$$m = 0^h\ 7^m.$$

Hence we find

$$i_m = +3^h\ 30^m \text{ or } -8^h\ 30^m.$$

The signs of the Lunar Tide show that the negative value of i_m is the proper one, hence

$$i_m = -8^h\ 30^m. \quad \ldots \ldots \quad \text{(c)}$$

We have also, from (a) and (b),

$$\text{M} \sin \overline{2\mu} = \sqrt{(2\ 223)^2 + (1\ 812)^2} = 2{\cdot}861,$$

and, finally,

$$\text{M} = 3\ 77 \text{ inches} \quad \ldots \ldots \quad \text{(d)}$$

$$\text{July 4, 5, 6} \quad \ldots \quad \overline{2\mu} = 49^\circ\ 30'\ \text{N}.$$

$$\tan \overline{m - i_m} = -\frac{2842}{1546},$$

$$m - i_m = -61^\circ\ 27' \text{ or } +118^\circ\ 33'$$

$$= -4^h\ 6^m \text{ or } +7^h\ 54^m,$$

$$i_m = +4^h\ 13^m \text{ or } -7^h\ 47^m, \quad \ldots \ldots \quad \text{(e)}$$

of which the latter value must be used.

We have also

$$\text{M} \sin \overline{2\mu} = \sqrt{(1\ 546)^2 + (2\ 842)^2} = 3\ 235,$$

and, finally,

$$\text{M} = 4\ 26 \text{ inches} \quad \ldots \ldots \quad \text{(f)}$$

The mean values of i_m and M, deduced from the preceding equations, are

$$i_m = -8^h\ 8^m, \quad \ldots \ldots \quad \text{(g)}$$

$$\text{M} = 4\ 00 \text{ inches} \quad \ldots \ldots \quad \text{(h)}$$

The Lunar Diurnal Tide is therefore expressed by the equation

$$\text{Lunar Tide} = 4 \sin \overline{2\mu} \cos(m + 8^h\ 8^m). \quad \ldots \ldots \quad \text{(i)}$$

XII. *On the Tides of the Arctic Seas.*

By the Rev. SAMUEL HAUGHTON, *M.D. Dubl., D.C.L Oxon., F.R.S., Fellow of Trinity College, Dublin.*

Part V. *On the Tides of Refuge Cove, Wellington Channel.*

Received July 11,—Read November 19, 1874.

THE following observations, like those at Northumberland Sound, were made on board H.M.S. 'Assistance,' under the command of SIR EDWARD BELCHER, R.N., K.C B. They were made from 16th September to 11th October 1853. Although the period of observation is so short, yet, owing to the fact that it was the time of Equinox, some useful information has been obtained as to the Lunar Diurnal Tide at this Station.

The position of Refuge Cove is

Lat. 75° 31′ N.
Long. 92° 10′ W.

The following Table contains the Height of each High and Low Water, and the Height of the Diurnal Tide, calculated by the second difference of the heights.

TABLE I.—Refuge Cove.

Time			High Water Height	Low Water Height	Diurnal Tide at High Water	Diurnal Tide at Low Water
1853		h m	ft. in	ft in	ft.	ft.
Sept	17	6 30 A.M.		6 0		
	17	12 30 P.M.	10 6½			
	17.	6 25 "		5 3	....	0·313
	18.	1 0 A.M.	11 8½		0·573	
	18	7 30 "	...	5 9		0·271
	18	12 50 P.M.	10 7	...	0·477	
	18	7 0 "	...	5 2	....	0·224
	19.	1 30 A.M.	11 4		0·393	
	19.	7 40 "	..	5 6		0·193
	19.	1 45 P.M.	10 6		0·271	
	19.	7 55 "	...	5 2		0·167
	20.	2 0 A.M.	10 11	..	0·135	
	20.	8 20 "	.....	5 8	.	0·094
	20.	2 10 P.M.	10 11	...	0·057	
	20.	8 0 "	..	5 10		0·005
	21.	2 5 A.M.	11 0	...	0·055	
	21.	8 30 "	....	6 0	.	0·083
	21.	2 25 P.M.	10 9	...	0·026	
	21.	8 40 "	......	6 5		0·203
	22.	2 54 A.M.	10 7½	....	0·046	
	22.	8 45 "	......	6 0	..	0·323
	22.	3 30 P.M.	10 8	...	0·120	
	22.	9 45 "	......	6 11	...	0·234

TABLE I. (continued).

Time		High Water Height.	Low Water Height	Diurnal Tide at High Water	Diurnal Tide at Low Water
1853	h m	ft in	ft in	ft	ft.
Sept. 23.	3 30 A.M.	10 2		0·112	
23.	9 50 ,,		6 2		0·439
23.	3 55 P.M.	10 4		0·005	
23.	10 30 ,,		7 6		0·516
24.	4 0 A.M.	10 7		0·041	
24.	10 30 ,,		6 9		0·594
24.	5 0 P.M.	10 3		0·125	
24.	11 10 ,,		8 0		0·604
25.	5 10 A.M.	9 4		0·266	
25.	10 20 ,,		6 10		0·687
25.	5 30 P.M.	9 10		0·234	
26.	12 30 A.M.		8 6		0·698
26.	5 45 ,,	9 6		0·240	
26.	12 30 P.M.		7 3		0·661
26.	6 30 ,,	9 10		0·401	
27.	1 15 A.M.		8 6		0·651
27.	7 0 ,,	8 6		0·531	
27	1 30 P.M.		7 1		0·635
27.	7 30 ,,	9 7		0·594	
28.	3 0 A.M.		8 3		0·578
28.	8 0 ,,	8 8		0·740	
28.	3 30 P.M.		7 1		0·516
28.	10 5 ,,	10 10		0·849	
29.	4 0 A.M.		7 11		0·455
29.	10 25 ,,	9 4		0·799	
29	4 0 P.M.		6 10		0·375
29	10 40 ,,	11 0		0·740	
30	5 0 A.M.		7 2½		0·328
30.	10 45 ,,	9 10½		0·734	
30	4 45 P.M.		6 5		0·344
30.	11 45 ,,	11 10		0·672	
Oct. 1.	6 0 A.M.		7 1		0·331
1.	11 0 ,,	11 0		0·565	
1	5 15 P.M.		6 4		0·208
1.	11 45 ,,	12 3		0·490	
2.	5 0 A.M.		6 7		0·182
2.	12 30 P.M.	11 5		0·354	
2.	6 15 ,,		6 3		0·115
3	12 20 A.M.	12 2		0·224	
3.	6 0 ,,		6 8		0·110
3.	1 0 P.M.	12 6		0·078	
3.	6 45 ,,		7 4		0·099
3	Midnight	13 0		0·088	
4	7 15 A.M.		7 4		0·188
4	Noon	13 0		0·094	
4.	8 0 P.M.		7 10		0·297
5.	1 0 A.M.	13 3		0·094	
5.	8 30 ,,		7 0		0·380
5.	1 28 P.M.	13 0		0·094	
5.	8 20 ,,		7 9		0·386
6	1 40 A.M.	13 0		0·135	
6.	8 30 ,,		6 11		0·380
6	1 0 P.M.	12 6		0·182	
6.	9 30 ,,		7 8		0·386
7	4 0 A.M.	12 11		0·125	
7.	9 0 ,,		7 1		0·484
7.	3 0 P.M.	12 9		0·026	
7.	9 45 ,,		8 5		0·615

TABLE I. (continued).

Time.			High Water Height.	Low Water Height.	Diurnal Tide at High Water	Diurnal Tide at Low Water
1853.		h m	ft. in	ft in	ft	ft.
Oct.	8.	4 0 A.M....	12 6	..	0·031	
	8.	10 0 „	....	7 2	...	0·688
	8.	5 0 P.M.	12 3	...	0·172	
	8.	10 50 „	..	8 8	.. ..	0·755
	9.	4 15 A.M. ..	11 5		0·370	
	9.	10 50 „	.	7 3	.	0 823
	9.	4 40 P.M	12 2	.	0·547	
	10.	12 10 A.M.. .	.	9 4		0 823
	10.	4 30 „	10 9	. .	0·651	
	10.	12 30 P.M.. .		8 0		0·709
	10.	6 45 „ . .	12 2		0·646	
	11	2 0 A.M.. ...		9 6		
	11.	6 40 „ ...	11 0	.. .		

A. *Diurnal Tide.*

The general expression for the Diurnal Tide is

$$D=M\sin 2\bar{\mu}\cos(m-i_m)+S\sin 2\bar{\sigma}\cos(s-i_s), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

which at the Equinoxes reduces simply to the Lunar Tide, viz.

$$D=M\sin 2\bar{\mu}\cos(m-i_m) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If the Tides be plotted carefully to scale, it appears that the Diurnal Tides in height vanish together at High Water and Low Water, when $\mu=0$, or nearly so

The mean interval from the time of the Moon's declination vanishing to the disappearance of the Diurnal Inequality is about 36 hours, which may be regarded as the age of the Lunar Diurnal Tide. It is evident from equation (2) that if h and l represent the range of Tide at High Water and Low Water respectively, since the phase changes by 90° from High Water to Low Water, we have the following equations to determine the unknown constants i_m and M —

$$\cot(m-i_m)=\pm\frac{h}{l}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$2M\sin 2(\overline{\text{max. value of }\mu})=\sqrt{h^2+l^2} \quad . \quad . \quad . \quad . \quad (4)$$

The mean maximum values of h and l were found to be

$$h=0{\cdot}849\text{ foot,}$$
$$l=0{\cdot}761\text{ foot,}$$

hence we find

$$\cot(m-i_m)=\pm\frac{849}{761},$$

$$m-i_m=41^\circ\,52'\text{ or }-138^\circ\,8'$$
$$=2^h\,53^m\text{ or }-9^h\,7^m \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The mean values of m at High Water and Low Water, as appears from the following Table, are:—

$$\text{High Water} \quad . \quad . \quad . \quad . \quad m = -0^{\mathrm{h}}\ 27^{\mathrm{m}}$$

$$\text{Low Water} \quad . \quad . \quad . \quad . \quad m = 6^{\mathrm{h}}\ 1^{\mathrm{m}}$$

or, reducing both to High-Water Standard,

$$\begin{array}{rr} m=-0 & 27 \\ -0 & 11 \\ \hline \text{Mean} \quad . \quad . \quad . \quad . \quad -0 & 19 \end{array}$$

Hence, by equation (5),

$$-0^{\mathrm{h}}\,19^{\mathrm{m}} - i_m = 2^{\mathrm{h}}\,53^{\mathrm{m}},$$

$$i_m = -3^{\mathrm{h}}\,12^{\mathrm{m}},$$

or

$$-0^{\mathrm{h}}\,19^{\mathrm{m}} - i_m = -9^{\mathrm{h}}\ \ 7^{\mathrm{m}},$$

$$i_m = +8^{\mathrm{h}}\,48^{\mathrm{m}}.$$

An examination of the signs of the Diurnal Tide shows that we must select the value

$$i_m = +8^{\mathrm{h}}\,48^{\mathrm{m}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 5\ (bis)$$

From equation (4) we find

$$\mathrm{M} = \frac{\sqrt{(0\cdot849)^2 + (0\cdot761)^2}}{2 \sin 49^\circ}$$

$$= 0\cdot76 \text{ foot} = 9\cdot06 \text{ inches.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

If we plot the Lunitidal Intervals at High Water and Low Water to scale, from the following Table we obtain the Diurnal Inequality in time. It produces a maximum acceleration or retardation in the time of Tide, amounting to 39 minutes.

The following Table gives the Lunitidal Intervals at High Water and Low Water.

TABLE II.—Refuge Cove. Lunitidal Intervals.

	High Water	Low Water		High Water	Low Water
1853 h m	h m	h m	1853 h m	h m	h m
Sept. 17. 6 30 A.M.		6 35	Sept. 21. 8 30 A.M.		6 19
17. 12 30 P.M.	+0 11		21. 2 25 P.M.	−0 48	
17. 6 25 „		5 54	21. 8 40 „		6 33
18. 1 0 A.M.	+0 20		22. 2 54 A.M.	−0 39	
18. 7 30 „		5 10	22. 8 45 „		6 48
18. 12 50 P.M.	−0 14		22. 3 30 P.M.	−0 27	
18. 7 0 „		6 4	22. 9 45 „		6 12
19 1 30 A.M.	+0 7		23 3 30 A.M.	−0 49	
19. 7 40 „		5 43	23. 9 50 „		6 29
19. 1 45 P.M.	−0 2		23. 3 55 P.M.	−0 48	
19. 7 55 „		5 52	23. 10 30 „		6 13
20. 2 0 A.M.	−0 6		24. 4 0 A.M.	−1 6	
20. 8 20 „		5 46	24. 10 40 „		6 26
20. 2 10 P.M.	−0 20		24. 5 0 P.M.	−0 30	
20. 8 0 „		6 30	24. 11 10 „		6 20
21. 2 5 A.M.	−0 44		25. 5 10 A.M.	−0 45	

TABLE II. (continued).

1853 h m	High Water h m	Low Water. h m
Sept. 25. 10 20 A.M.	..	7 35
25. 5 30 P.M.	−0 49	
26. 12 30 A.M.	...	5 49
26. 5 45 „	−1 1	
26. 12 30 P.M.	..	6 16
26. 6 30 „	−0 40	
27. 1 15 A.M.	..	5 55
27. 7 0 „	−0 37	
27. 1 30 P.M.	..	6 7
27. 7 30 „	−0 31	
28. 3 0 A.M.	..	5 1
28. 8 0 „	−0 29	
28. 3 30 P.M.	..	4 59
28. 10 5 „	+1 12	
29. 4 0 A.M.	...	4 53
29. 10 25 „	+1 6	
29. 4 0 P.M.		5 19
29. 10 40 „	+0 57	
30. 5 0 A.M.	.	4 43
30. 10 45 „	+0 36	
30. 4 45 P.M.	.	5 24
30. 11 45 „	+1 12	
Oct. 1. 6 0 A.M.	.	4 33
1. 11 0 „	+0 2	
1. 5 15 P.M.		6 17
1. 11 45 „	+0 23	
2. 5 0 A.M.	.	6 22
2. 12 30 P.M.	+0 18	
2. 6 15 „	..	5 57
3. 12 20 A.M.	−0 16	
3. 6 0 „		6 36
3. 1 0 P.M.	+0 24	
Oct. 3. 6 45 P.M.	.	5 51
3. Midnight	−1 0	
4. 7 15 A.M.	...	5 45
4. Noon	−1 27	
4. 8 0 P.M.		5 27
5. 1 0 A.M.	−0 51	
5. 8 30 „		5 21
5. 1 28 P.M.	−0 53	
5. 8 20 „		6 1
6. 1 40 A.M.	−1 5	
6. 8 30 „	....	6 15
6. 1 0 P.M.	−2 17	
6. 9 30 „	.	5 47
7. 4 0 A.M.	+0 19	
7. 9 0 „	..	6 41
7. 3 0 P.M.	−1 16	
7. 9 45 „	...	6 31
8. 4 0 A.M.	−0 40	
8. 10 0 „	...	6 40
8. 5 0 P.M.	−0 17	
8. 10 50 „	.	6 27
9. 4 15 A.M.	−1 26	
9. 10 50 „	.	6 51
9. 4 50 P.M.	−1 28	
10. 12 10 A.M.	...	6 8
10. 4 30 „	−2 12	
10. 12 30 P.M.		6 12
10. 6 45 „	−0 31	
11. 2 0 A.M.	.	5 16
11. 6 40 „	−1 0	
Mean	−0 26·7	6 1·1

If we compare the mean Lunitidal Intervals here found with the corresponding intervals at Northumberland Sound, we find.—

Mean Lunitidal Interval.

	High Water h m	Low Water h m
Refuge Cove	−0 26·7	6 1·1
Northumberland Sound	+0 7·05	6 35 35
Difference	33·75	34 25

These differences represent the time of the Atlantic Tide-wave passing from Refuge Cove to Northumberland Sound (uncorrected for longitude), and their agreement is a proof of the accuracy of the observations at both places.

B. *Semidiurnal Tide* (Heights).

When Table I. is plotted to scale, it is easy to correct the tide for the Diurnal Inequality, or to do so by the Diurnal Tide at High Water and Low Water given in

that Table. When this correction is made we find the following Spring and Neap Ranges:—

		ft.	in.
Springs.—19th September, 1.30 A.M.	. . . .	5	8
5th October, 1.28 P.M.	. . . .	5	$8\frac{1}{2}$
Neaps.— 27th September, 7.0 A.M.	. . . .	1	4

Using the formula for the Semidiurnal Tide,

$$T=M'\cos 2(m-\iota_m)+S'\cos 2(s-i_s), \quad . \; . \; . \; . \; . \; . \; . \; . \; . \qquad (7)$$

$$T=A\cos 2(m-B),$$

we find at Springs

$$(M'+S')=34{\cdot}1 \text{ inches},$$

and at Neaps

$$(M'-S')=\ 8 \text{ inches},$$

from which we obtain

$$2\,M'=42{\cdot}1 \text{ inches},$$

$$2\,S'\ =26{\cdot}1 \quad ,,$$

$$\frac{S'}{M'}=\ 0{\cdot}621.$$

C. *Semidiurnal Tide* (Intervals).

When Table II. is plotted to scale, and the Tide corrected for the Diurnal Inequality, we obtain the following results, making use of the formulæ given in discussing the Tide at Northumberland Sound:—

Maximum Value of 2 B.

Range of Lunitidal Interval at High Water.

		h	m
29th September, 10.25 A.M.	. . . .	+1	6
9th October, 4.50 P.M.	. . . .	−1	24
	2 B=	2	30

Range of Lunitidal Interval at Low Water.

		h	m
25th September, 10.20 A.M.	. . . .	+6	48
28th September, 3.30 P.M.	. . . .	+4	59
	2 B=	1	49

The approximate value of i_m, taken from the mean of the observations, is,

		h	m
At High Water	. . . .	−0	27
Low Water	. . . .	−0	11

Hence we have

		$2(B-i_m)$.
	h m	
High Water . . .	3 24	. . . . . . $49° 30'$
Low Water	2 3	. . . . . . $30 \quad 0$

Hence we obtain

$$\frac{S'}{M'}=\sin 2(B-i_m)=0\ 76 \text{ High Water}$$
$$=0{\cdot}50 \text{ Low Water.}$$
$$\text{Mean} \quad . \quad . \quad \overline{0\ 63}$$

Collecting the several constants, we obtain :—

Diurnal Tide.

$i_m=+8^h\ 48^m$,
$M'=\ 9{\cdot}06$ inches.

Semidiurnal Tide

$\frac{S'}{M'}=0\ 62$ (Heights),

$\frac{S'}{M'}=0{\cdot}63$ (Intervals).

	High Water	Low Water.
Mean Lunitidal Interval=	$-0^h\ 26^m\ 7$	$6^h\ 1^m{\cdot}1$

XIII. *On the Tides of the Arctic Seas.*

By the Rev. SAMUEL HAUGHTON, *M.D Dubl., D.C.L. Oxon., F.R.S., Fellow of Trinity College, Dublin.*

Part VI. *Tides of Port Kennedy, in Bellot Strait.*

Received January 20,—Read March 4, 1875.

THE following observations were made by Admiral Sir LEOPOLD M'CLINTOCK (then Captain) in the month of July 1859, at Port Kennedy, in Bellot Strait, on board the yacht 'Fox,' in eleven fathoms depth, lat. 72° 01′ N, long. 94° 15′ W. The observations were made hourly, and, judging from the consistent and highly interesting results obtained from them, they must have been made and recorded with unusual care

In Table I. the first column contains the Solar Hour, the second contains the Height of Tide, the third contains the Diurnal Tide, and the fourth the Semidiurnal Tide.

The Diurnal Tide was calculated as follows:—Let h_1, h_2, h_3 be the heights of the water at three periods, separated by intervals of twelve hours each, then the Diurnal Tide corresponding to the height h_2 is

$$D=\frac{h_1-2h_2+h_3}{4}. \qquad \ldots \qquad \ldots \qquad (1)$$

The third column was calculated from the second by this formula, and the fourth column, containing the Semidiurnal Tide only, is the algebraical sum of the second and third columns.

TABLE I.—Hourly Values of the Diurnal and Semidiurnal Tide at Port Kennedy in July 1859.

5th July.

Time	Height	Diurnal Tide	Semidiurnal Tide
	ft in	ft. in.	ft in
Noon.			
1.	5 2		
2.	5 11		
3	6 2½		
4.	6 3		
5.	6 0½		
6.	5 7		
7.	4 9½		
8	4 2½		
9.	3 9½		
10.	3 7		
11	4 0½		
Midnight.	5 2		
13.	6 10	−0 8	6 2
14.	8 1	−1 2	6 11
15.	9 3	−1 5¾	7 9¼
16	10 0	−1 8¾	8 3¼
17.	10 0½	−1 9½	8 3
18	9 3½	−1 6¼	7 8¼
19.	7 11½	−1 3	6 8
20.	6 8	−0 10¾	5 9¼
21.	5 5	−0 6½	4 10½
22.	4 8	−0 3¼	4 4¼
23.	4 1	+0 1	4 2
		Mean	6 8¾

6th July

Time	Height	Diurnal Tide	Semidiurnal Tide
Noon.	4 2	+0 5	4 7
1.	4 9½	+0 6	5 3½
2	5 8½	+0 11	6 7½
3	6 4½	+1 2	7 6½
4.	6 10	+1 3½	8 1½
5.	6 11	+1 5½	8 4½
6.	6 9	+1 2½	7 11½
7.	6 2	+0 11¼	7 1¼
8	5 6½	+0 7¼	6 2¼
9	4 11	+0 4	5 3
10	4 8	+0 1	4 9
11.	4 5½	−0 2	4 3½
Midnight.	4 10½	−0 5¼	4 5¼
13	5 9	−0 7½	5 1½
14.	7 0	−0 10½	6 1½
15.	8 1½	−1 0¾	7 0¾
16.	8 10	−1 1	7 9
17.	9 8	−1 4½	8 3½
18	9 1	−1 4	7 9
19	8 1½	−0 9	7 4½
20	7 0	−0 5	6 7
21.	5 9½	−0 1½	5 8
22.	5 0	+0 1½	5 1½
23.	4 2	+0 5¼	4 7¼
		Mean .	6 4

7th July.

Time	Height.	Diurnal Tide	Semidiurnal Tide
	ft in	ft. in	ft in
Noon	3 10	+0 9¼	4 7¼
1	4 0	+0 11¼	4 11¼
2.	4 9½	+1 0¾	5 10¼
3	5 8	+1 1¾	6 9¾
4.	6 5½	+1 1	7 6½
5.	6 11	+1 2¼	8 1¼
6	7 1	+1 0½	8 1½
7	7 0½	+0 8¾	7 9½
8.	6 9	+0 5	7 2
9.	6 2	+0 2	6 4
10	5 10	−0 1½	5 8½
11	5 8	−0 5	5 3
Midnight	5 10½	−0 9½	5 1
13	6 0	−0 10¼	5 1¾
14.	6 10	−1 0	5 10
15	7 9	−1 0¾	6 8¼
16	8 5	−0 11½	7 5½
17.	8 11	−0 11	8 0
18.	9 2½	−0 10¼	8 4¼
19	8 10½	−0 7½	8 3
20	8 2½	−0 4¼	7 10¼
21	7 2½	+0 1	7 3½
22.	6 2	+0 3¼	6 5¼
23.	5 6	+0 6½	6 0½
		Mean	6 7¾

8th July

Time	Height	Diurnal Tide	Semidiurnal Tide
Noon.	4 9½	+0 9½	5 7
1.	4 7	+0 10½	5 5½
2.	4 11	+1 0¼	5 11¼
3.	5 7	+1 0¼	6 7¼
4.	6 6	+0 10	7 4
5.	7 2½	+0 9	7 11½
6	7 10½	+0 6¾	8 5¼
7	8 2½	+0 3¼	8 6¼
8	8 3	+0 1	8 4
9.	7 11	−0 2¼	7 8¾
10.	7 7	−0 5¾	7 1¼
11	7 2	−0 8	6 6
Midnight	6 11	−0 10½	6 0½
13	6 8	−0 11¼	5 8¾
14.	7 1½	−1 1	6 0½
15.	7 6	−1 0	6 6
16	7 11½	−0 9½	7 2
17.	8 6	−0 8	7 10
18.	8 9½	−0 4½	8 5
19.	8 9½	−0 1¾	8 7¾
20.	8 7	+0 1	8 8
21	7 10½	+0 5	8 3½
22.	7 1	+0 8¼	7 9½
23.	6 2½	+0 11	7 1½
		Mean ...	7 2¾

TABLE I. (continued).

9th July.

Time	Height	Diurnal Tide	Semidiurnal Tide
	ft in	ft in.	ft. in
Noon.	5 7	+1 1	6 8
1.	5 0	+0 11	5 11
2.	5 0½	+0 10¾	5 11¼
3.	5 5	+1 0	6 5
4	6 2½	+0 11½	7 2
5.	7 1	+0 8½	7 9½
6.	8 2	+0 3¼	8 5¼
7.	8 9½	±0 0	8 9½
8	9 2½	−0 3	8 11½
9.	9 5½	−0 11	8 6½
10.	9 4	−0 10¾	8 5¼
11.	8 11	−1 1	7 10
Midnight.	8 7	−1 2¼	7 4¾
13.	7 0	−0 9	6 3
14.	6 7	−0 7½	5 11½
15.	7 3½	−0 10¼	6 5¼
16.	8 4	−1 1	7 3
17.	8 6	−0 9½	7 8½
18.	8 7½	−0 1	8 6½
19	8 10	+0 1	8 11
20.	8 9½	+0 3½	9 1
21.	8 4	+0 6	8 10
22.	8 0	+0 8¼	8 8¼
23.	7 3	+0 11¾	8 2¼
		Mean	7 8

10th July

Time	Height	Diurnal Tide	Semidiurnal Tide
Noon.	6 10	+1 0¾	7 10¾
1.	6 0	+0 11	6 11
2.	5 7½	+0 10¼	6 5¾
3	5 9	+1 1	6 10
4.	6 1	+1 2	7 3
5.	6 9	+0 10	7 7
6.	7 8½	+0 4¼	8 0¾
7	8 9½	−0 0¾	8 8¾
8.	9 6½	−0 6	9 0½
9.	9 2	−0 5	8 9
10	9 5	−0 8¼	8 8¾
11.	9 6	−1 0¾	8 5¼
Midnight.	9 4	−1 2	8 2
13	8 9½	−1 1½	7 8
14	8 1½	−0 10¼	7 2¾
15	8 7	−1 1¼	7 5¾
16.	8 6	−1 0	7 6
17.	8 4	−0 9	7 7
18.	8 4	−0 6	7 10
19.	8 4	−0 2½	8 1½
20.	8 4	+0 3	8 7
21.	8 4	+0 3¾	8 7¾
22.	8 1	+0 9½	8 10½
23.	7 6	+1 1¼	8 7¾
		Mean	7 11½

11th July.

Time	Height.	Diurnal Tide	Semidiurnal Tide
	ft. in.	ft in	ft in
Noon.	7 2	+1 3½	8 5½
1.	7 1	+1 1	8 2
2.	7 0½	+0 9	7 9½
3.	7 0	+0 10	7 10
4.	6 11½	+0 9¼	7 8¾
5.	6 11½	+0 8¾	7 8¼
6.	7 0	+0 8½	7 8½
7.	7 1	+0 8	7 9
8.	8 1	+0 2	8 3
9	8 9	−0 4	8 5
10.	9 11	−0 11¾	8 11¼
11	10 1	−1 4	8 9
Midnight	10 1½	−1 6½	8 7
13.	9 8	−1 6	8 2
14.	9 0	−1 1½	7 10½
15.	8 9	−1 5	7 4
16.	8 6	−1 4¼	7 1¼
17.	8 6	−1 4½	7 1½
18.	8 6	−0 11¼	7 6¾
19.	8 6	−0 11	7 7
20.	8 6	−0 4½	8 1½
21.	7 9½	+0 2	7 11½
22.	7 10	+0 8½	8 6½
23.	7 4½	+1 6¼	8 10¾
		Mean	7 11¾

12th July

Time	Height	Diurnal Tide	Semidiurnal Tide
Noon.	6 10½	+1 8¼	8 6¾
1	6 2½	+1 8¾	7 11¼
2	6 5½	+1 3¼	7 8¾
3.	4 9½	+1 11	6 8½
4	4 5	+1 8¼	6 1¼
5.	4 6	+1 9¼	6 3¼
6.	5 0½	+1 5	6 5½
7.	6 3	+0 9½	7 0½
8	7 5	+0 2½	7 7½
9.	7 6½	+0 0½	7 7
10	8 7½	−0 5¼	8 2¼
11.	10 9	−1 7½	9 1½
Midnight.	10 5	−1 7½	8 9½
13.	9 8	−1 7½	8 0½
14.	9 0½	−1 4½	7 8
15	8 5½	−1 8	6 9½
16.	8 1	−1 10¾	6 2¼
17.	7 7	−1 6¼	6 0¾
18.	7 3	−1 1½	6 1½
19.	7 1½	−0 6½	6 7
20.	7 2	+0 0½	7 2½
21.	7 6	+0 9¼	8 3¼
22.	7 8	+0 9	8 5
23.	7 8	+1 5¼	9 1¼
		Mean ..	7 1¼

TABLE I. (continued)

13th July

Time	Height	Diurnal Tide	Semidiurnal Tide
	ft in.	ft in	ft in
Noon.	7 5	+1 7¾	9 0¾
1.	6 7	+2 0	8 7
2.	6 2	+1 11¼	8 1¼
3.	5 6	+1 7	7 1
4.	4 2	+2 2¼	6 4¼
5.	4 7	+1 7	6 2
6	4 11	+1 2¼	6 1¾
7.	5 10	+0 8	6 6
8.	7 1	+0 0¼	7 1¼
9.	8 7	−0 7¼	7 11¼
10.	9 6	−0 11¼	8 6¼
11.	10 4	−1 4	9 0
Midnight	11 0	−1 9	9 3
13	11 6½	−2 3½	9 3
14.	11 1	−2 4¼	8 8¼
15.	10 2	−2 2¼	7 10¼
16.	9 0	−2 2½	6 9½
17.	7 11	−1 8¼	6 2¼
18.	7 6	−1 4¼	6 1¼
19	7 2½	−0 11½	6 3
20.	7 1	−0 3¼	6 9¾
21	7 3½	+0 4¼	7 8¼
22.	7 7	+0 10¼	8 5¼
23.	7 8½	+1 2½	8 11
		Mean	7 7½

14th July

Time	Height	Diurnal Tide	Semidiurnal Tide
Noon.	7 7½	+1 8¾	9 4¼
1.	7 3½	+2 2	9 5½
2.	6 5	+2 5½	8 10½
3	5 7	+2 4¾	7 11¼
4	5 0	+2 2¼	7 2¼
5	4 6	+1 10½	6 4½
6.	4 5	+1 6¾	5 11¼
7.	4 9	+1 2½	5 11½
8	6 1	+0 5½	6 6½
9.	7 7	−0 2½	7 4½
10.	9 1	−0 9	8 4
11.	9 11	−1 5½	8 5½
Midnight	11 2	−1 5	9 9
13	11 8	−1 9½	9 10½
14	11 6½	−2 4¼	9 2¼
15.	10 7	−2 3¾	8 3¼
16.	9 9	−2 2½	7 6½
17.	8 7	−2 0¼	6 6¾
18.	7 7	−1 7½	5 11½
19.	7 2	−1 3	5 11
20	7 0	−0 7	6 5
21.	7 1	+0 0¾	7 1¾
22.	7 6½	+0 6¾	8 1¼
23.	7 11½	+0 10¾	8 10¼
		Mean .	8 1¾

15th July.

Time	Height.	Diurnal Tide	Semidiurnal Tide
	ft in.	ft. in	ft in
Noon.	8 1	+1 3½	9 4½
1.	7 10	+1 5½	9 3½
2.	7 3	+1 9	9 0
3.	6 4	+1 11½	8 3½
4	5 8	+1 10½	7 6½
5	4 7	+1 10¼	6 5¼
6	4 2½	+1 7¼	5 10¼
7.	4 6½	+1 3½	5 10
8.	5 5	+0 9½	6 2½
9.	6 10	+0 2	7 0
10.	8 3	−0 4	7 11
11.	9 7	−0 9¾	8 9¼
Midnight.	9 10	−0 11	8 11
13.	9 10	−0 11½	8 10½
14.	9 11	−1 3	8 8
15.	9 11	−1 7½	8 3½
16.	9 1	−1 7½	7 5¼
17.	8 2	−1 7½	6 6½
18.	7 5	−1 6	5 11
19.	7 0½	−1 3½	5 9
20	7 0½	−0 11½	6 1
21.	7 2½	−0 3¼	6 11¼
22.	7 7½	+0 2¼	7 10¼
23.	7 11½	+0 6¼	8 5¼
		Mean	7 6¼

16th July.

Time	Height	Diurnal Tide	Semidiurnal Tide
Noon.	7 11½	+0 10½	8 10
1	8 0½	+1 1¼	9 2¼
2.	7 7	+1 7	9 2
3.	7 0½	+1 10½	8 11
4.	6 1	+1 10	7 11
5	5 3	+1 9¾	7 0¾
6.	4 7	+1 7¾	6 2¼
7.	4 5	+1 5½	5 10½
8.	4 10	+1 1½	5 11½
9.	6 6	+0 3¾	6 9¾
10	7 11	−0 0½	7 10½
11	8 5	−0 4	8 1
Midnight.	9 7	−0 9½	8 9½
13.	10 10	−1 1	9 9
14.	11 7	−1 10½	9 8½
15.	11 8	−2 2½	9 5½
16.	10 5	−1 11¾	8 5¼
17.	9 7	−1 11½	7 7½
18	8 4	−1 8¼	6 7¾
19.	7 8	−1 3½	6 4½
20.	7 2	−1 2½	5 11½
21.	7 0½	−0 6¼	6 6¼
22.	7 0	+0 2¼	7 2¼
23.	7 6	+0 4¼	7 10¼
		Mean ...	7 9

TABLE I (continued).

17th July.

Time	Height.	Diurnal Tide	Semidiurnal Tide
	ft in	ft in	ft in
Noon.	8 0	+0 8½	8 8½
1.	8 3	+1 1	9 4
2	8 1	+1 7¼	9 8¾
3.	7 6	+2 0¼	9 6¼
4.	6 10	+1 11½	8 9½
5.	6 0½	+1 10¾	7 11¼
6.	5 4	+1 7¼	6 11¼
7.	4 9	+1 5¼	6 2¼
8.	4 8½	+1 2½	5 11
9.	5 6	+0 11½	6 5½
10.	6 10	−0 0½	6 9½
11	8 0	−0 4½	7 7½
Midnight.	9 3	−0 8½	8 6½
13	10 0	−0 11	9 1
14.	11 2	−1 5	9 9
15.	11 5	−1 10	9 7
16.	11 1	−1 11½	9 1½
17.	10 1	−1 10¼	8 2¾
18	8 9	−1 7¼	7 1¾
19.	7 9	−1 4½	6 4½
20.	7 1	−1 1½	5 11½
21.	6 9½	−0 8¾	6 0¾
22.	6 6	±0 0	6 6
23.	7 0	+0 4¾	7 11¼
		Mean .	7 10

18th July

Time	Height.	Diurnal Tide	Semidiurnal Tide
Noon.	7 8	+0 10¾	8 6¾
1.	8 1½	+0 10¼	8 11¾
2.	8 2½	+1 2½	9 5
3	8 0½	+1 3½	9 4
4.	7 6	+1 6¼	9 0¼
5	6 8	+1 7¼	8 3¼
6.	5 9	+1 7¼	7 4¼
7.	5 3	+1 3½	6 6½
8.	5 0	+0 11¼	5 11¼
9.	5 2	+0 8	5 10
10.	6 2	+0 2	6 4
11.	7 5	−0 2½	7 2½
Midnight	8 5½	−0 6½	7 11
13.	9 8	−0 11	8 9
14.	10 1	−1 0	9 1
15.	10 2	−1 0½	9 1½
16.	10 0	−1 2½	8 9½
17.	9 8	−1 4½	8 3½
18.	9 1½	−1 7	7 6½
19.	7 11	−1 3¼	6 7¾
20.	6 8	−0 9¼	5 10¾
21.	6 3	−0 6¾	5 8¼
22.	6 6	−0 4¼	6 1¾
23.	7 0½	−0 2	6 10½
		Mean ..	7 7¾

19th July.

Time	Height	Diurnal Tide	Semidiurnal Tide
	ft in	ft in	ft in
Noon	7 0½	+0 6¼	7 6¾
1.	7 6	+0 10¾	8 4¾
2.	8 0	+0 11¼	8 11¼
3.	8 1	+1 0	9 1
4.	7 8	+1 2¼	8 10¼
5.	7 2	+1 4	8 6
6.	6 2	+1 6¾	7 8¾
7	5 6	+1 3	6 9
8.	5 3	+0 10	6 1
9.	5 1	+0 7¼	5 8¼
10	5 5	+0 5½	5 10½
11	6 5	+0 0½	6 5½
Midnight.	7 9	−0 6	7 3
13	8 11	−0 9¼	8 1¼
14.	9 10	−0 11¾	8 10¼
15.	10 0	−0 11¾	9 0¼
16.	10 1	−1 1¾	8 11¾
17.	10 0	−1 4	8 8
18.	9 3½	−1 4½	7 11
19.	8 1	−1 1½	6 11½
20.	7 2	−0 10	6 4
21.	6 5	−0 7	5 10
22.	6 2	−0 4	5 10
23	6 0	+0 2	6 2
		Mean	7 5½

20th July

Time	Height	Diurnal Tide	Semidiurnal Tide
Noon	6 6	+0 5¾	6 11¾
1	7 1	+0 9½	7 10½
2	7 9	+0 11½	8 8½
3.	8 0	+0 11¾	8 11¾
4	7 11	+1 1	9 0
5.	7 6	+1 3½	8 9½
6	6 11	+1 3¼	8 2¼
7	6 2	+1 0¼	7 2¼
8.	5 9	+0 9	6 6
9	5 5	+0 6½	5 11½
10	5 6½	+0 3	5 9½
11.	6 3	−0 2½	6 0½
Midnight	7 2	−0 5¾	6 8¼
13.	8 5	−0 9¾	7 7¼
14.	9 6	−1 0	8 6
15	9 11	−1 1	8 10
16.	10 1	−1 2	8 11
17.	10 2	−1 3¾	8 10¼
18.	9 8	−1 3¼	8 4¾
19.	8 6	−1 0¼	7 5¾
20	7 4	−0 8	6 8
21.	6 5	−0 4¼	6 0¾
22.	5 11	−0 1	5 10
23.	5 8	+0 2¾	5 10¾
		Mean	7 6

TABLE I. (continued).

21st July

Time	Height	Diurnal Tide	Semidiurnal Tide
	ft in	ft in	ft in
Noon	5 11	+0 5	6 4
1.	6 6	+0 8½	7 2½
2.	7 2½	+0 9½	8 0
3.	7 6	+0 10¾	8 4¾
4.	7 7	+1 0½	8 7½
5.	7 7	+1 1½	8 8½
6	7 4	+1 0	8 4
7.	6 9	+0 9¼	7 6¼
8	6 2½	+0 6¼	6 8¼
9.	6 0	+0 2¼	6 2¼
10.	5 11	−0 1	5 10
11	6 0	−0 3½	5 8½
Midnight	6 4½	−0 5¼	5 11¼
13.	7 4½	−0 6¾	6 9¾
14	8 1	−0 7	7 6
15	8 8	−0 8	8 0
16.	9 3	−0 10½	8 4½
17.	9 6	−1 0	8 6
18.	9 0	−0 10½	8 1½
19.	8 1	−0 8	7 5
20.	7 2	−0 5¼	6 8¾
21.	6 4	−0 1½	6 2½
22.	5 7	+0 2¼	5 9¼
23	5 2	+0 6	5 7
		Mean	7 2¼

23rd July

Time	Height	Diurnal Tide	Semidiurnal Tide
	ft in	ft. in	ft in
Noon.	4 11	+0 9¼	5 8¼
1.	5 1	+0 8¼	5 9¼
2.	5 7	+0 10¼	6 5¼
3.	5 8	+1 2	6 10
4.	5 8	+1 3½	6 11½
5.	7 7	+0 5½	8 0½
6.	7 7½	+0 5	8 0½
7.	7 9	+0 1¾	7 10¾
8.	7 7	−0 0¾	7 6¼
9.	7 8	−0 6¼	7 1¾
10.	7 0	−0 6½	6 5½
11	6 8	−0 8¼	5 11¾
Midnight	6 2	−0 7	5 7
13.	5 10	−0 5¾	5 4¼
14.	7 1	−0 10½	6 2½
15	7 10	−1 1¼	6 8¾
16.	8 0½	−1 0½	7 0
17	8 3	−0 5¼	7 9¾
18.	8 3	−0 3¾	7 11¼
19.	8 0	−0 1	7 11
20.	7 6	+0 3	7 9
21	6 9½	+0 8	7 5½
22	6 0	+0 9½	6 9½
23.	5 5	+0 8½	6 1½
		Mean	6 11¾

22nd July

Time	Height	Diurnal Tide	Semidiurnal Tide
Noon	5 1	+0 9	5 10
1	6 0	+0 7¼	6 7¾
2	6 8	+0 6¼	7 2¼
3	7 2	+1 1½	8 3½
4	7 4½	+0 9	8 1½
5	7 4½	+0 10½	8 3
6	7 1½	+0 10	7 11½
7.	6 9	+0 8	7 5
8.	6 4	+0 5¾	6 9¾
9.	6 2	+0 1¼	6 3¼
10.	6 2	−0 2¾	5 11¼
11	6 4	−0 7	5 9
Midnight.	6 9	−0 10½	5 10½
13	7 2	−0 9¾	6 4¼
14	7 6	−0 8¼	6 9¾
15	8 2	−0 10½	7 3½
16.	8 6	−1 0	7 6
17.	8 9	−0 7½	8 1½
18	8 7½	−0 7½	8 0
19	8 1	−0 5	7 8
20.	7 5	−0 2¾	7 2¼
21	6 5	+0 3	6 8
22.	5 10	+0 4½	6 2½
23.	5 2	+0 8	5 10
		Mean .	6 11

24th July

Time	Height	Diurnal Tide	Semidiurnal Tide
Noon	5 1	+0 7¾	5 8¾
1.	4 10	+0 11	5 9
2.	5 1	+1 4½	6 5½
3.	5 7	+1 1	6 8
4.	6 3	+0 10	7 1
5.	7 2	+0 5¾	7 7¾
6	7 8	+0 3	7 11
7.	7 11	+0 0½	7 11½
8.	8 5	−0 4¾	8 0¼
9.	8 3	−0 7½	7 7½
10.	8 1½	−0 11¼	7 2¼
11.	7 0	−0 8	6 4
Midnight.	6 7	−0 7¾	5 11¼
13.	7 6	−1 3½	6 2½
14.	7 7	−1 4	6 3
15.	7 8	−1 2	6 6
16.	7 10	−0 11¾	6 10¼
17	8 0	−0 7	7 5
18.	8 1	−0 3	7 10
19.	8 0	+0 0½	8 0½
20	7 9	+0 5	8 2
21	7 2	+0 8¼	7 10¼
22.	6 6	+1 0	7 6
23.	5 11	+1 0¾	6 11¾
		Mean...	7 1

TABLE I. (continued).

25th July.

Time	Height	Diurnal Tide	Semidiurnal Tide
	ft in.	ft in	ft in
Noon.	5 6	+1 1	6 7
1	5 0	+1 0	6 0
2.	4 9	+1 6	6 3
3.	5 1	+1 4	6 5
4.	5 4	+1 2¾	6 6¾
5.	6 6	+0 8¼	7 2¼
6.	7 6½	+0 2½	7 9
7.	8 3½	−0 2½	8 1
8.	8 9	−0 6	8 3
9.	8 10	−0 8¼	8 1¾
10.	8 11	−1 1¼	7 9¼
11.	9 0	−1 5½	7 6½
Midnight	8 9	−1 5¼	7 3¼
13.	8 6	−1 8	6 10
14.	7 11	−1 6¼	6 4¾
15.	7 10	−1 2	6 8
16.	7 9	−1 3	6 6
17.	7 9	−0 9	7 0
18.	7 10	−0 1¼	7 8¼
19	7 9	+0 1¼	7 10¼
20.	7 9	+0 5¼	8 2¼
21.	7 9	+0 6	8 3
22.	6 11	+1 4	8 2
23.	6 3	+1 8¼	7 11¼
		Mean	7 3¾

26th July

Time	Height	Diurnal Tide	Semidiurnal Tide
Noon.	6 1	+1 8¼	7 9¼
1.	5 4	+1 10¼	7 2¼
2.	4 11½	+1 3	6 2½
3	4 11	+1 8	6 7
4.	5 2	+1 5¼	6 7¼
5.	6 0	+1 1¾	7 1¾
6.	7 8½	+0 2½	7 11
7.	7 9½	+0 1¼	7 10¾
8.	8 8	−0 3½	8 4½
9	9 7½	−0 10¾	8 8¾
10.	10 2½	−1 5¼	8 9¼
11.	10 3	−1 8¾	8 6¼
Midnight	10 2	−1 10¾	8 3¼
13.	9 7	−1 11½	7 7½
14.	7 0	−0 11½	6 0½
15.	8 8	−1 11	6 9
16.	8 4	−1 8¾	6 7¼
17.	8 10	−1 7½	7 2½
18	8 5	−0 9½	7 7½
19.	8 5	−0 2½	8 2½
20.	8 5	+0 1	8 6
21.	7 11	+0 10¾	8 9¾
22.	7 9	+1 2½	8 11½
23.	7 4	+1 6½	8 10½
		Mean...	7 8½

27th July.

Time	Height.	Diurnal Tide.	Semidiurnal Tide
	ft in	ft. in	ft in
Noon.	6 9	+2 4	9 1
1.	6 0	+2 6¾	8 6¾
2.	5 2	+2 0	7 2
3.	4 9	+2 4½	7 1½
4.	4 7	+2 2½	6 9½
5.	5 2	+1 10¾	7 0¾
6	6 0	+1 4½	7 4½
7.	7 2	+0 10¼	8 0¼
8	8 6	+0 2½	8 8½
. 9	9 10	−0 10	9 0
10	10 2	−0 10	9 4
11	10 6	−1 1½	9 4½
Midnight	12 8	−2 5¼	10 2¼
13	12 8	−2 10¼	9 9¼
14	11 4	−2 7½	8 8½
15	10 4	−2 5¼	7 10¼
16.	9 8	−2 3¼	7 4¼
17.	9 1	−1 10½	7 2½
18.	9 1	−1 6½	7 6½
19.	9 2	−0 10½	8 3½
20	9 1	−0 2	8 11
21.	9 3	+0 6	9 9
22.	9 3	+0 10½	10 1½
23.	9 1½	.	
		Mean	8 5

28th July

Time	Height	Diurnal Tide	Semidiurnal Tide
Noon.	8 8		
1	7 11		
2.	7 0		
3.	6 2		
4.	5 6		
5	5 6		
6.	6 1		
7.	7 8		
8.	9 0		
9	10 8		
10	11 10		

The mean level of all the Semidiurnal Tide-heights, 538 in number, =7 ft 5·414 in

A. *Diurnal Tide.*

Having obtained the hourly values of the Diurnal Tide in height, I plotted them to scale, and readily obtained the following Table, showing the chief phases of the Diurnal Tide each day

TABLE II.—Times of the Principal Phases of the Diurnal Tide at Port Kennedy in July 1859.

Time	High Water	Half-Ebb	Low Water	Half-Flood.
	h m	h m	h m	h m
July 5. .		.	16 40	22 46
6. .	5 0	10 20	17 0	21 30
7	5 0	·10 26	15 0	20 48
8	2 30	8 20	14 0	19 40
9	1 30	7 0	12 0	18 30
10.	1 30	6 48	13 30	19 25
11.. ..	Noon	8 20	14 30	20 20
12.	2 40	9 0	14 30	20 0
13.	2 30	8 0	15 0	20 30
14.	2 30	8 40	15 30	21 0
15 .	3 30	9 20	16 0	21 30
16 ..	4 0	9 48	16 0	21 45
17. ..	4 30	10 0	16 30	22 0
18. .	4 30	9 30	18 0	23 45
19...	5 40	11 0	17 30	22 40
20. .	5 0	10 30	17 0	22 15
21. .	5 0	9 40	17 0	21 20
22.	3 0	9 15	16 0	20 30
23	3 0	7 40	15 0	19 15
24.	2 0	7 10	14 0	18 8
25	2 0	6 30	13 0	18 30
26.	1 0	7 12	13 30	19 40
27 ..	1 0	8 10	14 0	20 15

If we extract from this Table the Maximum and Minimum values of the apparent Solitidal Interval for each Phase, and reduce all to the Phase of High Water, we find—

TABLE III.—Maximum apparent Diurnal Solitidal Interval at Port Kennedy in July 1859.

High Water	Half-Ebb	Low Water	Half-Flood	Apparent Solitidal Interval reduced to High-Water Phase.
d h m	d h m	d h m	d h m	h m s
. .	.	....	5 22 46	4 46 0
6 5 0		...	..	5 00 0
.		6 17 0	...	5 00 0
.	7 10 26	....		4 26 0
		18 18 0	.	6 00 0
		..	18 23 45	5 45 0
19 5 40		....	.	5 40 0
....	19 11 0	....	....	5 00 0
			Mean ..	5 12 7½

The Diurnal Tide is represented by the formula

$$D=S'\sin 2\bar{\sigma}\cos(s-i_s)+M'\cos 2\bar{\mu}\cos(m-i_m), \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where the letters have the meaning stated in my former papers, viz.—

S', M' the Solar and Lunar coefficients uncorrected for Parallax,

$\bar{\sigma}$, $\bar{\mu}$ the declinations of the Sun and Moon at an interval preceding the observation called the Age of the Tide,

s, m the Solar and Lunar Hour-angles at the time of observation,

i_s, i_m the true Solitidal and Lunitidal Diurnal Intervals.

The Lunar Tide vanishes when $\mu=0$, and this corresponds with Table III., which contains the Maximum value of the apparent Solitidal Interval not influenced by the Moon, but representing the full effect of the Sun

The Moon's declination vanished twice

From N to S at $5^d\,5^h\,11^m$, and
From S to N. at $19^d\,17^h\,14^m$,

which correspond fairly with the times of Maximum retardation of Solidiurnal Interval. The age of the Lunidiurnal Tide may be found from the interval between the Moon's declination vanishing and the Lunar Tide vanishing, as shown by the Maximum value of the Solitidal Interval From the first time of tide vanishing we have

	d	h	m
	5	22	46
	6	5	0
	6	17	0
	7	10	26
Mean=	6	13	48
$\mu=0$ at	5	5	11
Age of Lunar Diurnal Tide	1	6	37

From second time of tide vanishing we have

	18	18	0
	18	23	45
	19	5	40
	19	11	0
Mean=	19	2	36
$\mu=0$ at	19	17	14
Age of Lunar Diurnal Tide	00	14	38

From Table III. it appears that the value of i_s, the true Diurnal Solitidal Interval, is

$$i_s=5^h\,12^m\,7\tfrac{1}{2}^s. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The Minimum values of the apparent Solitidal Intervals, caused by the Maximum influence of the Lunar Tide, are contained in Table IV

TABLE IV.—Minimum apparent Diurnal Solitidal Interval at Port Kennedy in July 1859.

High Water	Half-Ebb	Low Water	Half-Flood	Apparent Solitidal Interval reduced to High-Water Phase
d h m	d h m	d h m	d h m	h m s
		9 12 0		0 0 0
			9 18 30	0 30 0
	10 6 48			0 48 0
11 0 0				0 0 0
			24 18 8	0 8 0
	25 6 30			0 30 0
		25 13 0		0 1 0
26 1 0				0 1 0
			Mean	0 29 30

This Table corresponds with the Maximum effect of the Lunar Tide, and the age of Lunar Diurnal Tide may be found by comparing the results of this Table with the times of Maximum of Moon's Declination.

The Moon's Declination attained its Maximum value twice, viz.

	d	h	m		
July	8	10	0	. . . . . .	$\mu = 27^\circ\ 43'\ 33''$ S.
„	22	23	0	. .	$\mu = 27\ 43\ 21$ N.

From the first time of Minimum Solitidal Interval in Table IV. we have

	d	h	m
	9	12	0
	9	18	30
	10	6	48
	11	0	0
Mean=	10	3	$19\frac{1}{2}$
μ=Max	8	10	0
Age of Lunar Diurnal Tide . .	1	17	$19\frac{1}{2}$

From the second time of Minimum Solitidal Interval we have

	24	18	8
	25	6	30
	25	13	0
	26	1	0
Mean=	25	9	$39\frac{1}{2}$
μ=Max.	22	23	0
Age of Lunar Diurnal Tide	2	10	$39\frac{1}{2}$

The Mean Age of Lunar Diurnal Tide is therefore

	+1	6	37
	−0	14	38
	+1	17	$19\frac{1}{2}$
	+2	10	$39\frac{1}{2}$
Mean =	1	4	$14\frac{1}{2}$

or, Age of Lunar Diurnal Tide,

$$= \overset{\mathrm{d}}{1}\ \overset{\mathrm{h}}{4}\ \overset{\mathrm{m}}{14\tfrac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

We now proceed to determine the value of the Solar Coefficient S′, which may be readily found as follows.—

We may throw the expression (2) for the Diurnal Tide into the following form, writing

$$S'' = S' \sin 2\bar{\sigma},$$

$$M'' = M' \sin 2\bar{\mu},$$

$$D = A \cos(s - B), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (5)$$

where

$$A = \sqrt{S''^2 + M''^2 + 2S''M'' \cos\overline{(s - m - \iota_s - \iota_m)}}, \quad . \quad . \quad . \quad . \quad . \qquad (6)$$

$$\tan B = \frac{S'' \sin \iota_s + M'' \sin(s - m + \iota_m)}{S'' \cos \iota_s + M'' \cos(s - m + \iota_m)} \quad . \quad . \quad . \qquad (7)$$

The Solar Diurnal Tide will occur alone when $M'' = 0$ or $\bar{\mu} = 0$

The values of A are given from Table I., and are as follows, in Table V.

TABLE V.—Heights of High and Low Water of Diurnal Tide at Port Kennedy in July 1859.

Time	High Water	Low Water	Time	High Water	Low Water
h m	ft in	ft in	h m	ft in	ft in
July 5. 17 0		1 9½	July 17. 4 0	2 0¼	
6. 5 0	1 5½		17 16 0		1 11½
6. 17 0		1 4½	18 5 0	1 7¼	
7. 5 0	1 2¼		18. 18 0		1 7
7. 15 0		1 0¾	19 6 0	1 6¼	
8 2 30	1 0¼		19 17 30		1 4½
8. 14 0		1 1	20 5 0	1 3½	
9. 1 30	1 1		20. 17 0		1 3¼
9 12 0		1 2¼	21 5 0	1 1½	
10. 3 0.	1 2		21. 17 0		1 0
10. 13 0		1 2	22 3 0	1 1½	
11. 0 0	1 3½		22 16 0		1 0
11. 13 0		1 6½	23. 3 30	1 3½	
12. 3 0	1 11		23 15 30		1 1¼
12. 16 0		1 10¾	24 2 20	1 4½	
13. 4 0	2 2¼		24. 14 0		1 4
13. 14 0		2 4¼	25. 2 30	1 6	
14. 2 30	2 5½		25 13 0		1 8
14 14 30		2 4¼	26. 1 0	1 10¼	
15. 4 0.	1 11½		26 15 30		1 11
15 16 0		1 7½	27. 1 0	2 6¼	
16. 4 0	1 10½		27. 13 0		2 10¼
16. 15 30		2 2½			

If we add the age of the Lunar Diurnal Tide to the times of the Moon's Declination vanishing, we shall have the times when $M'' = 0$.—

	d h m	d h m
$\mu=0$ . .	5 5 11	19 17 14
Age= .	1 4 14	1 4 14
	6 9 25	20 21 28

If we take the values of A nearest to these times, from Table V. we find

$$A=S''=1^{\text{ft}}\ 5^{\text{in}} \text{ at } 6^{\text{d}}\ 9^{\text{h}}\ 25^{\text{m}},$$
$$A=S''=1\ 3 \text{ at } 20\ 21\ 28,$$

and using the Sun's declination at noon of the day before, we find

$$S'=\frac{S''}{\sin 2\bar{\sigma}},$$

or

$$S'=\frac{17}{\sin(45^\circ 40')}=23\cdot 8 \text{ inches},$$

and

$$S'=\frac{15}{\sin(41^\circ 40')}=22\cdot 6 \text{ inches}.$$

The mean of these values is

$$S'=23\cdot 4 \text{ inches} \quad \ldots \ldots \quad (8)$$

We can obtain the ratio of M' to S' from Tables III. and IV., and thus calculate M' as follows. Differentiating (7) so as to make B a Maximum or Minimum, we find the equation of condition

$$M''+S''\cos(\overline{s-m}-\overline{\iota_s-\iota_m})=0. \quad \ldots \ldots \quad (9)$$

Substituting in (7), we find at the Maximum and Minimum

$$\tan B=\frac{\sqrt{S''^2-M''^2}\sin\iota_s+M''\cos\iota_s}{\sqrt{S''^2-M''^2}\cos\iota_s-M''\sin\iota_s}, \quad \ldots \ldots \quad (10)$$

when $\bar{\mu}=0$, $M''=0$, and the equation reduces to

$$\tan B=\tan\iota_s, \text{ or } B=\iota_s,$$

as we assumed in determining the value of the true Diurnal Solitidal Interval from Table III.

If we write

$$\sin\theta=\frac{M''}{S''}, \quad \ldots \ldots \quad (11)$$

we can reduce (10) to the following form,

$$\tan B=\tan(\iota_s+\theta),$$

or

$$B=\iota_s+\theta. \quad \ldots \ldots \quad (12)$$

The Maximum and Minimum values of B are found from Tables III and IV.

$$B=\text{Maximum}=5^{\text{h}}\ 12^{\text{m}}\ 7\tfrac{1}{2}^{\text{s}},$$
$$B=\text{Minimum}=0\ 29\ 30;$$

when B is a Maximum, $M''=0$ and $\theta=0$; therefore (12) reduces to

$$B=\iota_s=5^{\text{h}}\ 12^{\text{m}}\ 7\tfrac{1}{2}^{\text{s}};$$

when B is a Minimum, equation (12) reduces to

$$\begin{array}{ccc} \text{h} & \text{m} & \text{s} \\ 0 & 29 & 30 \end{array} = \begin{array}{ccc} \text{h} & \text{m} & \text{s} \\ 5 & 12 & 7\tfrac{1}{2} \end{array} + \theta,$$

or

$$\theta = -\ 4\ 42\ 37\tfrac{1}{2},$$

or

$$19\ 17\ 22\tfrac{1}{2}$$

$$\frac{M''}{S''} = \sin(4^{\text{h}}\ 42^{\text{m}}\ 37\tfrac{1}{2}^{\text{s}}) = \sin(70^\circ\ 39\tfrac{1}{2}') = 0{\cdot}943,$$

but

$$\frac{M''}{S''} = \frac{M' \sin \overline{2\mu}}{S' \sin 2\sigma},$$

or

$$\frac{M'}{S'} = \frac{M''}{S''}\ \frac{\sin \overline{2\sigma}}{\sin 2\mu} = 0{\cdot}943 \times \frac{\sin 43^\circ 34'}{\sin 55^\circ 27'},$$

or

$$\frac{M'}{S'} = 0{\cdot}788 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

From (8) and (13) we find

$$M' = 18{\cdot}4 \text{ inches.} \quad . \quad . \quad . \quad . \quad (14)$$

From the values already found for the constants of the Solar Diurnal Tide, it was easy to calculate its value, for every hour, from the formula

$$D = S' \sin \overline{2\sigma} \cos(s - \iota_s).$$

These values, if subtracted from the Diurnal Tide in Table I., would leave the Lunar Diurnal Tide, the principal phases of which are given in the following Table.

TABLE VI.—*Times* of Half-Flood and Half-Ebb, and *Heights* of High Water and Low Water of the Lunar Diurnal Tide at Port Kennedy in July 1859.

	Half-Ebb	Low Water	Half-Flood	High Water
	h m	ft in	h m	ft in
July 7. . . .	2 0	0 5½	13 40	0 6½
8 .	1 50	0 9½	14 10	1 0
9. . .	1 30	1 1½	13 50	1 2¼
10.	2 20	1 2¼	14 0	1 0
11. . .	1 30	1 3	15 40	1 3½
12.	6 15	1 1½	17 25	1 4
13	5 50	1 6¼	18 25	1 5¼
14	7 10	1 5¼	19 15	0 8
15	7 30	0 8¼	20 10	0 8
16.	8 30	0 9	20 45	0 9½
17 .	9 35	0 7½	21 30	0 4
18. .	10 0	——	——	——
19. . .	——	——	——	——
	Half Flood	High Water	Half-Ebb	Low Water
20. . .	——	——	23 35	0 4
21. .	11 50	0 6	——	——
22. .	13 10	0 9	0 40	0 5½
23.	13 0	1 2	1 30	0 11¾
24. . . .	14 30	1 4	2 50	1 2½
25.	15 40	1 6	3 30	1 4
26 . .	17 0	1 7½	4 15	1 8¼
27. . .	——	——	6 15	1 7

I have used the Times of Half-Flood and Half-Ebb in this Table in preference to the Times of High Water and Low Water, as the vertical motion of the water is a maximum at Half-Flood and Half-Ebb.

Table VI. contains the Solar Hours of Half-Flood and Half-Ebb. These are reduced in the following Table to Lunar Hours.

TABLE VII.—Moon's Hour-Angle at times of Half-Flood and Half-Ebb of Lunar Diurnal Tide at Port Kennedy in July 1859.

Day	Moon's Hour-Angle at Half-Flood	Moon's Hour-Angle at Half-Ebb
	h m	h m
July 7.	7 15	19 15
8	6 45	18 48
9	5 38	17 43
10.	4 59	17 44
11	4 9	16 6
12	6 36	19 50
13.	6 43	18 36
14.	6 34	19 14
15.	6 51	18 34
16.	6 42	18 46
17.	6 46	19 6
18	—	18 51
19.	—	—
20.	Half-Ebb	Half-Flood
21.	6 10	18 40
22.	6 29	19 17
23	6 29	18 23
24	6 53	19 3
25	6 34	19 17
26.	6 15	19 37
27.	7 9	
	Mean 6 23 10s	18 44 30s

Hence the mean value of the true Diurnal Lunitidal Interval is at High Water

$$\iota_m = 0^h\ 33^m\ 50^s. \qquad (15)$$

The coefficient M′, of the Lunar Diurnal Tide, may be found from Heights from Table VI.

The Lunar Diurnal Tide reached its maximum—

	d	h	m		ft.	in.
July	13	11	38	. . . .	1	$6\frac{1}{4}$
July	26	10	3	. . . . .	1	$8\frac{1}{4}$

The Moon's Maximum declination occurred—

	d	h	m		
July	8	10	0	. . . .	$\bar{\mu} = 27^\circ\ 43'\ 33''$ S.
July	22	23	0	. . . .	„ 27 43 21 N.

These values give for the age of the Lunar Tide deduced from Heights,

	d	h	m
Age of Lunidiurnal Tide . . .	5	1	38
" " . .	3	11	3
Mean . .	4	6	$20\frac{1}{2}$

This result differs considerably from the age deduced from Times, and agrees with what I have noticed in several tidal observations, viz. that the age of the Tide deduced from Heights is greater than that deduced from Times

Taking the mean of the Maximum Heights, we have

$$M'=\frac{M''}{\sin 2\mu}=\frac{19\cdot 25}{\sin 55^\circ\ 27'}=23\cdot 37 \text{ inches},$$

$$M'=23\cdot 37 \text{ inches}, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (16)$$

and finally, from (8) and (16),

$$\frac{M'}{S'}=\frac{23\cdot 37}{23\cdot 4}=0\cdot 994 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (17)$$

If we collect together all the preceding results, we obtain the following —

Constants of the Diurnal Tide at Port Kennedy in July 1859.

Solar Diurnal Tide.

Age	Unknown.
True Solitidal Interval	$\iota_s=5^h\ 12^m\ 7\frac{1}{2}^s$
Coefficient (uncorrected for Parallax) .	$S'=23\cdot 4$ inches

Lunar Diurnal Tide

Age .	$1^d\ 4^h\ 14\frac{1}{2}^m$ (Times)
" .	$4^d\ 6^h\ 20\frac{1}{2}^m$ (Heights).
True Lunitidal Interval	$\iota_m=0^h\ 33^m\ 50^s$
Coefficient (uncorrected for Parallax)	$M'=18\cdot 4$ inches (Times)
" " " .	$M'=23\cdot 37$ " (Heights)
	$\frac{M'}{S'}=0\cdot 788$ (Times)
	" $=0\cdot 994$ (Heights)

B. *Semidiurnal Tide*

From the column of Semidiurnal Tides in Table I it is easy to construct the following Table:—

TABLE VIII.—Heights and Lunitidal Intervals of Semidiurnal Tide at Port Kennedy in July 1859.

Time	Heights		Lunitidal Intervals	
	High Water	Low Water	High Water	Low Water
d h m	ft in	ft in	h m	h m
5 16 30	8 3¼		11 17	
5 23 0		4 2		17 34
6 5 0	8 4½		23 24	
6 11 0		4 3½		5 12
6 17 0	8 3½		11 0	
6 23 30		4 7¼		17 17
7 5 40	8 1½		23 18	
7 12 0		5 1		5 15
7 18 0	8 4¼		11 14	
8 1 0		5 5½		18 0
8 7 0	8 6¼		23 49	
8 13 0		5 8¼		5 37
8 19 30	8 8		12 7	
9 1 30		5 11		17 55
9 8 0	8 11½		24 0	
9 14 0		5 11½		5 48
9 20 0	9 1		11 36	
10 2 0		6 5¾		17 24
10 8 0	9 0½		23 11	
10 14 0		7 2¾		4 59
10 22 0	8 10½		12 43	
11 5 30		7 8¼		19 58
11 10 0	8 11¼		23 44	
11 16 20		7 1¼		5 51
11 23 0	8 10¾		13 11	
12 4 0		6 1½		18 1
12 11 0	9 1½		23 35	
12 17 0		6 0¾		5 23
12 23 30	9 1¼		12 28	
13 5 30		6 1¾		18 16
13 12 20	9 3		24 50	
13 18 0		6 1¼		6 19
14 1 0	9 5½		13 5	
14 6 30		5 11½		18 24
14 12 0	9 10½		23 42	
14 18 40		5 11		6 9
15 0 0	9 4½		11 18	
15 6 40		5 10		17 45
15 12 0	8 11		22 57	
15 19 0		5 9		5 43
16 1 30	9 2¼		12 0	
16 7 0		5 10½		17 19
16 13 0	9 9		23 12	
16 20 0		5 11½		5 58
17 2 0	9 8¾		11 47	
17 8 0		5 11		17 35
17 14 0	9 9		23 32	
17 20 0		5 11½		5 20
18 2 30	9 5		11 36	
18 9 0		5 10		17 53
18 14 40	9 1½		23 30	
18 21 0		5 8¼		5 37
19 3 0	9 1		11 26	
19 9 0		5 8¼		17 14
19 15 0	9 0¼		23 11	

TABLE VIII (continued).

Time	Heights.		Lunitidal Intervals	
	High Water	Low Water	High Water	Low Water
d h m	ft in	ft in	h m	h m
19 21 30	...	5 10	...	5 28
20 3 0	9 0		10 47	
20 10 0		5 9½		17 33
20 16 0	8 11		23 30	
20 22 0		5 10		5 18
21 5 0	8 8½		12 4	
21 11 0		5 8½		17 52
21 17 0	8 6		23 47	
21 23 0		5 7		5 35
22 5 0	8 3½		11 24	
22 11 0		5 9		17 12
22 17 30	8 1½		23 33	
23 0 0		5 8¼		5 50
23 5 30	8 0½		11 9	
23 13 0		5 4½		18 24
23 18 20	7 11¼		23 33	
24 0 20		5 8¼		5 21
24 8 0	8 0¼		12 46	
24 12 0		5 11¼		16 38
24 20 0	8 2		24 17	
25 1 0		6 0		5 7
25 7 0	8 3		10 55	
25 14 0		6 4¼		17 41
25 21 20	8 3		24 36	
26 2 0		6 2½		5 7
26 9 40	8 9¼		12 31	
26 14 0		6 0½		16 51
26 21 0	9 1		24 46	
27 4 0		6 9½		7 32
27 12 0	10 2¼		13 45	
27 17 0		7 2½	..	18 35

From this Table we find :—

Mean of Lunitidal Intervals

	h	m	s
43 High Waters	23	48	58½
43 Low Waters reduced to phase of High Water .	23	43	1½
Mean Lunitidal Interval . . .	23	46	0

Mean Height.

	ft	in.
High Water	8	10·55
Low Water	5	10·98

The Maximum and Minimum Ranges in height were as follows :—

	d	h	m	ft.	in.
Maximum Range	6	2	0	4	1½
„ „	14	15	20	3	11½
				4	0½

	d	h	m	ft.	in.
Minimum Range	11	3	45	1	$2\frac{1}{4}$
" "	25	17	40	2	2
				1	$8\frac{1}{8}$

Hence we have, if M″ and S″ represent the Lunar and Solar Semidiurnal Coefficients, *uncorrected for Declination and Parallax,*

$$2\,(M''+S'')=48.$$
$$2\,(M''-S'')=20.$$
$$M''=17 \text{ inches.}$$
$$S''=7 \quad \text{"}$$
$$\frac{M''}{S''}=0\,412 \quad . \qquad . \qquad . \qquad . \quad (18)$$

Table IX. contains the Times and Lunitidal Intervals of Half-Flood and Half-Ebb, determined from Table I., from the moment at which the water-level passed the mean height of Tide-level, as noted at foot of column 4, for each day

TABLE IX.—Times and Lunitidal Intervals of Half-Flood and Half-Ebb of Semidiurnal Tide at Port Kennedy in July 1859

	Times		Lunitidal Intervals	
	Half-Flood	Half-Ebb	Half-Flood	Half-Ebb
	h m	h m	h m	h m
July 5	13 45		8 37	
5		18 56		13 38
6	1 46		20 16	
6.		7 51		1 12
6	14 14		8 29	
6		20 17		14 11
7	2 50		20 35	
7.		8 38		2 11
7.	14 56		8 17	
7.		21 45		14 52
8	3 50		20 50	
8		9 48		2 37
8	16 7		8 43	
8		22 49		14 47
9	4 48		20 54	
9.		11 22		3 15
9	16 54		8 37	
10		0 13		15 40
10	5 47		21 3	
10.		12 25		3 27
10	18 26		9 8	
11		1 22		15 58
11	7 29		21 49	
11.		13 33		3 41
11	21 3		10 57	
12.		2 37		16 9
12.	7 4		21 36	

TABLE IX. (continued)

	Times.		Lunitidal Intervals.	
	Half-Flood	Half-Ebb.	Half-Flood.	Half-Ebb
	h m	h m	h m	h m
July 12		14 41		3 57
12	19 48		8 53	
13		2 28		15 20
13	8 36		21 14	
13		15 13		3 37
13	20 56		9 10	
14		2 47		14 49
14	9 48		21 34	
14		15 12		2 48
14	22 2		9 24	
15		3 58		15 8
15	9 37		20 39	
15		15 53		2 42
15	21 41		8 19	
16		4 12		14 36
16	9 53		20 13	
16		16 50		2 55
16	22 50		8 52	
17		5 7		14 48
17	11 14		20 50	
17		17 21		2 45
17	22 56		8 9	
18		5 42		14 41
18	11 36		20 33	
18		17 53		2 36
18	23 52		8 24	
19		6 16		14 35
19	12 15		20 32	
19		18 28		2 32
20	0 25		8 17	
20		6 43		14 23
20	12 53		20 29	
20		18 58		2 22
21	0 58		8 10	
21		7 24		14 24
21	13 37		20 27	
21		19 23		2 5
22	1 28		7 58	
22		7 59		14 16
22	14 13		20 22	
22		20 31		2 28
23	4 1		9 43	
23		9 13		14 44
23	15 50		21 8	
23		21 43		2 50
24	4 0		8 54	
24		10 9		14 51
24	16 24		20 49	
24		22 48		3 1
25	5 13		9 13	
25		11 50		15 37
25	17 27		19 49	
26		0 7		3 17
26	5 44		8 44	
26		12 53		16 1
26	18 8		20 29	
27		1 6		3 12
27	7 34		9 28	
27		14 21		16 1
27	19 11		20 37	

From this Table we find:—

	h m s
Mean Lunitidal Interval of 44 Half-Floods .	20 45 19
" " 43 Half-Ebbs . . .	2 54 24

Reducing the Lunitidal Intervals found from Tables VIII and IX to the phase of High Water, we have

Mean Lunitidal Interval $=i_m$.

	h m s
From High Waters	23 48 58
" Low Waters	23 43 1
" Half-Floods	23 45 19
" Half-Ebbs	23 54 46
Mean	23 48 1

$$\left.\begin{array}{rl} i_m = & 23^{\text{h}}\ 48^{\text{m}}\ 1^{\text{s}} \\ \text{or} - & 0\ \ 11\ \ 59 \end{array}\right\} \quad . \quad . \quad . \qquad (19)$$

We may calculate the ratio of the Solar and Lunar Semidiurnal Tides from Tables VIII. and IX. by the following method —

$$\text{Let } M''=M\left(\frac{P}{P_m}\right)^3 \cos^2\bar{\mu},$$

$$S''= S\left(\frac{p}{p_m}\right)^3 \cos^2\bar{\sigma},$$

where P, p are the parallax of the Moon and Sun, taken at an interval before the observation equal to the age of the respective Tides, and P_m, p_m are the mean values of same

Then if the Semidiurnal Tide be

$$T=M'' \cos 2\,(m-i_m)+S'' \cos 2\,(s-i_s), \quad . \quad . \qquad (20)$$

we may write (20) thus,

$$T=A \cos 2\,(m-B), \quad . \quad . \quad . \qquad (21)$$

where

$$A=\sqrt{M''^2+S''^2+2M''\,S'' \cos 2\,[\overline{m-s}-\overline{i_m-i_s}]}, \quad . \quad . \qquad (22)$$

$$\tan 2B=\frac{M'' \sin 2i_m+S'' \sin 2\,[m-s+i_s]}{M'' \cos 2i_m+S'' \cos 2\,(m-s+i_s)}. \quad . \quad . \quad . \qquad (23)$$

The Maximum and Minimum values of A are $M''+S''$ and $M''-S''$, as used in finding (18); and the Maximum and Minimum values of B are found by differentiating (23), which gives, as the equation of condition,

$$S''+M'' \cos 2\,(\overline{m-s}-\overline{i_m-i_s})=0. \quad . \quad . \quad . \quad . \qquad (24)$$

Combining (23) and (24) we find, after a few reductions,

$$\tan 2B = \frac{\sqrt{M''^2 - S''^2}\sin 2\imath_m + S''\cos 2\imath_m}{\sqrt{M''^2 - S''^2}\cos 2\imath_m - S''\sin 2\imath_m}. \quad (25)$$

If we assume

$$\frac{S''}{M''} = \sin 2\phi,$$

the equation (25) will reduce to the following —

$$\tan 2B = \tan 2(\phi + \imath_m),$$

or

$$B = \phi + \imath_m. \quad (26)$$

The Maximum and Minimum values of B, or of the Lunitidal Interval, are found from Tables VIII and IX., and are as follows.—

Maximum Values of Lunitidal Interval

	h	m	s		d	h	m
High Water	25	5	0	at	14	1	0
,,	24	46	0	,,	26	21	0
Low Water	25	58	0	,,	11	5	30
,,	24	35	0	,,	27	17	0
Half-Flood	25	57	0	,,	11	21	3
,,	24	28	0	,,	27	7	34
Half-Ebb	25	9	0	,,	12	2	37
,,	25	1	0	,,	27	14	21
Mean=	25	7	22½				

Minimum Values of Lunitidal Interval

	h	m	s		d	h	m
High Water	23	0	0	at	6	17	0
,,	22	55	0	,,	25	7	0
Low Water	23	12	0	,,	6	11	0
,,	22	38	0	,,	24	12	0
Half-Flood	23	16	0	,,	6	1	46
,,	22	58	0	,,	22	1	28
Half-Ebb	22	12	0	,,	6	7	51
,,	23	5	0	,,	21	19	23
Mean=	22	54	0				

From equation (26) we see that the value of B ranges above and below that of $\imath_m$ by

a quantity equal to ϕ, which is half the difference between the maximum and minimum values of B. Hence we find

$$\begin{array}{lrrr} & \mathrm{h} & \mathrm{m} & \mathrm{s} \\ \text{Maximum value of B} = & 25 & 7 & 22\tfrac{1}{2} \\ \text{Minimum} \quad ,, \quad ,, \quad = & 22 & 54 & 0 \\ \hline 2\phi = & 2 & 13 & 22\tfrac{1}{2} \\ 2\phi = & \multicolumn{3}{l}{33^\circ\ 20\tfrac{1}{2}'.} \end{array}$$

$$\frac{M''}{S''} = \sin 2\phi = 0.549 \quad . \qquad (27)$$

Collecting together the foregoing results, we have the following

Constants of the Semidiurnal Tide at Port Kennedy in July 1859

Lunar Semidiurnal Tide

True Lunitidal Interval	$23^{h}\ 48^{m}\ 1^{s}$.
Ratio of $\frac{S''}{M''}$ . . .	0·412 (Heights) 0·549 (Times)

(Uncorrected for Declination or Parallax.)

XIV. *On the Mathematical Expression of Observations of Complex Periodical Phenomena; and on Planetary Influence on the Earth's Magnetism.* *By* CHARLES CHAMBERS, *F.R.S.*, *and* F. CHAMBERS.

Received May 26,—Read June 19, 1873 *.

THE writers purpose in the following pages to determine, by BESSEL's method, a mathematical expression for a periodical phenomenon from observations which are affected by one or more other periodical phenomena, and to find criteria for judging of the extent to which the expression is affected by these other phenomena, also, having found an expression for a period of known approximation to the truth, to find from it the expression for the true period. In the course of these inquiries, certain ambiguities which affect similarly BESSEL's expression for a single periodical phenomenon and the results here arrived at will be remarked upon, and, finally, the results will be applied to determine the nature of periodic planetary magnetic influence in particular cases.

2. In BESSEL's paper "On the Determination of the Law of a Periodic Phenomenon" (a translation of which has been published by the Meteorological Committee in the Quarterly Weather Report, part iv. 1870), the author describes, in Section VII, how periodical phenomena which depend on two or more angles can be developed from observations of the same, and he remarks upon the simplicity of a certain class of cases in which both angles are exact measures of 2π, and one is a multiple of the other. In the description of the process occur the following words —

"If we designate the two angles by x, x', then in the expression

$$y=p+p_1\cos x+q_1\sin x+p_2\cos 2x+q_2\sin 2x+ \text{\&c.}$$

the p, p_1, q_1, &c. which occur are not constant, but depend on x', and as they are periodic functions of x', each of them has an expression of the form

$$a+a_1\cos x'+b_1\sin x'+a_2\cos 2x'+b_2\sin 2x'+ \text{\&c.}$$

It is therefore necessary to deduce this development of p, p_1, q, &c. from the observations. If the available series of observations gives the values of y, not only for values of x $(0, z, 2z, \ldots\ldots (n-1)z)$, which are in arithmetical progression and fill up the period, but also for the combination of each of these values of x with n' values of $x'(0, z', 2z', \ldots\ldots (n'-1)z')$, fulfilling the same conditions, the development has no difficulties." After a perfect elucidation of a type of these cases follow remarks upon comparatively difficult cases, which require more cumbrous methods for eliminating the several constants.

* Subsequently revised by the authors.

3. There is a yet simpler case, the importance of which possibly did not press itself on BESSEL's attention, but which the present writers (having occasion to describe in connexion with actual observations) find it convenient thus to introduce. It is that of two or more combined phenomena, each of which separately recurs after a certain period, which is of different duration for each phenomenon; and the object of this inquiry will be to determine under what circumstances, and with what degree of accuracy, may the coefficients of the expression, according to BESSEL's form, of each separate phenomenon be found from a series of observed values of the combined phenomena.

4. As the result will be equally applicable to any number of combined phenomena, we will consider the case of only two, whose periods are respectively $\varkappa$ and $\varkappa'$.

Let $\frac{\varkappa}{\varkappa'}=\frac{f}{g}$, f and g being the least integral numbers that will satisfy this condition*, and x being the interval in time (supposed to be constant) between every two consecutive observations, let the series of observations extend over the time $g\varkappa$ or $f\varkappa'$, of which x is a measure, then will the number (r) of observations be $\frac{g\varkappa}{x}$, and the angles corresponding to the time x under the respective periods $\frac{2\pi}{\varkappa}x=z$ (say) and $\frac{2\pi}{\varkappa'}x=\frac{f}{g}z$; further, the angles corresponding to the time rx or $g\varkappa$ will be $2g\pi$ and $2f\pi$ respectively. If α_m represent the observed value of the combined phenomena at the time mx, and β_m and γ_m be the separate phenomena of which it is composed, β_m recurring after the period $\varkappa$, and γ_m after the period $\varkappa'$, we shall have

$$\beta_m=p_0+p_1\cos mz+q_1\sin mz+p_2\cos 2mz+q_2\sin 2mz+\text{\&c.}, \quad \ldots \quad (1)$$

$$\gamma_m=P_0+P_1\cos\frac{f}{g}mz+Q_1\sin\frac{f}{g}mz+P_2\cos 2\frac{f}{g}mz+Q_2\sin 2\frac{f}{g}mz+\text{\&c.}, \quad \ldots \quad (2)$$

$$\alpha_m=\beta_m+\gamma_m. \quad \ldots \quad (3)$$

The observations will furnish r equations of the following form:—

$$\alpha_m=\left\{\begin{array}{l}+p_0+p_1\cos mz+q_1\sin mz+p_2\cos 2mz+q_2\sin 2mz+\text{\&c.}\\ +P_0+P_1\cos\frac{f}{g}mz+Q_1\sin\frac{f}{g}mz+P_2\cos 2\frac{f}{g}mz+Q_2\sin 2\frac{f}{g}mz+\text{\&c.}\end{array}\right\} \quad . \quad (4)$$

and the most probable values of p_0, p_1, q_1, &c., P_0, P_1, Q_1, &c. will be those which give a minimum value to

$$\sum_{m=0}^{m=r-1}\left[-\alpha_m=\left\{\begin{array}{l}+p_0+p_1\cos mz+q_1\sin mz+p_2\cos 2mz+q_2\sin 2mz+\text{\&c.}\\ +P_0+P_1\cos\frac{f}{g}mz+Q_1\sin\frac{f}{g}mz+P_2\cos 2\frac{f}{g}mz+Q_2\sin 2\frac{f}{g}mz+\text{\&c.},\end{array}\right\}\right]^2. \quad (5)$$

the sum being taken between the limits $m=0$ and $m=r-1$ for integral values of m.

This will have such a value when the differential coefficients of it with respect to

* It will be shown later that it will suffice that f and g *very nearly* satisfy this condition; but it is convenient in what immediately follows to regard them as doing so rigidly.

each of the quantities p_0, p_1, q_1, &c., P_0, P_1, Q_1, &c. vanish*; or, dividing out the factor 2, when

$$\left.\begin{aligned}
0&=\sum_{m=0}^{m=r-1}\left[-\alpha_m+\beta_m+\gamma_m\right],\\
0&=\sum_{m=0}^{m=r-1}\left[\cos\ mz(-\alpha_m+\beta_m+\gamma_m)\right],\\
0&=\sum_{m=0}^{m=r-1}\left[\sin\ mz(-\alpha_m+\beta_m+\gamma_m)\right],\\
0&=\sum_{m=0}^{m=r-1}\left[\cos 2mz(-\alpha_m+\beta_m+\gamma_m)\right],\\
0&=\sum_{m=0}^{m=r-1}\left[\sin 2mz(-\alpha_m+\beta_m+\gamma_m)\right],\\
&\quad\&c.\qquad\qquad \&c.,\\
0&=\sum_{m=0}^{m=r-1}\left[\cos tmz(-\alpha_m+\beta_m+\gamma_m)\right],\\
0&=\sum_{m=0}^{m=r-1}\left[\sin\ tmz(-\alpha_m+\beta_m+\gamma_m)\right],\\
0&=\sum_{m=0}^{m=r-1}\left[\cos\frac{f}{g}mz(-\alpha_m+\beta_m+\gamma_m)\right],\\
0&=\sum_{m=0}^{m=r-1}\left[\sin\frac{f}{g}mz(-\alpha_m+\beta_m+\gamma_m)\right],\\
0&=\sum_{m=0}^{m=r-1}\left[\cos 2\frac{f}{g}mz(-\alpha_m+\beta_m+\gamma_m)\right],\\
0&=\sum_{m=0}^{m=r-1}\left[\sin 2\frac{f}{g}mz(-\alpha_m+\beta_m+\gamma_m)\right],\\
&\quad\&c.\qquad\qquad \&c.,\\
0&=\sum_{m=0}^{m=r-1}\left[\cos t\frac{f}{g}mz(-\alpha_m+\beta_m+\gamma_m)\right],\\
0&=\sum_{m=0}^{m=r-1}\left[\sin t\frac{f}{g}mz(-\alpha_m+\beta_m+\gamma_m)\right]
\end{aligned}\right\}\quad . \quad . \quad . \quad . \quad (6)$$

Representing by s the suffix of a p, q, P, or Q in a type term of $(\beta_m+\gamma_m)$ in each of the equations (6) in turn, and by t the integral numerical factor of the angle in a type of the sine or cosine which multiplies $(-\alpha_m+\beta_m+\gamma_m)$, let us note that

$$\sum_{m=0}^{m=r-1}\cos s\frac{f}{g}mz\cos tmz=\tfrac{1}{2}\left[\frac{\sin\frac{1}{2}rz\left\{s\frac{f}{g}+t\right\}}{\sin\frac{1}{2}z\left\{s\frac{f}{g}+t\right\}}\cos\tfrac{1}{2}(r+1)z\left\{s\frac{f}{g}+t\right\}+\frac{\sin\frac{1}{2}rz\left\{s\frac{f}{g}-t\right\}}{\sin\frac{1}{2}z\left\{s\frac{f}{g}-t\right\}}\cos\tfrac{1}{2}(r+1)z\left\{s\frac{f}{g}-t\right\}\right],\quad \text{(a)}$$

* The second differential coefficients being all squares, and therefore positive, there is no ambiguity as to whether equations (6) correspond to a maximum or minimum value of (5).

which, since $rz=2g\pi$, and if $z\left\{s\frac{f}{g}\pm t\right\}$ be not a multiple of 2π

or $(sf\pm tg)$ not a multiple of r,

$$=0. \qquad \text{(b)}$$

Similarly, under the same conditions,

$$\sum_{m=0}^{m=r-1}\sin s\frac{f}{g}mz\sin tmz=0; \qquad \text{(c)}$$

and

$$\left.\begin{array}{ll}\displaystyle\sum_{m=0}^{m=r-1}\sin s\frac{f}{g}mz\cos tmz=0, & \text{(d)}\\ \displaystyle\sum_{m=0}^{m=r-1}\cos s\frac{f}{g}mz\sin tmz=0, & \text{(e)}\end{array}\right\}\text{ invariably.}$$

If, now, we define as follows,

$a_0=$	$+\{$	the sum of all the values of	p_s	for which	s is 0, or	such that	sg	is a multiple of $r\}$
	$+\{$	„	P_s	„	s is 0, or	„	sf	„ $\}$,
$a_1=$	$+\{$	„	p_s	„	s is 1, or	„	$(s\mp1)g$	„ $\}$
	$+\{$	„	P_s	„	s is	„	$(sf\mp g)$	„ $\}$,
$b_1=$	$+\{$	„	q_s	„	s is 1, or	„	$(s-1)g$	„ $\}$
	$-\{$	„	q_s	„	s is	„	$(s+1)g$	„ $\}$
	$+\{$	„	Q_s	„	s „	„	$(sf-g)$	„ $\}$
	$-\{$	„	Q_s	„	s „	„	$(sf+g)$	„ $\}$,
$a_2=$	$+\{$	„	p_s	„	s is 2, or	„	$(s\mp2)g$	„ $\}$
	$+\{$	„	P_s	„	s is	„	$(sf\mp2g)$	„ $\}$,
$b_2=$	$+\{$	„	q_s	„	s is 2, or	„	$(s-2)g$	„ $\}$
	$-\{$	„	q_s	„	s is	„	$(s+2)g$	„ $\}$
	$+\{$	„	Q_s	„	s „	„	$(sf-2g)$	„ $\}$
	$-\{$	„	Q_s	„	s „	„	$(sf+2g)$	„ $\}$,
&c.			&c.		&c.		&c.	
$a_t=$	$+\{$	„	p_s	„	s is t, or	„	$(s\mp t)g$	„ $\}$
	$+\{$	„	P_s	„	s is	„	$(sf\mp tg)$	„ $\}$,
$b_t=$	$+\{$	„	q_s	„	s is t, or	„	$(s-t)g$	„ $\}$
	$-\{$	„	q_s	„	s is	„	$(s+t)g$	„ $\}$
	$+\{$	„	Q_s	„	s „	„	$(sf-tg)$	„ $\}$
	$-\{$	„	Q_s	„	s „	„	$(sf+tg)$	„ $\}$,
$A_1=$	$+\{$	„	P_s	„	s is 1, or	„	$(s\mp1)f$	„ $\}$
	$+\{$	„	p_s	„	s is	„	$(sg\mp f)$	„ $\}$,
$B_1=$	$+\{$	„	Q_s	„	s is 1, or	„	$(s-1)f$	„ $\}$
	$-\{$	„	Q_s	„	s is	„	$(s+1)f$	„ $\}$
	$+\{$	„	q_s	„	s „	„	$(sg-f)$	„ $\}$
	$-\{$	„	q_s	„	s „	„	$(sg+f)$	„ $\}$,
$A_2=$	$+\{$	„	P_s	„	s is 2, or	„	$(s\mp2)f$	„ $\}$
	$+\{$	„	p_s	„	s is	„	$(sg\mp2f)$	„ $\}$,

(7)

$$B_2=\left[\begin{array}{lllllllll} +\{\text{the sum of all the values of } Q_s & \text{for which} & s \text{ is } 2, & \text{or such that} & (s-2)f & \text{is a multiple of } r\} \\ -\{\quad ,, \quad ,, \quad Q_s & ,, & s \text{ is} & ,, & (s+2f) & ,, \quad \} \\ +\{\quad ,, \quad ,, \quad q_s & ,, & s \;\; ,, & ,, & (sg-2f) & ,, \quad \} \\ -\{\quad ,, \quad ,, \quad q_s & ,, & s \;\; ,, & ,, & (sg+2f) & ,, \quad \} \end{array}\right],$$

&c. &c. &c. &c.

$$A_t=\left[\begin{array}{llllll} +\{\quad ,, \quad ,, \quad P_s & ,, & s \text{ is } t, \text{ or} & ,, & (s\mp t)f & ,, \quad \} \\ +\{\quad ,, \quad ,, \quad p_s & ,, & s \text{ is} & ,, & (sg\mp tf) & ,, \quad \} \end{array}\right],$$

$$B_t=\left[\begin{array}{llllll} +\{\quad ,, \quad ,, \quad Q_s & ,, & s \text{ is } t, \text{ or} & ,, & (s-t)f & ,, \quad \} \\ -\{\quad ,, \quad ,, \quad Q_s & ,, & s \text{ is} & ,, & (s+t)f & ,, \quad \} \\ +\{\quad ,, \quad ,, \quad q_s & ,, & s \;\; ,, & ,, & (sg-tf) & ,, \quad \} \\ -\{\quad ,, \quad ,, \quad q_s & ,, & s \;\; ,, & ,, & (sg+tf) & ,, \quad \} \end{array}\right],$$

it is easy by means of (a), (b), (c), (d), (e), and other similar formulæ, to convert the equations (6) into

$$0=\sum_{m=0}^{m=r-1}\left[-\alpha_m+\beta_m+\gamma_m\right]=\sum_{m=0}^{m=r-1}\left[-\alpha_m\right]+ra_0,$$

whence
$$a_0=\frac{1}{r}\sum_{m=0}^{m=r-1}\left[\alpha_m\right],$$

$$0=\sum_{m=0}^{m=r-1}\left[\cos mz(-\alpha_m+\beta_m+\gamma_m)\right]=\sum_{m=0}^{m=r-1}\left[\cos mz(-\alpha_m)\right]+\frac{r}{2}a_1,$$

whence
$$a_1=\frac{2}{r}\sum_{m=0}^{m=r-1}\left[\alpha_m\cos mz\right],$$

$$0=\sum_{m=0}^{m=r-1}\left[\sin mz(-\alpha_m+\beta_m+\gamma_m)\right]=\sum_{m=0}^{m=r-1}\left[\sin mz(-\alpha_m)\right]+\frac{r}{2}b_1,$$

whence
$$b_1=\frac{2}{r}\sum_{m=0}^{m=r-1}\left[\alpha_m\sin mz\right],$$

$$0=\sum_{m=0}^{m=r-1}\left[\cos 2mz(-\alpha_m+\beta_m+\gamma_m)\right]=\sum_{m=0}^{m=r-1}\left[\cos 2mz(-\alpha_m)\right]+\frac{r}{2}a_2,$$

whence
$$a_2=\frac{2}{r}\sum_{m=0}^{m=r-1}\left[\alpha_m\cos 2mz\right],$$

$$0=\sum_{m=0}^{m=r-1}\left[\sin 2mz(-\alpha_m+\beta_m+\gamma_m)\right]=\sum_{m=0}^{m=r-1}\left[\sin 2mz(-\alpha_m)\right]+\frac{r}{2}b_2,$$

whence
$$b_2=\frac{2}{r}\sum_{m=0}^{m=r-1}\left[\alpha_m\sin 2mz\right];$$

&c. &c. &c.;

$$0=\sum_{m=0}^{m=r-1}\left[\cos tmz(-\alpha_m+\beta_m+\gamma_m)\right]=\sum_{m=0}^{m=r-1}\left[\cos tmz(-\alpha_m)\right]+\frac{r}{2}a_t,$$

whence
$$a_t=\frac{2}{r}\sum_{m=0}^{m=r-1}\left[\alpha_m\cos tmz\right];$$

$$0=\sum_{m=0}^{m=r-1}[\sin tmz(-\alpha_m+\beta_m+\gamma_m)]=\sum_{m=0}^{m=r-1}[\sin tmz(-\alpha_m)]+\frac{r}{2}b_t,$$

whence
$$b_t=\frac{2}{r}\sum_{m=0}^{m=r-1}[\alpha_m \sin tmz];$$

$$0=\sum_{m=0}^{m=r-1}[\cos\tfrac{f}{g}mz(-\alpha_m+\beta_m+\gamma_m)]=\sum_{m=0}^{m=r-1}[\cos\tfrac{f}{g}mz(-\alpha_m)]+\frac{r}{2}A_1,$$

whence
$$A_1=\frac{2}{r}\sum_{m=0}^{m=r-1}[\alpha_m \cos\tfrac{f}{g}mz];$$

$$0=\sum_{m=0}^{m=r-1}[\sin\tfrac{f}{g}mz(-\alpha_m+\beta_m+\gamma_m)]=\sum_{m=0}^{m=r-1}[\sin\tfrac{f}{g}mz(-\alpha_m)]+\frac{r}{2}B_1,$$

whence
$$B_1=\frac{2}{r}\sum_{m=0}^{m=r-1}[\alpha_m \sin\tfrac{f}{g}mz];$$

$$0=\sum_{m=0}^{m=r-1}[\cos 2\tfrac{f}{g}mz(-\alpha_m+\beta_m+\gamma_m)]=\sum_{m=0}^{m=r-1}[\cos 2\tfrac{f}{g}mz(-\alpha_m)]+\frac{r}{2}A_2,$$

whence
$$A_2=\frac{2}{r}\sum_{m=0}^{m=r-1}[\alpha_m \cos 2\tfrac{f}{g}mz];$$

$$0=\sum_{m=0}^{m=r-1}[\sin 2\tfrac{f}{g}mz(-\alpha_m+\beta_m+\gamma_m)]=\sum_{m=0}^{m=r-1}[\sin 2\tfrac{f}{g}mz(-\alpha_m)]+\frac{r}{2}B_2,$$

whence
$$B_2=\frac{2}{r}\sum_{m=0}^{m=r-1}[\alpha_m \sin 2\tfrac{f}{g}mz];$$

&c. &c. &c.;

$$0=\sum_{m=0}^{m=r-1}[\cos t\tfrac{f}{g}mz(-\alpha_m+\beta_m+\gamma_m)]=\sum_{m=0}^{m=r-1}[\cos t\tfrac{f}{g}mz(-\alpha_m)]+\frac{r}{2}A_t,$$

whence
$$A_t=\frac{2}{r}\sum_{m=0}^{m=r-1}[\alpha_m \cos t\tfrac{f}{g}mz],$$

$$0=\sum_{m=0}^{m=r-1}[\sin t\tfrac{f}{g}mz(-\alpha_m+\beta_m+\gamma_m)]=\sum_{m=0}^{m=r-1}[\sin t\tfrac{f}{g}mz(-\alpha_m)]+\frac{r}{2}B_t,$$

whence
$$B_t=\frac{2}{r}\sum_{m=0}^{m=r-1}[\alpha_m \sin t\tfrac{f}{g}mz]. \qquad (8)$$

5 In any special inquiry, having found by (8) the numerical values of a_0, a_1, b_1, a_2, b_2, &c., A_1, B_1, A_2, B_2, &c., we may insert these in the equations (7), which it will now be desirable to consider the significance of. If our object was simply to find two periodical phenomena which would jointly satisfy the r observations, then this could be done with the same degree of closeness in an infinite variety of ways; for we might give to the several terms of the right-hand members of (7) any arbitrary values consistently with their sum being equal to the left-hand member, and so long as the same coefficient is taken of the same value in all the equations (7). But although all the varieties would agree in giving the same value of the combined phenomena *at any one*

of the r times of observation, they would all generally differ as to its value at any time intermediate between any consecutive two of the r observations. In the first of the equations (7), if we were to attribute the whole of a_0 to p_0 or P_0, it would imply that the phenomenon a_0 occurred at all times irrespective of any periodicity; but if we attribute it all to (say) $p\frac{r}{g}$, it would imply that the phenomenon a_0 occurred only at the times of observation, whilst at intermediate times the corresponding phenomenon would be represented by $a_0 \cos\frac{r}{g} mz$, which passes through a complete cycle of change during the interval between every two consecutive observations, or whilst m passes from one integral value to the next, and combined with this there may be a phenomenon represented by $q\frac{r}{g} \sin\frac{r}{g} mz$ of any arbitrary range. Similarly, the distinction between the different terms of the other equations is that they go through a full cycle of change in different periods, and graphically each term would be represented by a complete wave whose length corresponded to the period of that term.

6. As the mathematical theory of this process affords no criterion for selection, we ought to find reasons apart from it for preferring particular appropriations of a_0, a_1, b_1, &c to the several component parts of their equalities; otherwise it is clear from what has been said that no useful result will be attained. It may be remarked that an ambiguity, similar to the one under consideration, attaches to BESSEL'S treatment of a single periodical phenomenon, the values corresponding to our a_0, a_1, b_1, &c being given at the foot of page 26, Section III. of BESSEL'S paper. BESSEL remarks that if we compare a mathematical theory of any periodical phenomenon, based on physical principles, with the observations, his expression for the values of the phenomenon is more convenient for the purpose than the observations themselves—the reason of this being that, as the expression given by the mathematical theory is developed in the form in which the observations have been expressed, the two expressions may be compared term by term, or by equal subordinate periods. This is probably the most important use of the method; and as the most striking features of a variation will generally be those of long period, they may be examined apart from the others. The next most important use of this method is probably that which has for its object the elimination of casual irregularities from the observations; but this is served only when the subordinate variations of short period are rejected; and after such rejection, it must always be borne in mind that the remaining expression is incomplete: this does not, however, interfere with the comparison of the subordinate variations retained with other phenomena of nature involving variations of the same subordinate periods; indeed by indicating the periods followed by the subordinate variations which are of largest amount, it suggests a means of distinguishing other phenomena that on examination may be found to be related to the one which is the subject of the observations. The reason assigned by BESSEL for giving preference to the terms of long period, viz. that "the development of the

expression which represents the given values of y will in general only be interesting when it converges so rapidly that only a few of the first terms have appreciable values," had reference doubtless to the incompleteness of the partial expression—this being of no consequence when the rejected part, the absence of which makes the expression incomplete, is of inconsiderable amount. We may, however, be guided as to the validity of this reason by noting well whether the values of a_t, b_t, A_t, B_t, &c. do themselves become inappreciable whilst t is still small.

7 *Now in many special inquiries f, g, and r will have such values that $(s\mp t)g$, $(sf\mp tg)$, $(s\mp t)f$, $(sg\mp tf)$,* &c. will first become a multiple of r only when s or t has ceased to be small; in which case, following BESSEL, we may neglect as inappreciable all the terms on the right-hand side of equation (7), except p_0 and P_0 in the first equation and the first term of each of the others; we then have

$$p_0+P_0=a_0=\frac{1}{r}\sum_{m=0}^{m=r-1}[\alpha_m],$$

$$\left.\begin{aligned}
p_1=a_1&=\frac{2}{r}\sum_{m=0}^{m=r-1}[\alpha_m\cos mz],\\
q_1=b_1&=\frac{2}{r}\sum_{m=0}^{m=r-1}[\alpha_m\sin mz],\\
p_2=a_2&=\frac{2}{r}\sum_{m=0}^{m=r-1}[\alpha_m\cos 2mz],\\
q_2=b_2&=\frac{2}{r}\sum_{m=0}^{m=r-1}[\alpha_m\sin 2mz],\\
\&c \qquad \&c. &\qquad \&c,\\
p_t=a_t&=\frac{2}{r}\sum_{m=0}^{m=r-1}[\alpha_m\cos tmz],\\
q_t=b_t&=\frac{2}{r}\sum_{m=0}^{m=r-1}[\alpha_m\sin tmz],
\end{aligned}\right\}\text{which are the same values as those that would be found by applying BESSEL's method to the } r \text{ observations, on the supposition that they are unaffected by the phenomenon whose period is } x'.$$

$$\left.\begin{aligned}
P_1=A_1&=\frac{2}{r}\sum_{m=0}^{m=r-1}\left[\alpha_m\cos\frac{f}{g}mz\right],\\
Q_1=B_1&=\frac{2}{r}\sum_{m=0}^{m=r-1}\left[\alpha_m\sin\frac{f}{g}mz\right],\\
P_2=A_2&=\frac{2}{r}\sum_{m=0}^{m=r-1}\left[\alpha_m\cos 2\frac{f}{g}mz\right],\\
Q_2=B_2&=\frac{2}{r}\sum_{m=0}^{m=r-1}\left[\alpha_m\sin 2\frac{f}{g}mz\right],\\
\&c. \qquad \&c &\qquad \&c,\\
P_t=A_t&=\frac{2}{r}\sum_{m=0}^{m=r-1}\left[\alpha_m\cos t\frac{f}{g}mz\right],\\
Q_t=B_t&=\frac{2}{r}\sum_{m=0}^{m=r-1}\left[\alpha_m\sin t\frac{f}{g}mz\right];
\end{aligned}\right\}\text{which are the same values as those that would be found by applying BESSEL's method to the } r \text{ observations, on the supposition that they are unaffected by the phenomenon whose period is } x. \qquad (9)$$

8. If instead of applying BESSEL's process at once to each individual observation, we had begun by finding a mean value $\Sigma\frac{\beta_m}{g}$ (as affected by the other phenomenon) of the one phenomenon at a particular phase of its period $\varkappa$, and then proceeded to apply BESSEL's process to the $\frac{r}{g}$ mean values of this character, we should have arrived at precisely the same results.

We might also have regarded a hypothetical complex phenomenon of period $g\varkappa$ *as being produced solely by the recurrence of the phenomena whose periods are* $\varkappa$ and $\varkappa'$, and finding by BESSEL's process from the r observations the coefficients of its expression —from these determining the coefficients of the expressions for the component periodical phenomena; this, too, would have led to the same results.

9. To conclude this section, we draw from what has preceded the following practical rule for deducing from a series of observations of the combined effect of several independent phenomena (observations taken at equal intervals of time) the coefficients of BESSEL's series for each separate phenomenon —Find the least integral numbers f, g, h, &c. which are proportional (or nearly so) to the periods $\varkappa$, $\varkappa'$, $\varkappa''$, &c. of the several phenomena, and let v be the least common multiple of those numbers, choose then for treatment observations extending exactly over some multiple of the period $\frac{v}{f}\varkappa$, and note whether any values of p_s or q_s, P_s or Q_s, &c, for which s is small, other than the first terms, enter into the equations (7), if not, proceed to apply BESSEL's method to determine from the observations the coefficients of the expression of each phenomenon, just as would be done if the observations were unaffected by the other phenomena

II.

10. It will be useful further to estimate in what degree the phenomenon whose period is $\varkappa'$ affects the values of the constants p_1, q_1, &c., in the expression of the phenomenon whose period is $\varkappa$, when the number (R) of observations is greater than and not a multiple of r. And here, confining our attention to strictly and exclusively periodical phenomena, we must reject the constant term (p_0+P_0) in the expression for the combined phenomena: this is equivalent to substituting for the original observations $\alpha_0, \alpha_1, \alpha_2, \ldots \ldots \alpha_m$ the excesses of them respectively above their mean value $\Sigma\frac{\alpha_m}{R}$. Let $\frac{c\varkappa}{x}=R=\frac{2c\pi}{z}$, c being an integer, and let $c\varkappa=(d+e)f\varkappa'$, d being integral and e a proper fraction. If we represent β_m by the general term

$$[p_s \cos smz+q_s \sin smz],$$

and γ_m by the general term

$$\left[P_s \cos s\frac{f}{g}mz+Q_s \sin s\frac{f}{g}mz\right],$$

the first set of expressions of (6) may be put into the following typical form,

$$\left.\begin{aligned}&\sum_{m=0}^{m=r-1}[p_s\cos smz+q_s\sin smz]\cos tmz\\&\quad=\sum_{m=0}^{m=r-1}[\alpha_m\cos tmz]-\sum_{m=0}^{m=r-1}\left[\mathrm{P}_s\cos s\frac{f}{g}mz+\mathrm{Q}_s\sin s\frac{f}{g}mz\right]\cos tmz,\end{aligned}\right\}\quad .\quad (10)$$

which, in the case before us, is

$$\left.\begin{aligned}&\sum_{m=0}^{m=\mathrm{R}-1}[p_s\cos smz+q_s\sin smz]\cos tmz\\&\quad=\sum_{m=0}^{m=\mathrm{R}-1}[\alpha_m\cos tmz]-\sum_{m=0}^{m=\mathrm{R}-1}\left[\mathrm{P}_s\cos s\frac{f}{g}mz+\mathrm{Q}_s\sin s\frac{f}{g}mz\right]\cos tmz;\end{aligned}\right\}\quad .\quad (11)$$

and since $\mathrm{R}z=2c\pi$, and neglecting, with BESSEL, the terms of β_m and γ_m for which s is not small, and also dividing through by $\frac{\mathrm{R}}{2}$, this becomes

$$p_t=\frac{2}{\mathrm{R}}\sum_{m=0}^{m=\mathrm{R}-1}[\alpha_m\cos tmz]-\frac{2}{\mathrm{R}}\sum_{m=0}^{m=\mathrm{R}-1}\left[\mathrm{P}_s\cos s\frac{f}{g}mz+\mathrm{Q}_s\sin s\frac{f}{g}mz\right]\cos tmz.\quad .\quad (12)$$

But after each successive period $f\varkappa'$ the quantity

$$\left[\mathrm{P}_s\cos s\frac{f}{g}mz+\mathrm{Q}_s\sin s\frac{f}{g}mz\right]\cos tmz$$

passes again, in each of its terms, through the same identical values; it is therefore a proper periodical function, and passes at the same phase of each period $f\varkappa'$ through some maximum value, which cannot ever be of magnitude so great as the sum of all the P's and Q's disregarding their signs; much less can

$$\frac{1}{\mathrm{R}}\sum_{m=0}^{m=\mathrm{R}-1}\left[\mathrm{P}_s\cos s\frac{f}{g}mz+\mathrm{Q}_s\sin s\frac{f}{g}mz\right]\cos tmz$$

ever reach that sum; hence the last term of (12) can never be so great as twice the sum of all the P's and Q's regardless of signs. Suppose this to be its value at some time during the first period $f\varkappa'$, then at no time in the second period $f\varkappa'$ can it exceed the half of this, since R will have been at least doubled, whilst the part under the sign of summation cannot have increased; similarly, at no time during the nth period $f\varkappa'$ can its value be of greater magnitude than $\frac{2}{n}$ths of the sum of its P's and Q's regardless of signs. Hence if n be made large enough, *i. e.* if the observations be sufficiently extended, this quantity can always be reduced till its effect upon the value of p_t is inappreciable.

11. Now it has been shown in the preceding investigation that, as R increases and passes successively through the values $\frac{f\varkappa'}{x}$, $\frac{2f\varkappa'}{x}$, $\frac{3f\varkappa'}{x}$ . . . $\frac{df\varkappa'}{x}$, &c., the quantity

$$\sum_{m=0}^{m=\mathrm{R}-1}\left[\mathrm{P}_s\cos s\frac{f}{g}mz+\mathrm{Q}_s\sin s\frac{f}{g}mz\right]\cos tmz$$

vanishes at each passage; when, therefore, the series of observations is not sufficiently extensive to obliterate the effect of the last term of (12), it may be worth while, in the first place, to calculate approximately the values of P_1, Q_1, P_2, Q_2, &c., choosing for the purpose a number of observations R′ which very nearly completes an integral number of periods fx', and thence the value of

$$\Sigma_s\left[P_s \cos s\frac{f}{g}mz+Q_s \sin s\frac{f}{g}mz\right]\cos tmz$$

for the fractional part of a period fx' which is in excess of the last completed period.

Similar reasoning, with a similar result, may be applied to each of the expressions of (6), of which (10) is a type.

III.

12. The variations in a series of n observations (equidistant in time) are by hypothesis due to a periodical phenomenon whose true expression is

$$\alpha_m=p_0+p_1 \cos mz+q_1 \sin mz+p_2 \cos 2mz+q_2 \sin 2mz+\&c., \quad . \quad . \quad . \quad (13)$$

in relation to which $z=\frac{2c\pi}{x}$, c=a constant integer not small, $\varkappa$=the period of the phenomenon, $x=\varkappa\frac{c}{n}$=the interval of time corresponding to the angle z, t=the time reckoned from the commencement of the observations, and $m=\frac{t}{x}$. Let the interval between successive observations be $(x+\Delta x)$, so that the n observations will extend over a period $n(x+\Delta x)=c(\varkappa+\Delta\varkappa)$. The angle $(z+i)$ which corresponds to the interval of time $(x+\Delta x)$ will be equal to $z\frac{x+\Delta x}{x}=z+z\frac{\Delta x}{x}$ or $i=z\frac{\Delta x}{x}=z\frac{\Delta\varkappa}{\varkappa}$; let this be so small that sni is also a small angle, s being the suffix of a p or q. Under these conditions, to find the coefficients p_1, q_1, &c. Let it first be observed that the condition that sni is a small angle, implies that n has been so chosen that $(\varkappa+\Delta\varkappa)$ approximates as closely as possible to the known or assumed value of $\varkappa$. The phenomenon α_m occurring at the time mx, let that which occurs at the time $m(x+\Delta x)$ be called $\alpha_{m'}$; then we shall have

$$\alpha_{m'}=p_0+p_1 \cos m(z+i)+q_1 \sin m(z+i)+p_2 \cos 2m(z+i)+q_2 \sin 2m(z+i)+\&c., \quad (14)$$

the general term of which is $\{p_s \cos sm(z+i)+q_s \sin sm(z+i)\}$, where s represents the positive integral suffix of a p or q; and we may, for shortness, write

$$\alpha_{m'}=[p_s \cos sm(z+i)+q_s \sin sm(z+i)], \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

the square brackets indicating that the general term within them is to represent the sum of its series of values when for s is put 0, 1, 2, 3, &c. . . . . successively, whence

$$\alpha_{m'}=[p_s(\cos smz \cos smi-\sin smz \sin smi)+q_s(\sin smz \cos smi+\cos smz \sin smi)], \quad (16)$$

and smi being a small angle, we may write for its sine smi, and for its cosine $(1-\frac{1}{2}s^2m^2i^2)$, when we obtain

$$\left.\begin{array}{l}\alpha_{m'}=\Big[(p_s \cos smz+q_s \sin smz)+si(q_s m \cos smz-p_s m \sin smz) \\ \qquad \frac{s^2i^2}{2}(p_s m^2 \cos smz+q_s m^2 \sin smz)\Big].\end{array}\right\} \quad . \quad . \quad . \quad (17)$$

Multiplying both sides by $\cos tmz$, t being any positive integer,

$$\left.\begin{array}{l}\alpha_{m'} \cos tmz=\alpha_m \cos tmz+\Big[si(q_s m \cos smz \cos tmz-p_s m \sin smz \cos tmz) \\ \qquad -\frac{s^2i^2}{2}(p_s m^2 \cos smz \cos tmz+q_s m^2 \sin smz \cos tmz)\Big]\,;\end{array}\right\} \quad . \quad (18)$$

and taking the sum on both sides from $m=0$ to $m=(n-1)$,

$$\left.\begin{array}{l}\sum\limits_{m=0}^{m=n-1} \alpha_{m'} \cos tmz=\sum\limits_{m=0}^{m=n-1} \alpha_m \cos tmz \\ +\sum\limits_{m=0}^{m=n-1}\Big[\frac{si}{2}\{q_s(m \cos(s+t)mz+m \cos(s-t)mz)-p_s(m \sin(s+t)mz+m \sin(s-t)mz)\} \\ \quad -\frac{s^2i^2}{4}\{p_s(m^2\cos(s+t)mz+m^2\cos(s-t)mz)+q_s(m^2\sin(s+t)mz+m^2\sin(s-t)mz)\}\Big].\end{array}\right\} \quad (19)$$

Now observing, from the collected equations at the end of the first set of demonstrations in the Appendix, that when $nv=2c\pi$, and according as v is not or is 0 or a multiple of 2π,

$$\left.\begin{array}{ll}\sum\limits_{m=0}^{m=n-1} m \cos mv=-\frac{n}{2}, & \text{or } \frac{n^2}{2}-\frac{n}{2}, \\ \sum\limits_{m=0}^{m=n-1} m \sin mv=-\frac{n}{2}\cot\frac{v}{2}, & \text{or } 0, \\ \sum\limits_{m=0}^{m=n-1} m^2 \cos mv=-\frac{n^2}{2}+\frac{n}{2}\frac{1}{\sin^2\frac{v}{2}}, & \text{or } \frac{n^3}{3}-\frac{n^2}{2}+\frac{n}{6}, \\ \sum\limits_{m=0}^{m=n-1} m^2 \sin mv=-\frac{n^2}{2}\cot\frac{v}{2}, & \text{or } 0.\end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad (20)$$

And as in equation (19) $nz=2c\pi$, applying equations (20), equation (19) becomes

$$\begin{array}{l}\sum\limits_{m=0}^{m=n-1} \alpha_{m'} \cos tmz=\sum\limits_{m=0}^{m=n-1} \alpha_m \cos tmz+\Big[\frac{si}{2}\Big\{q_s\Big(-\frac{n}{2}-\frac{n}{2}\Big)-p_s\Big(-\frac{n}{2}\cot\frac{s+t}{2}z-\frac{n}{2}\cot\frac{s-t}{2}z\Big)\Big\} \\ \qquad -\frac{s^2i^2}{4}\Big\{p_s\Big(-\frac{n^2}{2}+\frac{n}{2}\frac{1}{\sin^2\frac{s+t}{2}z}-\frac{n^2}{2}+\frac{n}{2}\frac{1}{\sin^2\frac{s-t}{2}z}\Big) \\ \qquad +q_s\Big(-\frac{n^2}{2}\cot\frac{s+t}{2}z-\frac{n^2}{2}\cot\frac{s-t}{2}z\Big)\Big\}\Big],\end{array}$$

$$\left.\begin{aligned}
&\text{or } =\sum_{m=0}^{m=n-1}\alpha_m\cos tmz+\Big[\tfrac{si}{2}\Big\{q_s\Big(-\tfrac{n}{2}+\tfrac{n^2}{2}-\tfrac{n}{2}\Big)-p_s\Big(-\tfrac{n}{2}\cot\tfrac{s+t}{2}z+0\Big)\Big\}\\
&\qquad-\tfrac{s^2i^2}{4}\Big\{p_s\Big(-\tfrac{n^2}{2}+\tfrac{n}{2}\frac{1}{\sin^2\frac{s+t}{2}z}+\tfrac{n^3}{3}-\tfrac{n^2}{2}+\tfrac{n}{6}\Big)\\
&\qquad+q_s\Big(-\tfrac{n^2}{2}\cot\tfrac{s+t}{2}z+0\Big)\Big\}\Big],\\
&\text{or } =\sum_{m=0}^{m=n-1}\alpha_m\cos tmz+\Big[\tfrac{si}{2}\Big\{q_s\Big(\tfrac{n^2}{2}-\tfrac{n}{2}-\tfrac{n}{2}\Big)-p_s\Big(0-\tfrac{n}{2}\cot\tfrac{s-t}{2}z\Big)\Big\}\\
&\qquad-\tfrac{s^2i^2}{4}\Big\{p_s\Big(\tfrac{n^3}{3}-\tfrac{n^2}{2}+\tfrac{n}{6}-\tfrac{n^2}{2}+\tfrac{n}{2}\frac{1}{\sin^2\frac{s-t}{2}z}\Big)\\
&\qquad+q_s\Big(0-\tfrac{n^2}{2}\cot\tfrac{s-t}{2}z\Big)\Big\}\Big],\\
&\text{or } =\sum_{m=0}^{m=n-1}\alpha_m\cos tmz+\Big[\tfrac{si}{2}\{q_s n(n-1)\}-\tfrac{s^2i^2}{4}\Big\{p_s\Big(\tfrac{2}{3}n^3-n^2+\tfrac{n}{3}\Big)\Big\}\Big],
\end{aligned}\right\}\quad(21)$$

according as neither $(s+t)z$ nor $(s-t)z$ is 0 or a multiple of 2π; as $(s+t)z$ is not, and $(s-t)z$ is 0 or a multiple of 2π; as $(s+t)z$ is, and $(s-t)z$ is not 0 or a multiple of 2π; or as both $(s+t)z$ and $(s-t)z$ are multiples of 2π.

And multiplying both sides by $\frac{2}{n}$,

$$\left.\begin{aligned}
&\tfrac{2}{n}\sum_{m=0}^{m=n-1}\alpha_{m'}\cos tmz=\tfrac{2}{n}\sum_{m=0}^{m=n-1}\alpha_m\cos tmz+\Big[\tfrac{si}{2}\Big\{-2q_s+p_s\Big(\cot\tfrac{s+t}{2}z+\cot\tfrac{s-t}{2}z\Big)\Big\}\\
&\quad-\tfrac{s^2i^2}{4}\Big\{p_s\Big(-2n+\frac{1}{\sin^2\frac{s+t}{2}z}+\frac{1}{\sin^2\frac{s-t}{2}z}\Big)-q_s\Big(n\cot\tfrac{s+t}{2}z+n\cot\tfrac{s-t}{2}z\Big)\Big\}\Big],\\
&\qquad\text{or}=\tfrac{2}{n}\sum_{m=0}^{m=n-1}\alpha_m\cos tmz+\Big[\tfrac{si}{2}\Big\{q_s(n-2)+p_s\Big(\cot\tfrac{s+t}{2}z\Big)\Big\}\\
&\quad-\tfrac{s^2i^2}{4}\Big\{p_s\Big(\tfrac{2}{3}n^2-2n+\tfrac{1}{3}+\frac{1}{\sin^2\frac{s+t}{2}z}\Big)-q_s\Big(n\cot\tfrac{s+t}{2}z\Big)\Big\}\Big],\\
&\qquad\text{or}=\tfrac{2}{n}\sum_{m=0}^{m=n-1}\alpha_m\cos tmz+\Big[\tfrac{si}{2}\Big\{q_s(n-2)+p_s\Big(\cot\tfrac{s-t}{2}z\Big)\Big\}\\
&\quad-\tfrac{s^2i^2}{4}\Big\{p_s\Big(\tfrac{2}{3}n^2-2n+\tfrac{1}{3}+\frac{1}{\sin^2\frac{s-t}{2}z}\Big)-q_s\Big(n\cot\tfrac{s-t}{2}z\Big)\Big\}\Big],\\
&\qquad\text{or}=\tfrac{2}{n}\sum_{m=0}^{m=n-1}\alpha_m\cos tmz+\Big[\tfrac{si}{2}\Big\{q_s\,2(n-1)\Big\}-\tfrac{s^2i^2}{4}\{p_s(\tfrac{4}{3}n^2-2n+\tfrac{2}{3})\}\Big],
\end{aligned}\right\}\quad(22)$$

according as neither $(s+t)z$ nor $(s-t)z$ is 0 or a multiple of 2π; as $(s+t)z$ is not, but $(s-t)z$ is 0 or a multiple of 2π; as $(s+t)z$ is, but $(s-t)z$ is not 0 or a multiple of 2π; or as both $(s+t)z$ and $(s-t)z$ are multiples of 2π.

Now by BESSEL'S process, and assuming, as we shall, that only a few of the first terms of the expression for α_m have considerable coefficients,

$$\frac{2}{n}\sum_{m=0}^{m=n-1}\alpha_m \cos tmz = p_t, \quad \text{and } \frac{2}{n}\sum_{m=0}^{m=n-1}\alpha_m \sin tmz = q_t.$$

Also let

$$\frac{2}{n}\sum_{m=0}^{m=n-1}\alpha_{m'} \cos tmz = \mathrm{P}_t, \quad \text{and } \frac{2}{n}\sum_{m=0}^{m=n-1}\alpha_{m'} \sin tmz = \mathrm{Q}_t.$$

Therefore, writing a_t and a_t respectively for the coefficients of i and i^2 in (22), and transposing,

$$p_t = \mathrm{P}_t - a_t i - \mathrm{a}_t i^2. \quad \ldots \quad (23)$$

Proceeding in a similar manner, we find.—

$$\left.\begin{aligned}
&\alpha_{m'} \sin tmz = \alpha_m \sin tmz \\
&+\Big[si(q_s m \cos smz \sin tmz - p_s m \sin smz \sin tmz) - \frac{s^2 i^2}{2}(p_s m^2 \cos smz \sin tmz \\
&+ q_s m^2 \sin smz \sin tmz)\Big];
\end{aligned}\right\} \quad (24)$$

$$\left.\begin{aligned}
&\sum_{m=0}^{m=n-1}\alpha_{m'} \sin tmz = \sum_{m=0}^{m=n-1}\alpha_m \sin tmz \\
&+\sum_{m=0}^{m=n-1}\Big[\frac{si}{2}\Big\{q_s\big(m\sin(s+t)mz - m\sin(s-t)mz\big) - p_s\big(m\cos(s-t)mz - m\cos(s+t)mz\big)\Big\} \\
&-\frac{s^2 i^2}{4}\Big\{p_s\big(m^2\sin(s+t)mz - m^2\sin(s-t)mz\big) + q_s\big(m^2\cos(s-t)mz - m^2\cos(s+t)mz\big)\Big\}\Big];
\end{aligned}\right\} \quad (25)$$

$$\left.\begin{aligned}
&\sum_{m=0}^{m=n-1}\alpha_{m'} \sin tmz = \sum_{m=0}^{m=n-1}\alpha_m \sin tmz \\
&+\Big[\frac{si}{2}\Big\{q_s\Big(-\frac{n}{2}\cot\frac{s+t}{2}z + \frac{n}{2}\cot\frac{s-t}{2}z\Big) - p_s\Big(-\frac{n}{2}+\frac{n}{2}\Big)\Big\} - \frac{s^2 i^2}{4}\Big\{p_s\Big(-\frac{n^2}{2}\cot\frac{s+t}{2}z \\
&+\frac{n^2}{2}\cot\frac{s-t}{2}z\Big) + q_s\Big(-\frac{n^2}{2} + \frac{n}{2}\frac{1}{\sin^2\frac{s-t}{2}z} + \frac{n^2}{2} - \frac{n}{2}\frac{1}{\sin^2\frac{s+t}{2}z}\Big)\Big\}\Big], \\
&\qquad\text{or} = \sum_{m=0}^{m=n-1}\alpha_m \sin tmz \\
&+\Big[\frac{si}{2}\Big\{q_s\Big(-\frac{n}{2}\cot\frac{s+t}{2}z + 0\Big) - p_s\Big(\frac{n^2}{2} - \frac{n}{2} + \frac{n}{2}\Big)\Big\} - \frac{s^2 i^2}{4}\Big\{p\Big(-\frac{n^2}{2}\cot\frac{s+t}{2}z + 0 \\
&+q_s\Big(\frac{n^3}{3} - \frac{n^2}{2} + \frac{n}{6} + \frac{n^2}{2} - \frac{n}{2}\frac{1}{\sin^2\frac{s+t}{2}z}\Big)\Big\}\Big],
\end{aligned}\right\} \quad (26)$$

$$\text{or} = \sum_{m=0}^{m=n-1} \alpha_m \sin tmz$$

$$+\left[\frac{si}{2}\left\{q_s\left(0+\frac{n}{2}\cot\frac{s-t}{2}z\right)-p_s\left(-\frac{n}{2}-\frac{n^2}{2}+\frac{n}{2}\right)\right\}-\frac{s^2i^2}{4}\left\{p_s\left(0+\frac{n^2}{2}\cot\frac{s-}{2}z\right)\right.\right.$$

$$\left.\left.+q_s\left(-\frac{n^2}{2}+\frac{n}{2}\frac{1}{\sin^2\frac{s-t}{2}z}-\frac{n^3}{3}+\frac{n^2}{2}-\frac{n}{6}\right)\right\}\right],$$

$$\text{or} = \sum_{m=0}^{m=n-1} \alpha_m \sin tmz,$$

according as neither $(s+t)z$ nor $(s-t)z$ is 0 or a multiple of 2π; as $(s+t)z$ is not, but $(s-t)z$ is 0 or a multiple of 2π, as $(s+t)z$ is, but $(s-t)z$ is not 0 or a multiple of 2π; or as both $(s+t)z$ and $(s-t)z$ are multiples of 2π.

$$\frac{2}{n}\sum_{m=0}^{m=n-1} \alpha_m \sin tmz = \frac{2}{n}\sum_{m=0}^{m=n-1} \alpha_m \sin tmz$$

$$+\left[\frac{si}{2}\left\{q_s\left(\cot\frac{s-t}{2}z-\cot\frac{s+t}{2}z\right)\right\}-\frac{s^2i^2}{4}\left\{p_s\left(n\cot\frac{s-t}{2}z-n\cot\frac{s+t}{2}z\right)\right.\right.$$

$$\left.\left.+q_s\left(\frac{1}{\sin^2\frac{s-t}{2}z}-\frac{1}{\sin^2\frac{s+t}{2}z}\right)\right\}\right],$$

$$\text{or} = \frac{2}{n}\sum_{m=0}^{m=n-1} \alpha_m \sin tmz$$

$$+\left[\frac{si}{2}\left\{q_s\left(-\cot\frac{s+t}{2}z\right)-p_sn\right\}-\frac{s^2i^2}{4}\left\{p_s\left(-n\cot\frac{s+t}{2}z\right)+q_s\left(\frac{2n^2}{3}+\frac{1}{3}-\frac{1}{\sin^2\frac{s+t}{2}z}\right)\right\}\right], \quad (27)$$

$$\text{or} = \frac{2}{n}\sum_{m=0}^{m=n-1} \alpha_m \sin \alpha \sin tmz$$

$$+\left[\frac{si}{2}\left\{q_s\left(\cot\frac{s-t}{2}z\right)+p_sn\right\}-\frac{s^2i^2}{4}\left\{p_sn\cot\frac{s-t}{2}z+q_s\left(-\frac{2}{3}n^2-\frac{1}{3}+\frac{1}{\sin^2\frac{s-t}{2}z}\right)\right\}\right],$$

$$\text{or} = \frac{2}{n}\sum_{m=0}^{m=n-1} \alpha_m \sin tmz,$$

according as neither $(s+t)z$ nor $(s-t)z$ is 0 or a multiple of 2π, as $(s+t)z$ is not, but $(s-t)z$ is 0 or a multiple of 2π; as $(s+t)z$ is, but $(s-t)z$ is not 0 or a multiple of 2π; or as both $(s+t)z$ and $(s-t)z$ are multiples of 2π

And writing b_t and b_t respectively for the coefficients of i and i^2 in (27), and transposing,

$$q_t = \mathrm{Q}_t - b_ti - \mathrm{b}_ti^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

13. From the general expressions (23) and (28), for the coefficients p_t, q_t we may now write down the particular values p_1, p_2, p_3, q_1, q_2, q_3 for the particular case in which, whilst neither s nor t is taken above 3, neither $(s-t)z$ nor $(s+t)z$ is ever a multiple of 2π; and at the same operation we may substitute for the general terms in which a_t, a_t, b_t, b_t are expressed, the series of terms obtained by giving s the values, 1, 2, 3 ...... successively, observing also that when $s=0$ these terms vanish. We have then,

$$\left.\begin{aligned}
p_1=\mathrm{P}_1-\frac{i}{2}\Big\{&q_1(n-2)+p_1\cot z-4q_2+2p_2\left(\cot\tfrac{3}{2}z+\cot\tfrac{z}{2}\right)-6q_3+3p_3(\cot 2z+\cot z)\Big\}\\
&+\frac{i^2}{4}\Big\{p_1\left(\tfrac{2}{3}n^2-2n+\tfrac{1}{3}+\frac{1}{\sin^2 z}\right)-q_1 n\cot z+4p_2\left(-2n+\frac{1}{\sin^2\frac{3}{2}z}+\frac{1}{\sin^2\frac{z}{2}}\right)\\
&-4q_2\left(n\cot\tfrac{3}{2}z+n\cot\tfrac{z}{2}\right)+9p_3\left(-2n+\frac{1}{\sin^2 2z}+\frac{1}{\sin^2 z}\right)-9q_3(n\cot 2z+n\cot z)\Big\}.
\end{aligned}\right\}\quad(29)$$

$$\left.\begin{aligned}
q_1=\mathrm{Q}_1-\frac{i}{2}\Big\{&-q_1\cot z-p_1 n+2q_2\left(\cot\tfrac{z}{2}-\cot\tfrac{3}{2}z\right)+3q_3(\cot z-\cot 2z)\Big\}\\
&+\frac{i^2}{4}\Big\{-p_1 n\cot z+q_1\left(\tfrac{2}{3}n^2+\tfrac{1}{3}-\frac{1}{\sin^2 z}\right)+4p_2\left(n\cot\tfrac{z}{2}-n\cot\tfrac{3}{2}z\right)\\
&+4q_2\left(\frac{1}{\sin^2\frac{z}{2}}-\frac{1}{\sin^2\frac{3}{2}z}\right)+9p_3(n\cot z-n\cot 2z)+9q_3\left(\frac{1}{\sin^2 z}-\frac{1}{\sin^2 2z}\right)\Big\}.
\end{aligned}\right\}\quad(30)$$

$$\left.\begin{aligned}
p_2=\mathrm{P}_2-\frac{i}{2}\Big\{&-2q_1+p_1\left(\cot\tfrac{3}{2}z-\cot\tfrac{z}{2}\right)+2q_2(n-z)+2p_2(\cot 2z)-6q_3\\
&+3p_3\left(\cot\tfrac{5}{2}z+\cot\tfrac{z}{2}\right)\Big\}+\frac{i^2}{4}\Big\{p_1\left(-2n+\frac{1}{\sin^2\frac{3}{2}z}+\frac{1}{\sin^2\frac{z}{2}}\right)-q_1\left(n\cot\tfrac{3}{2}z-n\cot\tfrac{z}{2}\right)\\
&+4p_2\left(\tfrac{2}{3}n^2-2n+\tfrac{1}{3}+\frac{1}{\sin^2 2z}\right)-4q_2 n\cot 2z+9p_3\left(-2n+\frac{1}{\sin^2\frac{5}{2}z}+\frac{1}{\sin^2\frac{z}{2}}\right)\\
&-9q_3\left(n\cot\tfrac{5}{2}z+n\cot\tfrac{z}{2}\right)\Big\}.
\end{aligned}\right\}\quad(31)$$

$$\left.\begin{aligned}
q_2=\mathrm{Q}_2-\frac{i}{2}\Big\{&q_1\left(-\cot\tfrac{z}{2}-\cot\tfrac{3}{2}z\right)-2q_2\cot 2z-2p_2 n+3q_3\left(\cot\tfrac{z}{2}-\cot\tfrac{5}{2}z\right)\Big\}\\
&+\frac{i^2}{4}\Big\{p_1\left(-n\cot\tfrac{z}{2}-n\cot\tfrac{3}{2}z\right)+q_1\left(\frac{1}{\sin^2\frac{z}{2}}-\frac{1}{\sin^2\frac{3}{2}z}\right)-4p_2 n\cot 2z\\
&+4q_2\left(\tfrac{2}{3}n^2+\tfrac{1}{3}-\frac{1}{\sin^2 2z}\right)+9p_3\left(n\cot\tfrac{z}{2}-n\cot\tfrac{5}{2}z\right)+9q_3\left(\frac{1}{\sin^2\frac{z}{2}}-\frac{1}{\sin^2\frac{5}{2}z}\right)\Big\}.
\end{aligned}\right\}\quad(32)$$

$$\left.\begin{aligned}
p_3=\mathrm{P}_3-\frac{i}{2}\Big\{&-2q_1+p_1(\cot 2z-\cot z)-4q_2+2p_2\left(\cot\tfrac{5}{2}z-\cot\tfrac{z}{2}\right)+3q_3(n-z)\\
&+3p_3\cot 3z\Big\}+\frac{i^2}{4}\Big\{p_1\left(-2n+\frac{1}{\sin^2 2z}+\frac{1}{\sin^2 z}\right)-q_1(n\cot 2z-n\cot z)\\
&+4p_2\left(-2n+\frac{1}{\sin^2\frac{5}{2}z}+\frac{1}{\sin^2\frac{z}{2}}\right)-4q_2\left(n\cot\tfrac{5}{2}z-n\cot\tfrac{z}{2}\right)\\
&+9p_3\left(\frac{2}{3}n^2-2n+\frac{1}{3}+\frac{1}{\sin^2 3z}\right)-9q_3 n\cot 3z\Big\}.
\end{aligned}\right\}\tag{33}$$

$$\left.\begin{aligned}
q_3=\mathrm{Q}_3-\frac{i}{2}\Big\{&q_1(-\cot z-\cot 2z)+2q_2\left(-\cot\tfrac{z}{2}-\cot\tfrac{5}{2}z\right)-3q_3\cot 3z-3p_3 n\Big\}\\
&+\frac{i^2}{4}\Big\{p_1(-n\cot z-n\cot 2z)+q_1\left(\frac{1}{\sin^2 z}-\frac{1}{\sin^2 2z}\right)+4p_2\left(-n\cot\tfrac{z}{2}-n\cot\tfrac{5}{2}z\right)\\
&+4q_2\left(\frac{1}{\sin^2\frac{z}{2}}-\frac{1}{\sin^2\frac{5}{2}z}\right)-9p_3 n\cot 3z+9q_3\left(\frac{2}{3}n^2+\frac{1}{8}-\frac{1}{\sin^2 3z}\right)\Big\}.
\end{aligned}\right\}\tag{34}$$

For the values of p_1, q_1, &c. in the last terms of equations (29) to (34), we must now insert their first approximations, $p_1=\mathrm{P}_1$, $q_1=\mathrm{Q}_1$, $p_2=\mathrm{P}_2$, $q_2=\mathrm{Q}_2$, &c. (and in the last terms but one, second approximations), as follows:—

$$\left.\begin{aligned}
p_1&=\mathrm{P}_1-\frac{i}{2}\Big\{\mathrm{Q}_1(n-2)+\mathrm{P}_1\cot z-4\,\mathrm{Q}_2+2\mathrm{P}_2\left(\cot\tfrac{3}{2}z+\cot\tfrac{z}{2}\right)\\
&\qquad\qquad-6\mathrm{Q}_3+3\mathrm{P}_3(\cot 2z+\cot z)\Big\},\\
q_1&=\mathrm{Q}_1-\frac{i}{2}\Big\{-\mathrm{Q}_1\cot z-\mathrm{P}_1 n+2\mathrm{Q}_2\left(\cot\tfrac{z}{2}-\cot\tfrac{3}{2}z\right)+3\mathrm{Q}_3(\cot z-\cot 2z)\Big\};\\
p_2&=\mathrm{P}_2-\frac{i}{2}\Big\{-2\mathrm{Q}_1+\mathrm{P}_1\left(\cot\tfrac{3}{2}z-\cot\tfrac{z}{2}\right)+2\mathrm{Q}_2(n-2)+2\mathrm{P}_2\cot 2z-6\mathrm{Q}_3\\
&\qquad\qquad+3\mathrm{P}_3\left(\cot\tfrac{5}{2}z-\cot\tfrac{z}{2}\right)\Big\};\\
q_2&=\mathrm{Q}_2-\frac{i}{2}\Big\{\mathrm{Q}_1\left(-\cot\tfrac{z}{2}-\cot\tfrac{3}{2}z\right)-2\mathrm{Q}_2\cot 2z-2\mathrm{P}_2 n+3\mathrm{Q}_3\left(\cot\tfrac{z}{2}-\cot\tfrac{5}{2}z\right)\Big\},\\
p_3&=\mathrm{P}_3-\frac{i}{2}\Big\{-2\mathrm{Q}_1+\mathrm{P}_1(\cot 2z-\cot z)-4\mathrm{Q}_2+2\mathrm{P}_2\left(\cot\tfrac{5}{2}z-\cot\tfrac{z}{2}\right)\\
&\qquad\qquad+3\mathrm{Q}_3(n-2)+3\mathrm{P}_3\cot 3z\Big\};\\
q_3&=\mathrm{Q}_3-\frac{i}{2}\Big\{\mathrm{Q}_1(-\cot z-\cot 2z)+2\mathrm{Q}_2\left(-\cot\tfrac{z}{2}-\cot\tfrac{5}{2}z\right)-3\mathrm{Q}_3\cot 3z-3\mathrm{P}_3\Big\}.
\end{aligned}\right\}\tag{35}$$

These operations correspond to the rejection of terms involving i^3.

We thus obtain, in lieu of equations (29) to (34), which involve the unknown true coefficients on both sides, others of the form

$$\left.\begin{aligned}
p_1 &= P_1 + A_1 i + \mathit{A}_1 i^2, \\
q_1 &= Q_1 + B_1 i + \mathit{B}_1 i^2, \\
p_2 &= P_2 + A_2 i + \mathit{A}_2 i^2, \\
q_2 &= Q_2 + B_2 i + \mathit{B}_2 i^2, \\
p_3 &= P_3 + A_3 i + \mathit{A}_3 i^2, \\
q_3 &= Q_3 + B_3 i + \mathit{B}_3 i^2,
\end{aligned}\right\} \quad \ldots \ldots \ldots \quad (36)$$
$$\&c. = \&c.,$$

in which A_1, A_1, B_1, B_1, &c. are numerical quantities.

The true period (and therefore i) being known, these expressions give the values of the coefficients for the true period in terms of those for the approximate period; and these values being inserted in equation (13), it will then express the phenomenon for the true period in terms of the coefficients for the approximate period. The general expression for A_1 and A_1 &c. would be too lengthy to write in full, although the calculation of their numerical values in any particular case is not very tedious; the most convenient mode of procedure is to work out, by equation (35), the *numerical* values of the second approximations to p_1, q_1, &c., and insert these in equations (29) to (34).

14. To illustrate the application of the method described, and to show that advantage is gained by it, we have chosen, arbitrarily, the law of periodical variation

$$\alpha_m = -\cos(mz + 60°) + \cos 2mz - \cos 3mz,$$

or

$$\alpha_m = -\cdot 5 \cos mz + \cdot 86603 \sin mz + \cos 2mz - \cos 3mz,$$

where

$$p_1 = -\cdot 50000;\ q_1 = +\cdot 86603;\ p_2 = +1\cdot 00000;\ q_2 = \cdot 00000;\ p_3 = +1\cdot 00000;\ q_3 = \cdot 00000;$$

and taking $z=30$, $i=5'$, and $n=120$, we have calculated one hundred and twenty successive values of α_m, corresponding to the successive values of z—0° 0′, 30° 5′, 60° 10′, &c. . . . . . (3570°+9° 55′); then, treating these numbers as if they corresponded to values of z—0°, 30°, 60°, &c. . . . . 3570°, and applying to them BESSEL's method, the following values of the approximate coefficients were obtained:—

$$P_1 = -\cdot 42262;\quad Q_1 = +\cdot 90398;\quad P_2 = +\cdot 97128,$$
$$Q_2 = -\cdot 17383;\quad P_3 = -\cdot 96147;\quad Q_3 = +\cdot 25536.$$

With these values, and the other data which supplied them, equations (35) and (36) give as third approximations to the true values of the coefficients,

$$p_1 = -\cdot 50000;\quad q_1 = +\cdot 86604;\quad p_2 = +1\cdot 00011;$$
$$q_2 = +\cdot 00002;\quad p_3 = -\cdot 99995;\quad q_3 = +\cdot 00002;$$

and as second approximations, that is excluding terms involving e^2,

$$p_1=-\cdot 50089;\ q_1=+\cdot 86829;\ p_2=+1\cdot 01028;$$
$$q_2=-\cdot 00328;\ p_3=-1\cdot 02186;\ q_3=+\cdot 00416;$$

the degree of approximation is in the second case close, and in the first almost perfect.

IV.

Application of the processes described to determine whether or not there be any periodical variation of disturbances of Magnetic Declination and Horizontal Force at Bombay, due to the influence of the planets Mercury, Venus, and the Earth, in the periods of their respective orbital revolutions, and of Mercury, Venus, and Jupiter in their synodic periods.*

15. In view of the remarkably definite evidence of periodicity in sun-spots indicative of planetary influence, brought to light by the investigations of Messrs. De La Rue, Stewart, and Loewy, and having regard to the common subjection of sun-spots and terrestrial magnetism to the well-known decennial period, it seemed to the writers very desirable to examine whether a similar connexion was exhibited by the two phenomena in respect of the planetary periods The connexion was first shown to exist, by General Sir Edward Sabine, between the larger disturbances of terrestrial magnetism and sun-spots, but it has since been extended to include also the regular magnetic variations. The present inquiry will, however, be confined to the larger disturbances, and of these to the disturbances of Magnetic Declination and Horizontal Force at Bombay, of which a large body, extending over a period of twenty-six years, is available for use in the discussion.

16. A description of the Declinometer, and of the method adopted for separating disturbances, which is that of General Sabine, appears in the 'Philosophical Transactions,' 1869, pp. 363 to 368, and, like the Declinometer, the Horizontal-force Magnetometer is of the kind by Grubb of Dublin, originally supplied to the British Colonial Observatories. Disturbed observations of Declination (Easterly) may be defined as all those observations which give a value of the easterly declination *in excess* of the average of the remaining observations at the same hour during the same month by more than 1'·4, and the easterly disturbance is that excess; and disturbed observations of Declination (Westerly) are all those observations which give a value of easterly declination *in defect* of that average by more than 1'·4, and the westerly disturbance is that defect. In Table I. the aggregates of such excesses and defects are shown for each month in each of the twenty-six years from 1847 to 1872. Disturbed observations increasing the Horizontal Force are all those which give a value of the Horizontal Force *in excess* of the average of the remaining observations at the same hour during the same month by more

* The mathematical expression for the Earth's influence being analogous to the expressions for the influence of Mercury and Venus, the influences are here classed together indiscriminately, although doubtless they are not wholly of the same character in each case.

than ·00334 (metre-gramme-second) units of force, and the increasing disturbance is that excess, and the disturbed observations decreasing the Horizontal Force are all those which give a value of Horizontal Force *in defect* of that average by more than ·00334 units, and the decreasing disturbance is that defect In Table II. the aggregates of the excesses and defects of Horizontal Force are shown for each month in each of the twenty-six years from 1847 to 1872 The two Tables contain all the observational data used in the present inquiry.

TABLE I —Showing the Monthly Aggregates of Disturbances of Declination exceeding 1′·4 in amount, from January 1847 to December 1872.

Years	Jan	Feb	March	April	May	June	July	August	Sept	Oct	Nov	Dec
	Easterly Disturbance Aggregates											
1847	32′ 601	9′ 313	53′ 343	81′ 730	52′ 675	31′ 843	41′ 527	69′ 892	153′ 729	108′ 458	67′ 142	210′·564
1848	52 144	82 755	81 714	46 396	25 086	87 569	87 888	44 773	15 669	81 189	233 638	27 256
1849	66 268	94 313	55 685	30 832	16 206	16 850	18 278	32 054	19 469	53 973	92 298	20 561
1850	28 997	28 544	34 003	15 067	18 623	16 777	42 915	21 585	15 098	8 540	5 443	82 517
1851	74 814	21 741	13 847	20 075	20 990	22 818	35 098	21 894	90 523	49 580	18·995	72 240
1852	53 346	120 985	24 127	74 522	126 439	14 392	18 704	34 921	32 894	16 508	28 006	36 853
1853	32 650	24 427	42 940	25 270	67 160	39 312	27 949	36 882	94 948	19 424	17 374	38 480
1854	50 902	64 843	56 504	43 466	10 708	22 208	31 258	10 283	22 554	44 416	18 769	22 813
1855	13 100	7 185	36 425	19 414	8 449	14 416	13 078	10 702	16 942	14 624	14 734	14 338
1856	8 054	14 978	10 926	13 921	1 665	12 098	5 052	9 554	21 379	15 433	0 000	6 893
1857	8 118	6 752	20 732	8 924	10 489	13 311	33 118	12 057	25 447	20 650	40 744	96 264
1858	49 958	39 181	64 664	111 087	18 894	25 834	77 967	30 488	43 782	44 923	27 337	15 034
1859	57 819	69 377	48 189	97 001	52 618	38 394	57 820	114 162	169 905	198 777	15 435	49 717
1860	23 984	70 715	103 117	37 567	17 695	38 733	140 932	221 409	64 623	63 580	11 436	38 406
1861	60 495	56 850	37 678	33 454	19 002	41 033	39 174	32 787	47 383	40 953	35 443	69 321
1862	30 153	26 615	179 782	84 132	27 592	21 266	51 978	81 257	55 343	163 152	13 866	55 289
1863	62 003	39 771	60 408	56 068	56 448	19 299	27 583	32 972	27 227	22 576	12 177	0 000
1864	17 328	32 728	26 264	27 672	6 541	80 336	57 361	40 140	39 891	20 698	5 020	20 390
1865	52 565	27 247	9 906	39 858	27 211	25 372	53 231	70 072	30 130	41 462	50·878	13 940
1866	30 095	81 696	16 635	3 045	17 569	13 226	3·059	27 330	12 088	81 926	32 221	8 047
1867	30 424	15 881	4 904	2 875	14 832	11 655	16·841	25 108	21 224	40 817	27 571	0 000
1868	6 695	4 370	16 389	110 398	5 117	18 591	13 281	38 992	55 874	85·000	7 087	6 428
1869	55 697	43 308	60 588	75 824	67 852	52 912	17 095	30 644	116 304	50 840	17 692	59 819
1870	64 717	57 199	70 171	101 645	77 381	64 724	33 539	62 790	79 164	153 136	51 690	45 709
1871	8 155	10·946	9 078	15 883	3 705	5 598	14 513	11 408	6 347	6 087	12 137	2 614
1872	3 818	18 097	8 803	6 006	7 109	8 368	10 140	14 941	10 221	31 311	3 622	6 856
	Westerly Disturbance Aggregates											
1847	22′ 397	7′ 678	0′ 000	44′ 912	27′ 338	39′ 353	28′ 176	40′ 743	60′ 499	31′ 620	37′ 967	53′ 274
1848	58 531	54 526	11 946	27 527	22 894	33 295	54 331	53 568	6 745	46 017	81 873	17 325
1849	47 932	29 114	51 404	50 760	21 203	39 149	40 501	49 756	32 347	28 293	26 837	11 849
1850	11 865	62 015	13 427	33 615	39 415	11 933	41 324	24 019	23 114	10 068	4 772	16 425
1851	25 920	26 131	16 060	6 148	36 951	48 112	40 230	41 939	26 847	23 471	7 637	10 712
1852	31 093	43 683	19 581	42 771	21 890	11 452	39 386	24 997	35 918	24 430	28 512	17 105
1853	71 474	2 889	23 612	13 683	15 860	27 882	26 281	23 721	24 766	42 571	6 748	8 598
1854	49 328	6 672	26 553	9 388	12 748	31 806	16 006	8 158	8 245	18 580	12 791	4 506
1855	8 686	14 803	1 447	13 382	6 207	30 213	21 586	23 924	23 347	5 025	3 191	5 548
1856	16 100	9 312	0 000	7 282	9 634	6 864	27 199	6 581	25 866	3 252	8 740	3 050
1857	36 052	12 035	5 714	7 974	36 935	12 845	22 012	10 628	97 241	28 370	13 757	57 011
1858	48 212	14 885	36 986	12 387	18 998	24 157	31 895	35 153	61 432	38 204	32 387	21 273
1859	29 810	31 247	33 086	35 032	51 457	34 145	41 192	49 006	81 670	43 294	22 539	45·387
1860	45 737	24 530	59 494	53 723	46 127	39 309	171 465	59 967	31 190	35 831	16 268	23 089
1861	23 738	12 540	28 022	44 486	19 973	36 941	64 533	31 395	20 087	45 476	10 469	10 644
1862	24 502	11 317	30 491	99 613	20 131	35 380	52 467	36 508	32 164	96 376	18 179	19 360
1863	45 074	29 515	24 148	35 883	17 075	9 793	28 931	29·248	10·118	29 466	12 512	9 864
1864	16 426	21 826	13 412	17 374	17 771	19 065	25 817	37 766	20 984	9 253	8 052	6 418
1865	12 802	19 926	11 832	3 695	48 677	8 625	25 480	44 368	38·070	34 283	3 015	3 355
1866	36 845	38 828	17 246	20 491	205 290	2 929	17 898	41 593	6·695	70 411	3 457	2·902
1867	14 021	6 881	5 481	3 156	3 080	1 811	18 432	23 159	21 280	11 930	3 073	0·000
1868	1 523	14 104	25 252	34 648	12 828	20 381	16 917	19 469	56 602	149 506	1 482	0 000
1869	20 950	6 277	13 912	32 983	21 287	21 938	39 260	20 195	55 086	19 379	8 246	33 442
1870	23 818	19 064	32 187	41 901	54 201	63 784	90 833	72 874	40 028	30 685	42 759	28 250
1871	4 967	6 338	6 090	4 778	3 473	5 347	6·644	8 570	4 551	5 588	3 172	2 443
1872	2 720	4 309	5 640	6 245	4 697	3 534	2 892	6 209	4 559	2 592	4 422	3 794

Table II.—Showing the Monthly Aggregates of Disturbances of Horizontal Force exceeding 00334 (metre-gramme-second) units of force, from January 1847 to December 1872.

Years	January	February	March	April	May	June	July	August	September	October	November	December
	Disturbances increasing the Horizontal Force											
1847	00907	01775	01674	02732	00000	00000	00000	·04421	·01062	·08554	·01976	02142
1848	03658	06152	02606	03002	02851	00000	05591	05134	00000	14069	03931	00000
1849	15325	01714	·01512	·00709	01523	·02326	00745	01570	02459	02833	00659	·00418
1850	01577	·01544	01105	00000	01620	01051	·02412	00000	·00727	04946	00364	·00796
1851	02077	02365	04298	00000	04550	05728	00349	01577	04406	04360	00000	00749
1852	04536	24419	11981	01501	03380	01076	06502	01202	00403	10634	00832	·02668
1853	02796	·04004	03449	00369	04307	06008	09990	·02997	04000	01945	00431	04752
1854	00409	00756	02712	03405	07026	00000	01132	01723	01548	00708	00000	00701
1855	·00000	01460	00748	00000	01445	00000	04876	00000	00000	00339	·00361	·01106
1856	19274	02631	00000	00389	00349	01745	00000	00763	00348	00393	00000	00768
1857	00000	00000	00788	01888	02353	01179	01651	00000	07689	01406	04776	01950
1858	06084	03265	10996	04717	09666	06326	02741	00000	04577	07583	03001	00974
1859	07124	02244	06345	09157	02721	03326	03232	04560	13183	22485	08819	02266
1860	05302	03254	19500	14087	05049	06765	07483	12618	13884	01510	01262	01841
1861	08372	10390	02560	03140	02096	19531	03583	05782	01107	03264	09290	03801
1862	06612	09045	00368	02728	04650	03443	08801	10445	10420	08087	00769	05626
1863	08151	10808	10098	09064	02881	12260	02140	20006	05847	03759	03886	01101
1864	00372	02653	01799	06810	04201	13197	08710	03553	04273	06378	00788	00000
1865	07666	07456	03871	00721	00338	01185	04881	23497	09785	18215	27939	03966
1866	03058	15329	14081	00829	00746	00000	05271	00382	02139	15274	02206	·00685
1867	00492	00768	03120	01817	03156	00752	01156	00730	02848	11074	03429	03356
1868	01109	01866	00729	13932	00364	03920	02604	07877	11978	02688	00000	00000
1869	04751	04613	03694	16159	06674	11407	06630	06377	10358	02430	07031	03387
1870	09332	07091	08896	03309	04273	03987	04036	13553	18851	09500	25815	09626
1871	17428	01968	16671	29371	02320	12342	10727	08968	03199	04688	14534	08743
1872	08526	06741	01145	06270	07961	09175	06115	05030	09116	15568	20019	01625
	Disturbances decreasing the Horizontal Force.											
1847	13291	07978	33491	17664	41126	01228	·03791	24616	1 29545	83941	62093	1 32156
1848	41396	73566	52963	44082	25848	00839	12689	04676	14562	1 03536	1 31195	14882
1849	15361	34546	07466	04392	06023	11311	02902	01472	09302	15[illegible]31	42311	04392
1850	05317	12276	06599	02308	04435	07866	11441	02015	04442	22712	01508	08766
1851	50339	21535	06455	00000	18515	14615	06707	08168	1 34014	60210	06286	20423
1852	35381	1 05696	11290	41404	12053	19390	11268	02830	28188	11441	14018	08564
1853	04322	20955	15301	28258	44698	14385	28554	00369	68770	02894	16452	38029
1854	20900	15717	33748	42278	11508	03705	08220	01891	09457	26276	06979	04745
1855	01555	05654	08636	10179	08011	02081	07373	02464	01256	25156	00369	02274
1856	01195	05033	05268	03507	04994	00866	01681	09126	06655	10663	00917	12140
1857	00749	02415	00839	10126	47028	04741	02716	00816	24404	05347	35977	1 24577
1858	42793	16029	58995	99566	19593	43089	07337	02714	35608	20237	14029	32211
1859	13537	89391	11752	62330	18915	30410	44571	46413	1 12879	1 77052	46265	82666
1860	10656	38462	1 09309	38132	31114	08680	91931	1 00879	43830	67307	21485	33536
1861	56006	29277	31021	21158	05457	05590	01261	17087	25154	59525	29297	63135
1862	24050	23634	29742	29392	10651	00891	18965	56125	43924	88875	25039	59343
1863	25914	07879	07900	14163	11356	10037	19459	13433	30873	34188	11432	02252
1864	00339	06082	29916	22083	16067	63418	30811	31811	29252	20128	09667	02541
1865	26233	27368	14320	29201	19018	12277	07151	81050	09827	30819	33401	02432
1866	06729	90626	13067	07426	07078	01029	01222	20720	11476	29021	13876	09057
1867	04851	25234	05092	07295	12774	04279	04308	01123	13665	17425	08676	·02294
*1868	·00000	08876	50612	35348	15057	09218	32487	13639	34144	54636	00000	01414
1869	27907	44794	84134	61314	77931	38806	06795	46084	55401	29192	21115	26690
1870	73891	51550	47810	48246	64513	35981	21999	67844	1 28971	1·09525	52101	57811
1871	36799	1 12210	51099	72576	05740	27627	33183	44565	31528	37913	73249	04426
1872	05184	96353	15542	92697	21553	27526	51272	94517	53956	1 63515	17688	15274

17 The sidereal periods of revolution of Mercury, Venus, and the Earth are 87·97, 224·70, and 365·26 mean solar days respectively. Nine periods of Mercury are so nearly equal to $2\frac{1}{6}$ years (26 months) that the accumulated difference after ninety-nine periods is less than four days, or $\frac{1}{24}$ of one period of Mercury, and the time of thirteen periods of Venus differs from 8 years so little that after thirty-nine periods the accumu-

lated difference is less than three days, or $\frac{1}{80}$ of the period of Venus; we shall therefore, in the first place, find (in accordance with what has preceded) the coefficients of BESSEL's series expressing the variation of aggregate disturbance of Magnetic Declination, Easterly and Westerly, and the variation of aggregate disturbance of Horizontal Force (increasing and decreasing) with variation of the position of Mercury in its orbit, just as if the observations were wholly due to the action of that planet, and so for each planet in turn; and we shall afterwards examine to what extent the values of the coefficients thus found are affected by the influence of the other planets.

18. The ninety-nine periods of Mercury extend over 23 years and 10 months, and the observations treated commence with the aggregates of March 1847, and end with those of December 1870. The thirty-nine periods of Venus and twenty-four periods of the Earth extend over 24 years, the observations treated being those for January 1847 to December 1870.

The application of BESSEL's process to these observations, taken from Tables I. and II., gives the values of the coefficients for the sidereal periods of Mercury, Venus, and the Earth as shown below.*

TABLE III.—Values of the coefficients p_1, q_1, &c. for the sidereal periods of Mercury, Venus, and the Earth.

Declination.

Easterly Disturbance.						
Coefficients	p_1	q_1	p_2	q_2	p_3	q_3
Mercury	−0′·523	−1′·190	+3′·604	+3′·435	+3′·781	+2′·707
Venus	−4·591	−1·199	+1·428	+1·055	+3·422	−2·786
The Earth	+1·146	−4·054	−4·812	+7·217	+0·558	−0·700

Westerly Disturbance.						
Coefficients	p_1	q_1	p_2	q_2	p_3	q_3
Mercury	−4·302	−1·802	−1·314	+1·796	+4·533	−2·495
Venus	−1·516	+1·497	+1·156	−1·965	+2·440	+2·083
The Earth	−7·253	−1·168	+0·131	+2·584	+3·274	+0·684

Horizontal Force.

Increasing Disturbance.						
Coefficients	p_1	q_1	p_2	q_2	p_3	q_3
Mercury	−·00702	+·00037	+·00080	+·00326	+·00322	−·00428
Venus	−·00193	+·00438	−·00203	+·00111	−·00190	+·00004
The Earth	−·00112	−·00524	−·00442	+·00967	+·00056	+·00779

Decreasing Disturbance.						
Coefficients	p_1	q_1	p_2	q_2	p_3	q_3
Mercury	−·00928	−·01847	+·02370	+·03871	+·05141	+·00490
Venus	−·02002	−·00859	−·00353	−·00730	+·01617	−·02442
The Earth	+·03180	−·06212	−·07347	+·04314	+·02711	+·01362

* An example of the calculations of one of these sets of coefficients is given at the end of the Appendix.

With these coefficients (and neglecting the non-periodic part of the phenomena) have been calculated the ordinates for the construction of the thick curves (Plate 53. figs 1–12), the ordinates of which represent disturbance, and the abscissæ time.

19. It may be objected to the procedure thus far, that the application of BESSEL's method to any arbitrary series of periodical numbers would yield a smooth-flowing curve, although the numbers themselves were subject to no corresponding law: this, we reply, is a mistake; the law is inherent in the series of numbers. It is another question to what cause the law must, in a particular case, be attributed, but this is so also when a periodical law has been found in a series of observations, by applying the common method of finding average values at different phases of the period. It may be interesting to some of our readers to show that, where the circumstances allow of the application of the latter method, it leads to the same form of curve as BESSEL's process. We choose for this purpose the variations, with the sidereal period of Mercury, of disturbances of Declination (Easterly and Westerly) and of disturbances increasing and decreasing the Horizontal Force. If we take twenty-six equidistant times in the period of Mercury and twenty-six consecutive months, the several months will correspond to the twenty-six phases of Mercury's period, as shown below.

Twenty-sixths of the period of Mercury	0	1	2	3	4	5	6	7	8	9	10	11	12
Months	0	3	6	9	12	15	18	21	24	1	4	7	10
Twenty-sixths of the period of Mercury	13	14	15	16	17	18	19	20	21	22	23	24	25
Months	13	16	19	22	25	2	5	8	11	14	17	20	23

And arranging each successive twenty-six months' aggregates in this way and in successive lines, we get, for each phase, eleven observed disturbance-aggregates, of which averages are calculated. Means are then taken of each consecutive pair of these averages, forming twenty-six new averages, and this process is repeated six times; after this the means are taken of every consecutive three of the last averages, and these numbers are curved thin in figs. 1–4. It will be seen that they agree with the thick curves obtained by BESSEL's process, which are also constructed from twenty-six equidistant ordinates; but the agreement is closer, as it clearly should be, when the twenty-six calculated ordinates are treated in the same manner (described above) as the twenty-six average disturbance-aggregates were, to obtain the ordinates of the thin curves. In this way the ordinates of the dotted curves have been obtained; and although the thick curves must be taken as best representing the true law, the dotted ones are more directly comparable with the thin curves, having been obtained by a similar process. The slight disagreement that is observable must be attributed mainly to the omission of the fourth and higher pairs of terms of BESSEL's expression.

TABLE IV.—The observed and calculated values of Aggregate Disturbance of Declination are, for the sidereal period of Mercury, as follows:—

Easterly Disturbance Aggregates, diminished by the constant value 43′ 180										No and kind of corresponding figure.	
Twenty-sixths of the period	0	1	2	3	4	5	6	7	8	No.	Kind.
Observed	+3 588	+4 265	+4 259	+2 297	−1 197	−4 038	−4 880	−4 474	−3·926	2	Thin.
Calculated	+6 862	+8 622	+7 000	+2 746	−2 367	−6 326	−7 796	−6 628	−3 816	2	Thick.
Ditto rendered comparable with "observed"	+4 401	+5 533	+4 566	+1 911	−1 383	−4 104	−5 392	−5·055	−3·557	2	Dotted
Twenty-sixths of the period	9	10	11	12	13	14	15	16	17	No	Kind
Observed	−2 967	−0 968	+1 256	+2 217	+1 872	+1 455	+1·759	+2 841	+4 381	2	Thin
Calculated	−0 855	+1 004	+1 364	+0 780	+0 346	+0 952	+2 748	+4 948	+6 233	2	Thick
Ditto rendered comparable with "observed"	−1 712	−0 242	+0 552	+0 866	+1 175	+1 874	+2 976	+4 038	+4 375	2	Dotted
Twenty sixths of the period	18	19	20	21	22	23	24	25		No	Kind
Observed	+5·054	+2 977	−1 792	−6 194	−6 892	−8 767	+0 236	+2 629		2	Thin
Calculated	+5 482	+2 440	−2 012	−6 136	−8 127	−6 954	−2 924	+2 406		2	Thick
Ditto rendered comparable with "observed"	+3 451	+1 249	−1 632	−4 141	−5 245	−4 362	−1 759	+1 601		2	Dotted

Westerly Disturbance Aggregates, diminished by the constant value 28′ 557										No and kind of corresponding figure	
Twenty sixths of the period	0	1	2	3	4	5	6	7	8	No	Kind
Observed	−4 218	−4 580	−4 231	−3 894	−3 749	−3 182	−1 705	+0 474	+2 394	1	Thin
Calculated	−1 083	−3 195	−5 817	−7 421	−6 780	−3 889	+0 124	+3 516	+4 807	1	Thick
Ditto rendered comparable with "observed"	−2 531	−3 611	−4 867	−5 460	−4 804	−2 917	−0 442	+1 654	+2 567	1	Dotted
Twenty sixths of the period	9	10	11	12	13	14	15	16	17	No	Kind
Observed	+2·972	+1 923	+0 078	−1 252	−0 909	+1 793	+6 177	+ 9 662	+ 9 641	1	Thin
Calculated	+3 554	+0 607	−2 277	−3 301	−1 545	+2 539	+7 311	+10 669	+11 070	1	Thick
Ditto rendered comparable with "observed"	+2 078	+0 688	−0 614	−0 838	+0 499	+3 103	+6 041	+ 8 140	+ 8 415	1	Dotted
Twenty-sixths of the period	18	19	20	21	22	23	24	25		No	Kind
Observed	+6 123	+1 553	−1 755	−3 086	−2 880	−2 164	−2 108	−3·072		1	Thin
Calculated	+8 239	+3 286	−1 822	−5 221	−5 980	−4 491	−2 171	−0 695		1	Thick
Ditto rendered comparable with "observed"	+6 573	+3 248	−0 302	−2 887	−3 955	−3 692	−2 828	−2 252		1	Dotted

And for the sidereal periods of Venus and the Earth, the calculated values are —

Easterly Disturbance Aggregates, diminished by the constant value 43'·026

Twenty-fourths of the period	0	1	2	3	4	5	6	7
Venus	+0 259	−2 523	−5 752	−7 482	−6 555	−3 543	+0·159	+2 681
The Earth	−3 108	−0 602	+2 109	+4 271	+5 160	+4 256	+1 458	−2 764

Twenty-fourths of the period	8	9	10	11	12	13	14	15
Venus	+3 049	+ 1 846	+ 0 408	+ 0 445	+2 597	+6 061	+9 010	+ 9 592
The Earth	−7 370	−10·993	−12 375	−10 822	−6 516	−0 516	+5 579	+10 163

Twenty-fourths of the period	16	17	18	19	20	21	22	23
Venus	+ 6 957	+ 2 113	−3·015	−6 219	−6 307	−3 956	−0 810	+0 985
The Earth	+12 152	+11 294	+8 166	+3 882	−0 318	−3 441	−4 937	−4 728

Westerly Disturbance Aggregates, diminished by the constant value 28' 463.

Twenty-fourths of the period	0	1	2	3	4	5	6	7
Venus	+2 080	+2 118	+0 373	−2 227	−4 220	−4 173	−1 742	+2 117
The Earth	−3 848	−3 105	−3 878	−5 201	−5 738	−4 627	−1 983	+1 177

Twenty-fourths of the period	8	9	10	11	12	13	14	15
Venus	+5 640	+7 313	+6 483	+3 638	+0 232	−2 118	−2·653	− 1 703
The Earth	+3 586	+4 517	+4 208	+3 693	+4 110	+5 915	+8 484	+10 369

Twenty-fourths of the period	16	17	18	19	20	21	22	23
Venus	− 0 400	+0 133	−0 570	−2 117	−3 360	−3 383	−1 863	+0 402
The Earth	+10 084	+6 985	+1 721	−3 987	−8 192	−9 685	−8 554	−6 051

TABLE V.—The observed and calculated values of Aggregate Disturbance of Horizontal Force are, for the sidereal period of Mercury, as follows.—

Aggregates of Disturbances increasing the Horizontal Force, diminished by the constant value 0·4634										No and kind of corresponding figure	
Twenty-sixths of the period	0	1	2	3	4	5	6	7	8	No	Kind.
Observed	− 00309	− 00525	− 00613	− 00500	− 00228	− 00003	+ 00097	+ 00168	+ 00214	4	Thin
Calculated	− 00300	− 00193	− 00677	− 00702	− 00507	− 00144	+ 00239	+ 00479	+ 00492	4	Thick
Ditto rendered comparable with 'observed'	− 00416	− 00481	− 00537	− 00508	− 00361	− ·00128	+ 00114	+ 00278	+ 00320	4	Dotted
Twenty-sixths of the period	9	10	11	12	13	14	15	16	17	No	Kind
Observed	+ 00206	+ 00177	+ 00240	+ 00410	+ 00627	+ 00840	+ 00957	+ 00890	+ 00666	4	Thin
Calculated	+ 00307	+ 00068	− 00049	+ 00085	+ 00160	+ 00939	+ 01303	+ 01370	+ 01061	4	Thick.
Ditto rendered comparable with "observed	+ 00261	+ 00178	+ 00168	+ 00292	+ 00540	+ ·00826	+ 01023	+ 01024	+ 00788	4	Dotted
Twenty sixths of the period	18	19	20	21	22	23	24	25		No	Kind
Observed	+ 00369	+ 00021	− 00380	− 00725	− 00852	− 00735	− 00523	− 00387		4	Thin
Calculated	+ 00456	− 00239	− 00791	− 01044	− 00973	− 00696	− 00397	− 00247		4	Thick
Ditto rendered comparable with observed"	+ 00369	− ·00114	− ·00520	− 00748	− ·00776	− ·00664	− 00313	− 00417		4	Dotted

Aggregates of Disturbances decreasing the Horizontal Force, diminished by the constant value ·27647										No and kind of corresponding figure	
Twenty-sixths of the period	0	1	2	3	4	5	6	7	8	No	Kind
Observed	+ 03273	+ 03976	+ 03006	+ 00077	− 03088	− 04195	− 03283	− 02474	− 02833	3	Thin
Calculated	+ ·06583	+ 06731	+ 03956	− 00306	− 04144	− 06043	− ·05604	− 03582	− 01417	3	Thick
Ditto rendered comparable with "observed"	+ 03822	+ 04076	+ 02546	− 00006	− 02481	− 03974	− 04158	− 03354	− 02265	3	Dotted.
Twenty sixths of the period	9	10	11	12	13	14	15	16	17	No	Kind
Observed	− 02996	− 01902	− 00496	− 00039	+ 00022	+ 01175	+ 03648	+ 06147	+ 07153	3	Thin
Calculated	− 00344	− 00764	− 02008	− 02769	− 01843	+ 01063	+ 05108	+ 08568	+ 09700	3	Thick
Ditto rendered comparable with 'observed'	− 01541	− 01401	− 01532	− 01297	− 00155	+ 01955	+ ·04468	+ 06395	+ 06782	3	Dotted
Twenty sixths of the period	18	19	20	21	22	23	24	25		No	Kind.
Observed	+ 05848	+ 02410	− 01900	− 05101	− 05656	− ·03673	− 00745	+ 01676		3	Thin
Calculated	+ 07625	+ 02852	− 02870	− 07265	− 08576	− 06350	− 01672	+ 03363		3	Thick
Ditto rendered comparable with 'observed	+ 05200	+ 02030	− 01640	− 04451	− 05359	− ·04103	− 01315	+ 01760		3	Dotted

And for the sidereal periods of Venus and the Earth, the calculated values are —

Aggregates of Disturbances increasing the Horizontal Force, diminished by the constant value 04611								
Twenty fourths of the period	0	1	2	3	4	5	6	7
Venus	− 00886	− 00615	−·00209	+ 00209	+·00520	+ 00657	+ 00637	+ 00535
The Earth	− 00498	+ 00147	+ 01036	+ 01029	+ 00492	− 00260	− 00861	−·01088
Twenty fourths of the period	8	9	10	11	12	13	14	15
Venus	+ 00440	+ 00417	+ 00453	+ 00495	+ 00480	+ 00373	+ 00199	+ 00013
The Earth	− 00958	− 00667	− 00441	− 00383	− 00386	− 00247	+ 00196	+ 00905
Twenty-fourths of the period	16	17	18	19	20	21	22	23
Venus	− 00126	− 00195	− 00231	− 00293	− 00430	− 00639	− 00847	− 00957
The Earth	+ 01624	+ 01992	+ 01745	+ 00888	− 00274	− 01267	− 01672	− 01349

Aggregates of Disturbances decreasing the Horizontal Force, diminished by the constant value 27529								
Twenty-fourths of the period	0	1	2	3	4	5	6	7
Venus	− 00738	− 03359	− 05313	− 05180	− 03615	− 00632	+ 02136	+ 03422
The Earth	− 01447	+ 00146	+ 01081	+ 01223	+ 00912	+ 00465	− 00227	− 01067
Twenty fourths of the period	8	9	10	11	12	13	14	15
Venus	+ 02855	+ 01096	− 00581	− 01047	+ 00032	+ 02017	+ 03697	+ 04020
The Earth	− 04326	− 08081	− 11915	− 14164	− 13247	− 08558	− 00955	+ 07405
Twenty fourths of the period	16	17	18	19	20	21	22	23
Venus	+ 02733	+ 00514	− 01430	− 02080	− 01239	+ 00364	+ 01493	+ 01165
The Earth	+ 13906	+ 16575	+ 14921	+ 10079	+ 04200	− 00547	− 02903	− 02876

20 Eighty-two synodic periods of Mercury extend over 26 years, and the observations treated are those for the years 1847 to 1872 Fifteen synodic periods of Venus and twenty-two of Jupiter extend over 24 years, the years treated being 1847 to 1870.

TABLE VI.—Values of the coefficients p_1, q_1, &c. for the synodic periods of Mercury, Venus, and Jupiter

Declination.

	Easterly Disturbance					
Coefficients	p_1	q_1	p_2	q_2	p_3	q_3
Mercury	+0·719	−4·541	+0·776	−1·628	−3·122	+0·443
Venus	+0·356	−0·364	+3·591	−4·112	−7·920	−3·402
Jupiter	−3·334	−4·883	+1·361	−0·741	+1·916	−1·676
	Westerly Disturbance					
Coefficients	p_1	q_1	p_2	q_2	p_3	q_3
Mercury	+1·197	−2·889	+1·775	+0·439	+2·553	−1·300
Venus	−1·604	−1·021	+1·160	+1·287	+0·278	−1·630
Jupiter	+0·251	+0·212	+0·189	−1·534	+0·808	+2·264

TABLE VI. (continued).

Horizontal Force.

Coefficients	Increasing Disturbance p_1	q_1	p_2	q_2	p_3	q_3
Mercury	−·00553	− 00804	+ 00127	−·00037	− 00335	+·00320
Venus	+·00672	− 00384	− 00075	− 00255	+·00537	−·00360
Jupiter	+·00150	− 00065	− 00556	+ 00522	− 00146	− 00272

Coefficients	Decreasing Disturbance p_1	q_1	p_2	q_2	p_3	q_3
Mercury	− 01019	−·02015	+·01047	+ 00771	− 02165	− 00065
Venus	+ 02607	−·02805	+·05284	− 03936	− 04866	−·02049
Jupiter	− 03509	− 03944	− 00751	+ 02636	+ 04341	+ 01217

With these coefficients have been calculated the ordinates for the construction of the thick curves, Plate 54 figs. 13 to 24.

21. TABLE VII —The calculated values of Aggregate Disturbance of Declination for the synodic periods of Mercury, Venus, and Jupiter are as follows —

Easterly Disturbance Aggregates, diminished by the constant value shown in the last column of the Table

Twenty-fourths of the period	0	1	2	3	4	5	6	7	8
Mercury	−1 627	−2 517	−2 226	−1 810	−2 250	−3 793	−5 760	−6 951	−6 392
Venus	−4 043	−6 762	−5 077	−0 922	+2 462	+2 640	−0 183	−4·632	−6 612
Jupiter	−0·057	−3 507	−6 966	−9 090	−9 134	−7 300	−4 568	−2 122	− ·684
Twenty-fourths of the period	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**
Mercury	−3 984	−0 652	+2 135	+ 3 179	+2 233	+0 182	−1 446	− 1 346	+ 0 821
Venus	4 402	+1 429	+7 862	+11 085	+8 748	+1 475	−7 302	−13 104	−12 850
Jupiter	− 184	+0 092	+ 965	+ 2 779	+5 125	+7 042	+7 608	+ 6 490	+ 4 202
Twenty-fourths of the period	**18**	**19**	**20**	**21**	**22**	**23**	**p_0**		
Mercury	+4 208	+7 235	+ 8 436	+ 7 240	+4 248	+0 837	40′ 472		
Venus	−6 559	+2 646	+10 214	+12 626	+9 213	+2 348	43′ 026		
Jupiter	+1 846	+0 504	+ 0 608	+ 1 666	+2 552	+2 133	43′ 026		

Westerly Disturbance Aggregates diminished by the constant value shown in the last column of the Table

Twenty-fourths of the period	0	1	2	3	4	5	6	7	8
Mercury	+5 525	+3 050	− 0 440	−3 482	− 4 964	−4 685	−3 364	−2 133	−1 814
Venus	− 0 166	−1 120	−1 844	−1 918	−1 429	−0 808	−0 551	−0·870	−1 499
Jupiter	+1 246	+1 865	+1 353	−0 176	−1 919	−2 832	−2 241	−0 288	+2 099
Twenty-fourths of the period	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**
Mercury	−2 442	−3 274	−3 310	−1 975	+0 462	+2 974	+4 360	+3 950	+2 049
Venus	− 1 831	−1 286	+0 298	+2 486	+4 416	+5 221	+4 492	+2 499	+0 084
Jupiter	+3 678	+3 575	+1 775	−0 868	−3 071	−3 821	−2 892	−0 925	+0 970
Twenty-fourths of the period	**18**	**19**	**20**	**21**	**22**	**23**	**p_0**		
Mercury	−0 186	− 1 379	−0 720	+1 564	+4 288	+5 940	26′ 638		
Venus	−1 769	2 426	−1 891	−0 744	+0 216	+0 426	28′ 463		
Jupiter	+1 863	+ 1 494	+0 369	−0 610	−0 731	+0 087	28′ 463		

TABLE VIII.—The calculated values of Aggregate Disturbance of Horizontal Force for the synodic periods of Mercury, Venus, and Jupiter are as follows:—

Aggregates of Disturbances increasing the Horizontal Force, diminished by the constant value shown in the last column of the Table

Twenty-fourths of the period	0	1	2	3	4	5	6	7	8
Mercury	− ·00761	− ·00661	− 00530	− 00533	− ·00732	− 01037	− 01251	− 01189	− 00786
Venus	+ 01134	+ 00483	− 00228	− ·00686	− ·00718	− ·00384	+ 00051	+ 00282	+ ·00126
Jupiter	− ·00552	− 00387	00000	+ ·00493	+ 00895	+ ·01013	+ 00763	+ 00207	− 00451
Twenty-fourths of the period	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**
Mercury	− 00151	+ 00492	+ 00917	+ ·01015	+ 00845	+ 00592	+ 00459	+ 00542	+ 00781
Venus	− 00366	− 00950	− 01321	− 01284	− 00867	− 00288	+ 00176	+ 00350	+ 00260
Jupiter	− 00969	− ·01164	− 00993	− 00560	− 00053	+ 00348	+ ·00551	+ 00565	+ 00471
Twenty-fourths of the period	**18**	**19**	**20**	**21**	**22**	**23**	p_0		
Mercury	+ 00997	+ 01005	+ 00724	+ 00225	00302	− 00661	05000		
Venus	+ 00099	+ 00102	+ 00390	+ 00876	+ 01318	+ 01445	04611		
Jupiter	+ 00349	+ 00233	+ 00103	− 00075	− 00296	− ·00491	04611		

Aggregates of Disturbances decreasing the Horizontal Force, diminished by the constant value shown in the last column of the Table

Twenty fourths of the period	0	1	2	3	4	5	6	7	8
Mercury	− 02137	− 01791	− 00763	+ 00111	+ 00056	− 01155	− ·02997	− 04459	− 04592
Venus	+ 03025	− 00489	− 01960	− 02085	− 02311	− 03690	− 06010	− 07984	− 07831
Jupiter	+ 00061	00674	− 01886	− 04842	− 06853	− 06680	− 04410	− 01360	+ 00771
Twenty fourths of the period	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**
Mercury	− 03053	− 00135	+ 02460	+ 04231	+ 04375	+ 03145	+ 01431	+ 00234	+ 00111
Venus	− 04779	+ 00342	+ 05291	+ 07543	+ 05705	+ 00426	− 05787	− 09791	− 09398
Jupiter	+ 00986	− 00374	− ·01808	− ·01583	+ 01150	+ 05702	+ 10114	+ 12169	+ 10616
Twenty-fourths of the period	**18**	**19**	**20**	**21**	**22**	**23**	p_0		
Mercury	+ 00903	+ 01875	+ 02210	+ 01511	+ 00045	− 01425	29213		
Venus	− 04528	+ 02768	+ 09365	+ 12651	+ 11760	+ 07797	27529		
Jupiter	+ 05912	+ 00024	− 04587	− 06258	− 04942	− 02128	27529		

22. Let us now estimate the errors in the coefficients for the Earth due to the sidereal period of Venus, and those of the coefficients for the sidereal period of Venus due to the Earth's period. The periods have the ratio of 13 to 8, so that in equations (7) $f=13$, $g=8$, and $r=96$, that is, in 96 months eight periods of the Earth and thirteen of Venus have been just completed. The least value of s for which $(s\mp1)g$ or $(s\mp1)\,8$ is a multiple of r or 96 is 11*, and therefore p_{11} or q_{11} is the first coefficient (after p_1 or q_1) that affects the value of a_1 or b_1, the least value of s for which $(s\mp2)\,8$ is a multiple of 96 is 10, and therefore p_{10} or q_{10} is the first coefficient (after p_2 or q_2) that affects the value of a_2 or b_2, and similarly p_7 or q_7 is the first coefficient (after p_5 or q_5) that affects the value of a_5 or b_5. Hence if we may disregard as small those terms in the expression for the Earth's period which repeat themselves six or more times in a year, or whose period is less than two months, a_1, b_1, a_2, b_2, a_3, b_3, &c will

* See second set of demonstrations in the Appendix

each be affected by only one of the Earth's coefficients, viz. p_1, q_1, p_2, q_2, p_3, q_3, &c. respectively. Again, the least positive integral value of s for which $(sf \mp tg)$ or $(13s \mp 8t)$ is a multiple of r or 96 is 8, and therefore, if we may disregard as small those terms in the expression for the period of Venus which repeat themselves eight or more times in that period, the quantities a_1, b_1, a_2, b_2, a_3, b_3 &c., being unaffected by the disturbance due to the planet Venus, will sensibly represent the true coefficients of the expression for the Earth's disturbance variation. In a similar manner it may be shown that A_1, B_1, A_2, B_2, A_3, B_3 &c of equations (7) are sensibly equal to the true coefficients of the expression for the period of Venus, for the least integral value of s for which $(s \mp t)f$ or $13(s \mp t)$ is a multiple of 96 is $s = 96 \mp t$, so that only very high terms, in the expression for Venus, would affect the values of the coefficients of the earlier terms; and further, since the least positive integral values of s and t which make $(sg \mp tf)$ or $8s \mp 13t$ a multiple of 96 are eleven and eight respectively, and the corresponding terms repeating themselves eleven and eight times respectively in the periods of the Earth and Venus, they may, as before, be neglected.

23 But we have adopted for thirteen periods of Venus the approximate time 8 years or 2922 05 days, instead of the true time 2921·11 days, which is less by 0 94 of a day Having worked out the question in Section III for three pairs of coefficients only, we will confine the examination to that number and to the Easterly disturbance variation for the sidereal period of the planet, and it will suffice that we determine *the second* approximations to the true coefficients, rejecting terms involving $\imath^2$, i e that we apply equation (35)

The first approximations are

$P_1 = -4\,591$, $Q_1 = -1\,199$, $P_2 = +1\,428$, $Q_2 = +1\,055$, $P_3 = +3\,422$, $Q_3 = -2{\cdot}786$;

the angle

$$z = \frac{2c\pi}{n} = \frac{3 \times 13}{3 \times 96}\, 2\pi = 48^\circ\, 45',$$

and the angle

$$\imath = \frac{\Delta x}{x} z = \frac{c\Delta x}{cx} \cdot \frac{2c\pi}{n} = \frac{c\Delta x}{x} \cdot \frac{2\pi}{n} = \frac{3 \times 0{\cdot}94}{224\ 7} \cdot \frac{2\pi}{3 \times 96} = 0'{\cdot}94\,,$$

and the greatest value of $sn\imath$ is $2 \times (3 \times 96) \times 0'\,94 = 9^\circ\,2'$ ($c\Delta x$ being the error in time in thirty-nine periods of Venus). Consequently $sn\imath$ being a small angle, the case is one to which the investigation in Section III. applies, therefore

$$\left.\begin{aligned} p_1 &= P_1 + A_1\imath = -4\,591 + \ 044 = -4{\cdot}547, \\ q_1 &= Q_1 + B_1\imath = -1\,199 - \ 180 = -1\,379, \\ p_2 &= P_2 + A_2\imath = +1\,428 - \ 088 = +1\,340, \\ q_2 &= Q_2 + B_2\imath = +1\,055 + \ 115 = +1\,170, \\ p_3 &= P_3 + A_3\imath = +3\,422 + \ 330 = +3\,752, \\ q_3 &= Q_3 + B_3\imath = -2\,786 + \ 406 = -2\,380, \end{aligned}\right\}$$

from which has been constructed the interrupted curve (Plate 53 fig 6), which is seen at a glance to be almost identical with the thick curve constructed from the first approximations P_1, Q_1, &c.

24 We may now examine how the sidereal disturbance period of Mercury affects the

coefficients of that of the Earth or Venus, and *vice versâ*, for which purpose we must use equations (12) of Section II, viz —

$$p_1=\frac{2}{R}\sum_{m=0}^{m=R-1}[\alpha_m \cos mz]-\frac{2}{R}\sum_{m=0}^{m=R-1}\left[P_s \cos s\frac{f}{g}mz+Q_s \sin s\frac{f}{g}mz\right]\cos mz,$$

which we will suppose to give the Earth's coefficient p_1, P_s and Q_s being the coefficients of Mercury, and $\frac{f}{g}$ being the ratio of the periods of the Earth and Mercury, which we may take as near enough to $\frac{108}{26}$ or $\frac{54}{13}$, $z=30°$, and $R=288$, inserting these values (12) becomes

$$p_1=\frac{1}{144}\sum_{m=0}^{m=287}[\alpha_m \cos mz]-\frac{1}{144}\sum_{m=0}^{m=287}\left[P_s \cos s\frac{54}{13}m\times 30°+Q_s \sin s\frac{54}{13}m\times 30°\right]\cos m\times 30° \qquad (37)$$

But the time of 314 observations is equal to 26 years, or 108 periods of Mercury, therefore

$$\sum_{m=0}^{m=313}\left[P_s \cos s\frac{54}{13}m\times 30°+Q_s \sin\frac{54}{13}m\times 30°\right]\cos m\times 30°=0, \quad . \quad . \quad . \qquad (38)$$

and adding $\frac{1}{144}$ of this to (37), we have

$$p_1=\frac{1}{144}\sum_{m=0}^{m=287}[\alpha_m \cos m\, 30°]+\frac{1}{144}\sum_{m=288}^{m=313}\left[P_s \cos s\frac{54}{13}m\times 30°+Q_s \sin s\frac{54}{13}m\times 30°\right]\cos m\times 30°, \quad (39)$$

and calculating the last term from the approximate coefficients of Mercury given in paragraph 18, we find its value to be, for Easterly disturbance, $+0{\cdot}006$, therefore

$$p_1=\frac{1}{144}\sum_{m=0}^{m=287}[\alpha_m \cos m\, 30°]\ +0\ 006$$

Similarly we find

$$q_1=\frac{1}{144}\sum_{m=0}^{m=287}[\alpha_m \sin m\, 30°]\ -0\ 021,$$

$$p_2=\frac{1}{144}\sum_{m=0}^{m=287}[\alpha_m \cos 2m\, 30°]+0\ 071,$$

$$q_2=\frac{1}{144}\sum_{m=0}^{m=287}[\alpha_m \sin 2m\, 30°]-0\ 132,$$

and for Westerly disturbance

$$p_1=\frac{1}{144}\sum_{m=0}^{m=287}[\alpha_m \cos m\, 30°]\ +0\ 035,$$

$$q_1=\frac{1}{144}\sum_{m=0}^{m=287}[\alpha_m \sin m\, 30°]\ +0\ 061,$$

$$p_2=\frac{1}{144}\sum_{m=0}^{m=287}[\alpha_m \cos 2m\, 30°]-0\ 013,$$

$$q_2=\frac{1}{144}\sum_{m=0}^{m=287}[\alpha_m \sin 2m\, 30°]-0\ 101,$$

in all of which the last terms are small enough to be neglected, in comparison with the absolute range of any of the component periodical variations, as may be seen by simple inspection of the values of the several coefficients given in paragraph 18. And as these calculations are given more in illustration of the method than for any intrinsic value of the result, we need carry them no further

25 Similarly, to find the effects of the Earth's period upon the coefficients for the sidereal period of Mercury, we have in lieu of (37),

$$p_1=\frac{1}{144}\sum_{m=0}^{m=287}\left[\alpha_m\cos\left(\frac{54}{13}m\,30^\circ\right)\right]-\frac{1}{144}\sum_{m=0}^{m=287}\left[P_s\cos sm\,30^\circ+Q_s\sin sm\,30^\circ\right]\cos\left(\frac{54}{13}m\,30^\circ\right)$$

$$=\frac{1}{144}\sum_{m=0}^{m=287}\left[\alpha_m\cos\left(\frac{54}{13}m\,30^\circ\right)\right]+\frac{1}{144}\sum_{m=288}^{m=313}\left[P_s\cos sm\,30^\circ+Q_s\sin sm\,30^\circ\right]\cos\left(\frac{54}{13}m\,30^\circ\right);$$

calculating which for Easterly disturbance, we obtain

$$p_1=\frac{1}{144}\sum_{m=0}^{m=287}\left[\alpha_m\cos\left(\frac{54}{13}m\,30^\circ\right)\right]-0\ 016,$$

also

$$q_1=\frac{1}{144}\sum_{m=0}^{m=287}\left[\alpha_m\sin\left(\frac{54}{13}m\,30^\circ\right)\right]+0\ 004,$$

$$p_2=\frac{1}{144}\sum_{m=0}^{m=287}\left[\alpha_m\cos 2\left(\frac{54}{13}m\,30^\circ\right)\right]-0\ 022,$$

$$q_2=\frac{1}{144}\sum_{m=0}^{m=287}\left[\alpha_m\sin 2\left(\frac{54}{13}m\,30^\circ\right)\right]+0\ 022,$$

$$p_3=\frac{1}{144}\sum_{m=0}^{m=287}\left[\alpha_m\cos 3\left(\frac{54}{13}m\,30^\circ\right)\right]-0\ 031,$$

$$q_3=\frac{1}{144}\sum_{m=0}^{m=287}\left[\alpha_m\sin 3\left(\frac{54}{13}m\,30^\circ\right)\right]-0\ 014,$$

and for Westerly disturbance

$$p_1=\frac{1}{144}\sum_{m=0}^{m=287}\left[\alpha_m\cos\left(\frac{54}{13}m\,30^\circ\right)\right]-0{\cdot}084,$$

$$q_1=\frac{1}{144}\sum_{m=0}^{m=287}\left[\alpha_m\sin\left(\frac{54}{13}m\,30^\circ\right)\right]+0\ 000,$$

$$p_2=\frac{1}{144}\sum_{m=0}^{m=287}\left[\alpha_m\cos 2\left(\frac{54}{13}m\,30^\circ\right)\right]+0\ 063,$$

$$q_2=\frac{1}{144}\sum_{m=0}^{m=287}\left[\alpha_m\sin 2\left(\frac{54}{13}m\,30^\circ\right)\right]-0\ 032,$$

$$p_3=\frac{1}{144}\sum_{m=0}^{m=287}\left[\alpha_m\cos 3\left(\frac{54}{13}m\,30^\circ\right)\right]-0\ 025,$$

$$q_3=\frac{1}{144}\sum_{m=0}^{m=287}\left[\alpha_m\sin 3\left(\frac{54}{13}m\,30^\circ\right)\right]+0\ 046,$$

in all of which also the last terms are small enough to be neglected, in comparison with the range of each component variation, as may be seen from the values of the several coefficients given in paragraph 18.

26 To make a similar estimate of the reciprocal actions of Venus and Mercury would, with a month as the interval between successive observations, be extremely troublesome, but what has been done shows sufficiently the practicableness of the process, and we do not consider it necessary to apply it at present to this or any of the other cases we are dealing with

27. The principal features pointed out by Messrs DE LA RUE, STEWART, and LOEWY * of the growth and decadence of sun-spots were of a simple character, the spots acquired a minimum magnitude at a heliocentric longitude a little greater than that of the planet, and a maximum at a heliocentric longitude a little more than 180° greater than that of the planet, and there was a gradual progression in the change from minimum to maximum and *vice versâ* in the intervening periods

28 It must be admitted that the curves which we have found of magnetic variation in planetary periods do not possess the same simple character, but if we confine our attention to those of them which have been yielded by the largest number of individual observations of disturbance, viz to the curves of *Easterly* disturbance of Declination and to the curves of disturbances decreasing the Horizontal Force, we shall find in them definitiveness of character and some remarkable points of correspondence and difference that would seem to be deserving of attention We first note that, for the synodic period of Venus, the curves of Declination and Horizontal Force have their principal inflections alike, and that this likeness attaches, though in a less degree, to the curves for the synodic periods of Mercury and Jupiter, in common with those for Venus, secondly, that whilst the curves of Venus are strikingly bolder and more definite for the synodic period than for the sidereal period, there is no very marked difference in the case of the curves of Mercury Again, we note the close resemblance in the two curves of the Earth and in the two of Mercury for its sidereal period—in the latter case of so precise a kind that, keeping in mind that the curves are derived from independent observations with instruments of different construction, it is difficult to suppose that they do not indicate a real periodicity in nature

29 It is not claimed for these investigations that they account for any substantial part of the so-called decennial variation of magnetic disturbance, but only that there may be, and probably are, subordinate planetary variations of the kind described, which are superimposed upon the more strongly marked decennial variation, and that if they are, they are included with the variations that have been deduced from the observations

It must be allowed, too, that, until the character of the decennial variation be brought out more fully than as yet (by a great extension of the period of observation), doubt must remain as to whether these apparent variations which follow the periods of the planets may not be due, wholly or in part, to the imperfect elimination of the decennial variation The irregularities observed in the duration of the sun-spot period, with general correspondence in magnetic disturbance, as far as observation permits the comparison, would seem to indicate that the decennial period itself must be regarded as subordinate to some more extended period, in the recurrence of which the irregularities alluded to would be repeated in the same order. It is for this reason that we have not attempted, from the twenty-six years of observations available, to determine the duration and character of the decennial variation, considering that such an undertaking would, with present data, be to a great extent labour in vain.

* Proceedings of the Royal Society, vol xx. page 210

It is also because the decennial period would greatly affect the apparent variation of magnetic disturbance following the sidereal period of Jupiter, that no attempt has been made to apply these observations, extending over less than three such periods, to the determination of the character of that variation

Appendix

Demonstrations First set

To find the sum of each of the following series —

(1) $\sin 0\beta + \sin\beta + \sin 2\beta + \ldots + \sin(n-1)\beta$

(2) $\cos 0\beta + \cos\beta + \cos 2\beta + \ldots + \cos(n-1)\beta$

(3) $0\sin 0\beta + \sin\beta + 2\sin 2\beta + \ldots + (n-1)\sin(n-1)\beta.$

(4) $0\cos 0\beta + \cos\beta + 2\cos 2\beta + \ldots + (n-1)\cos(n-1)\beta$

(5) $0\sin 0\beta + \sin\beta + 2^2\sin 2\beta + \ldots + (n-1)^2\sin(n-1)\beta.$

(6) $0\cos 0\beta + \cos\beta + 2^2\cos 2\beta + \ldots + (n-1)^2\cos(n-1)\beta$

If $X = 1 - 2x\cos\beta + x^2,$ . . . (a)

$\frac{dX}{dx} = -2\cos\beta + 2x,$ . . . (b)

$\frac{d^2X}{dx^2} = +2,$ . . . (c)

$\frac{dX^2}{dx^2} = 4(x - \cos\beta)^2.$ . . . (d)

If $Y = x\sin(\alpha+\beta) - x^n\sin(\alpha+n\beta) + x^{n+1}\sin\{\alpha+(n-1)\beta\} - x^2\sin\alpha,$ . . . (e)

$\frac{dY}{dx} = \sin(\alpha+\beta) - nx^{n-1}\sin(\alpha+n\beta) + (n+1)x^n\sin\{\alpha+(n-1)\beta\} - 2x\sin\alpha,$ . . (f)

$\frac{d^2Y}{dx^2} = -n(n-1)x^{n-2}\sin(\alpha+n\beta) + (n+1)nx^{n-1}\sin\{\alpha+(n-1)\beta\} - 2\sin\alpha.$ . . (g)

And when $x=1$ and $\alpha=0$, these become respectively

$X = 2(1-\cos\beta),$ . . . (h)

$\frac{dX}{dx} = 2(1-\cos\beta),$ . . . (i)

$\frac{d^2X}{dx^2} = +2,$ . . . (j)

$\frac{dX^2}{dx^2} - 4(1-\cos\beta)^2,$ . . . (k)

$Y = \sin\beta - \sin n\beta + \sin(n-1)\beta,$ . . . (l)

$$\frac{dY}{dx}=\sin\beta-n\sin n\beta+(n+1)\sin(n-1)\beta, \quad \text{(m)}$$

$$\frac{d^2Y}{dx^2}=-n(n-1)\sin n\beta+(n+1)n\sin(n-1)\beta \quad \text{(n)}$$

When $x=1$ and $\alpha=\frac{\pi}{2}$, (a) to (d) have the same values as in (h) to (k) respectively, and (e) to (g) become as follows —

$$Y=\cos\beta-\cos n\beta+\cos(n-1)\beta-1, \quad \text{(o)}$$

$$\frac{dY}{dx}=\cos\beta-n\cos n\beta+(n+1)\cos(n-1)\beta-2, \quad \text{(p)}$$

$$\frac{d^2Y}{dx^2}=-n(n-1)\cos n\beta+(n+1)n\cos(n-1)\beta-2. \quad \text{(q)}$$

Let $S=x\sin(\alpha+\beta)+x^2\sin(\alpha+2\beta)+x^3\sin(\alpha+3\beta)+\ldots x^{n-1}\sin\{\alpha+(n-1)\beta\}$ (1)

$$\sin(\alpha+n\beta+\beta)+\sin(\alpha+n\beta-\beta)=2\sin(\alpha+n\beta)\cos\beta,$$

$$2x^{n+1}\sin(\alpha+n\beta)\cos\beta=x^{n+1}\sin\{(\alpha+n\beta)+\beta\}+x^{n+1}\sin\{(\alpha+n\beta)-\beta\}$$

Hence, by giving n the values 1, 2, 3 $\ldots(n-1)$,

$$2x^2\sin\{\alpha+\beta\}\cos\beta=x^2\sin\{\alpha+2\beta\}+x^2\sin\alpha,$$

$$2x^3\sin\{\alpha+2\beta\}\cos\beta=x^3\sin\{\alpha+3\beta\}+x^3\sin\{\alpha+\beta\},$$

$$2x^4\sin\{\alpha+3\beta\}\cos\beta=x^4\sin\{\alpha+4\beta\}+x^4\sin\{\alpha+2\beta\},$$

$$\&c. \quad = \quad \&c \quad \&c,$$

$$2x^n\sin\{\alpha+(n-1)\beta\}\cos\beta=x^n\sin\{\alpha+n\beta\}+x^n\sin\{\alpha+(n-2)\beta\}$$

Now adding

$$2xS\cos\beta=S-x\sin(\alpha+\beta)+x^n\sin\{\alpha+n\beta\}+x^2S-x^{n+1}\sin\{\alpha+(n-1)\beta\}+x^2\sin\alpha, \quad (2)$$

$$S(1-2x\cos\beta+x^2)=x\sin(\alpha+\beta)-x^n\sin(\alpha+n\beta)+x^{n+1}\sin\{\alpha+(n-1)\beta\}-x^2\sin\alpha, \quad (3)$$

which, when $x=1$ and $\alpha=0$, becomes

$$2S(1-\cos\beta)=\sin\beta-\sin n\beta+\sin(n-1)\beta, \quad (3a)$$

which, when $n\beta=2c\pi$,

$$\left.\begin{aligned}&=0, \text{ whether or not } \beta \text{ is 0 or a multiple of } 2\pi\\ &=\sin\beta+\sin 2\beta+\sin 3\beta+\ldots\ldots+\sin(n-1)\beta.\end{aligned}\right\} \quad (3b)$$

If in (3) x be made $=1$ and $\alpha=\frac{\pi}{2}$,

$$2S(1-\cos\beta)=\cos\beta-\cos n\beta+\cos(n-1)\beta-1, \quad (3c)$$

which, when $n\beta=2c\pi$,

$$=-2(1-\cos\beta) \quad (3d)$$

$$S=-1=\cos\beta+\cos 2\beta+\cos 3\beta+\ldots+\cos(n-1)\beta,$$

to which adding $\cos 0\beta$, we have

$$\left.\begin{array}{ll} & 0=\cos 0+\cos\beta+\cos 2\beta+\cos 3\beta+\ldots\ldots+\cos(n-1)\beta, \\ \text{or} & n= \qquad\qquad \text{do.} \end{array}\right\}$$

according as β is not or is 0 or a multiple of 2π,

$$\text{or (say) } SX=Y, \qquad \ldots \qquad (4)$$

$$S=YX^{-1}, \qquad \ldots \qquad (5)$$

$$\left.\begin{array}{l} \frac{dS}{dx}=\sin(\alpha+\beta)+2x\sin(\alpha+2\beta)+3x^2\sin(\alpha+3\beta)+\ldots\ldots \\ \qquad +(n-1)x^{n-2}\sin\{\alpha+(n-1)\beta\} \\ \quad =\frac{dY}{dx}X^{-1}-Y\frac{dX}{dx}X^{-2}, \end{array}\right\} \qquad \ldots \qquad (6)$$

which, when $x=1$ and $\alpha=0$ (see equations (h), (i), (l), and (m),

$$\left.\begin{array}{l} =2^{-1}(1-\cos\beta)^{-1}\{\sin\beta-n\sin n\beta+(n+1)\sin(n-1)\beta\} \\ \qquad -2^{-2}(1-\cos\beta)^{-2}[2(1-\cos\beta)\{\sin\beta-\sin n\beta+\sin(n-1)\beta\}] \end{array}\right\} \ldots \qquad (7)$$

$$=2^{-1}(1-\cos\beta)^{-1}[-(n-1)\sin n\beta+n\sin(n-1)\beta], \qquad \ldots \qquad (8)$$

which, when $n\beta=2c\pi$, c being an integer,

$$\left.\begin{array}{l} =-\frac{n\sin\beta}{4\sin^2\frac{\beta}{2}}=-\frac{n}{2}\cot\frac{\beta}{2} \\ =\sin\beta+2\sin 2\beta+3\sin 3\beta+\ldots\ldots+(n-1)\sin(n-1)\beta, \end{array}\right\} \qquad \ldots \qquad (9)$$

and as $0\sin 0\beta=0$,

$$\sum_{m=0}^{m=n-1} m\sin m\beta=-\frac{n}{2}\cot\frac{\beta}{2}, \text{ when } \beta \text{ is not 0 or a multiple of } 2\pi. \qquad \ldots \qquad (9a)$$

But when β is 0 or a multiple of 2π, each term of the series is 0, and

$$\sum_{m=0}^{m=n-1} m\sin m\beta=0 \qquad \ldots \qquad (9b)$$

Now let $x=1$ and $\alpha=\frac{\pi}{2}$, and (6) becomes

$$\left.\begin{array}{l} \frac{dS}{dx}=2^{-1}(1-\cos\beta)^{-1}\{-\cos\beta-n\cos n\beta+(n+1)\cos(n-1)\beta\} \\ \qquad -2^{-2}(1-\cos\beta)^{-2}[2(1-\cos\beta)\{-\cos\beta-\cos n\beta+\cos(n-1)\beta+1\}] \end{array}\right\} \qquad (10)$$

$$=2^{-1}(1-\cos\beta)^{-1}[-(n-1)\cos n\beta+n\cos(n-1)\beta-1],$$

which, when $n\beta=2c\pi$,

$$\left.\begin{array}{l} =2^{-1}(1-\cos\beta)^{-1}[-(n-1)+n\cos\beta-1]=-\frac{n}{2} \\ =\cos\beta+2\cos 2\beta+3\cos 3\beta+\ldots+(n-1)\cos(n-1)\beta; \end{array}\right\} \qquad \ldots \qquad (11)$$

and as $0 \cos 0\beta = 0$,

$$\sum_{m=0}^{m=n-1} m \cos m\beta = -\frac{n}{2}, \quad \ldots \quad (11a)$$

except when β is 0 or a multiple of 2π, in which case

$$\sum_{m=0}^{m=n-1} m \cos m\beta = \{0+1+2+3+\ldots \quad +(n-1)\} = \frac{n^2}{2} - \frac{n}{2}. \quad \ldots \quad (11b)$$

Again, multiplying equation (6) by x,

$$\left.\begin{aligned} x\frac{dS}{dx} &= x \sin(\alpha+\beta) + 2x^2 \sin(\alpha+2\beta) + 3x^3 \sin(\alpha+3\beta) + \ldots \\ &\quad + (n-1)\, x^{n-1} \sin\{\alpha+(n+1)\beta\} \\ &= x\left\{\frac{dY}{dx} X^{-1} - Y \frac{dX}{dx} X^{-2}\right\}, \end{aligned}\right\} \quad \ldots \quad (12)$$

$$\left.\begin{aligned} \frac{d}{dx}\left(x\frac{dS}{dx}\right) &= \sin(\alpha+\beta) + 4x \sin(\alpha+2\beta) + 9x^2 \sin(\alpha+3\beta) + \ldots \\ &\quad + (n-1)^2 x^{n-2} \sin\{\alpha+(n-1)\beta\} \\ &= \left\{\frac{dY}{dx} X^{-1} - Y\frac{dX}{dx} X^{-2}\right\} + x\left\{\frac{d^2Y}{dx^2} X^{-1} - \frac{dY}{dx}\,\frac{dX}{dx} X^{-2} - \frac{dY}{dx}\,\frac{dX}{dx} X^{-2} \right. \\ &\quad \left. - Y\frac{d^2X}{dx^2} X^{-2} + 2Y\frac{dX^2}{dx^2} X^{-3}\right\} \end{aligned}\right\} \quad (13)$$

$$= X^{-1}\left\{\frac{dY}{dx} + x\frac{d^2Y}{dx^2}\right\} - X^{-2}\left\{\left(Y + 2x\frac{dY}{dx}\right)\frac{dX}{dx} + xY\frac{d^2X}{dx^2}\right\} + 2X^{-3}\, Yx\frac{dX^2}{dx^2}, \quad (14)$$

which, when $x=1$ and $\alpha=0$,

$$\begin{aligned} &= 2^{-1}(1-\cos\beta)^{-1}[\sin\beta - n^2 \sin n\beta + (n+1)^2 \sin(n-1)\beta] \\ &\quad - 2^{-2}(1-\cos\beta)^{-2}[\{3\sin\beta - (2n+1)\sin n\beta + (2n+3)\sin(n-1)\beta\}2(1-\cos\beta) \\ &\qquad + 2\{\sin\beta - \sin n\beta + \sin n-1)\beta\}] \\ &\quad + 2^{-3}(1-\cos\beta)^{-3}[8(1-\cos\beta)^2\{\sin\beta - \sin n\beta + \sin(n-1)\beta\}], \quad \ldots \quad (15) \end{aligned}$$

which, when $n\beta = 2c\pi$,

$$\begin{aligned} &= -2^{-1}(1-\cos\beta)^{-1}[\{n^2+2n\}\sin\beta] \\ &\quad - 2^{-2}(1-\cos\beta)^{-2}[2(1-\cos\beta)\{-2n\sin\beta\}] \quad \ldots \quad (16) \end{aligned}$$

$$= -2^{-1}(1-\cos\beta)^{-1} n^2 \sin\beta = -\frac{2\sin\frac{\beta}{2}\cos\frac{\beta}{2}}{4\sin^2\frac{\beta}{2}}\, n^2 = -\frac{n^2}{2}\cot\frac{\beta}{2}$$

$$= \sin\beta + 4\sin 2\beta + 9\sin 3\beta + \ldots + (n-1)^2 \sin(n-1)\beta; \quad \ldots \quad (17)$$

and as $0^2 \sin 0\beta = 0$,

$$\sum_{m=0}^{m=n-1} m^2 \sin m\beta = -\frac{n^2}{2}\cot\frac{\beta}{2}, \text{ when } \beta \text{ is not 0 or a multiple of } 2\pi \quad \ldots \quad (17a)$$

But when β is 0 or a multiple of 2π, each term of the series is 0, and

$$\sum_{m=0}^{m=n-1} m^2 \sin m\beta=0. \quad \text{(17b)}$$

Now let $x=1$ and $\alpha=\frac{\pi}{2}$, and (14) becomes

$$\begin{aligned}\frac{d}{dx}\left(x\frac{dS}{dx}\right)&=2^{-1}(1-\cos\beta)^{-1}\left[\cos\beta-n^2\cos n\beta+(n+1)^2\cos(n-1)\beta-4\right]\\&-2^{-2}(1-\cos\beta)^{-2}\left[\{3\cos\beta-(2n+1)\cos n\beta+(2n+3)\cos(n-1)\beta-5\}\right.\\&\quad\left.\times 2(1-\cos\beta)+2\{\cos\beta-\cos n\beta+\cos(n-1)\beta-1\}\right]\\&+2^{-3}(1-\cos\beta)^{-3}\left[8(1-\cos\beta)^2\{\cos\beta-\cos n\beta+\cos(n-1)\beta-1\}\right],\end{aligned} \quad (18)$$

which, when $n\beta=2c\pi$,

$$\begin{aligned}&=2^{-1}(1-\cos\beta)^{-1}\left[(n^2+2n+2)\cos\beta-n^2-4\right]\\&\quad-2^{-1}(1-\cos\beta)^{-1}\left[(2n+6)\cos\beta-(2n+6)-2\right]\\&\quad+2^{-1}(1-\cos\beta)^{-1}\left[4\cos\beta-4\right] \quad (19)\\&=2^{-1}(1-\cos\beta)^{-1}\left[(n^2+2n+2-2n-6+4)\cos\beta-n^2-4+2n+6+2-4\right], \quad (20)\end{aligned}$$

$$\left.\begin{aligned}&=2^{-1}(1-\cos\beta)^{-1}\left[-n^2(1-\cos\beta)+2n\right]\\&=-\frac{n^2}{2}+\frac{n}{2}\cdot\frac{1}{\sin^2\frac{\beta}{2}}\\&=\cos\beta+4\cos2\beta+9\cos3\beta+\ldots\ldots+(n-1)^2\cos(n-1)\beta;\end{aligned}\right\} \quad (21)$$

and as $0^2\cos 0\beta=0$,

$$\sum_{m=0}^{m=n-1} m^2\cos m\beta=-\frac{n^2}{2}+\frac{n}{2}\frac{1}{\sin^2\frac{\beta}{2}}, \quad \text{(21a)}$$

except when β is 0 or a multiple of 2π, in which case

$$\sum_{m=0}^{m=n-1} m^2\cos m\beta=0^2+1^2+2^2+3^2+\ldots+(n-1)^2=\frac{n^3}{3}-\frac{n}{2}+\frac{n}{6} \quad \text{(21b)}$$

Collecting together equations (3b), (3d), (9a), (9b), (11a), (11b), (17a), (17b), (21a), and (21b), we have, according as β is not or is 0 or a multiple of 2π,

$$\sum_{m=0}^{m=n-1}\sin m\beta=0, \qquad \text{or } =0,$$

$$\sum_{m=0}^{m=n-1}\cos m\beta=0, \qquad \text{or } =n,$$

$$\sum_{m=0}^{m=n-1} m\sin m\beta=-\frac{n}{2}\cot\frac{\beta}{2}, \qquad \text{or } =0,$$

$$\sum_{m=0}^{m=n-1} m\cos m\beta=-\frac{n}{2}, \qquad \text{or } =\frac{n^2}{2}-\frac{n}{2},$$

$$\sum_{m=0}^{m=n-1} m^2 \sin m\beta = -\frac{n^2}{2}\cot\frac{\beta}{2}, \qquad \text{or} = 0,$$

$$\sum_{m=0}^{m=n-1} m^2 \cos mb = -\frac{n^2}{2}+\frac{n}{2}\frac{1}{\sin^2\frac{\beta}{2}}, \qquad \text{or} = \frac{n^3}{3}-\frac{n^2}{2}+\frac{n}{6}.$$

Demonstrations. Second set.

To find when $(s\mp t)b$, $(sa\mp tb)$, $(s\mp t)a$, and $(sb\mp ta)$ are multiples of r if $a=13$, $b=8$, $r=96$, and s and t are positive and integral

(1) $8(s\mp t)=96c$, c being a positive integer, when

$$s\mp t=12c,$$

$$s=12c\pm t.$$

(2) $(13s\mp 8t)=96c$, when

$$13s=8(12c\pm t)$$

$$=8\{13c-(c\mp t)\},$$

$s=8c-\frac{8}{13}(c\mp t)$, which can only be integral when $(c\mp t)$ is a multiple of 13

(3) $13(s\mp t)=96c$

$$=8(13-1)c,$$

$$s\mp t=8(1-\tfrac{1}{13})c,$$

$s=8\left(c-\frac{c}{13}\right)\pm t$, which can only be integral when c is a multiple of 13

(4) $(8s\mp 13t)=96c$, when

$$8s=96c\pm 13t,$$

$s=12c\pm\frac{13}{8}t$, which can only be integral when t is a multiple of 8.

Specimen Calculation of BESSEL'S Coefficients, for variation of Aggregate

Successive months	0	1	2	3	4	5	6	7	8	9	10	11	12
Twenty-sixths of the sidereal period of Mercury	0	9	18	1	10	19	2	11	20	3	12	21	4
Cosine	+1 000	− 567	− 355	+ 971	− 749	− 120	+ 885	− 885	+ 120	+ 749	− 971	+ 355	+ 568
Sine	0 000	+ 823	− 935	+ 239	+ 663	− 993	+ 465	+ 465	− 993	+ 663	+ 239	− 935	+ 823

Monthly Aggregates

1st 9 period*	53 343	81 730	52 675	31 843	41 527	69 892	153 729	108 458	67 142	210 364	52·444	82 755	84 714
2nd "	16 206	16 850	18 278	32 054	19 469	53 973	92 298	20 561	28 997	28 544	34 003	15 067	18·623
3rd "	35 098	21 894	90 523	49 580	18 995	72 240	53 346	129 985	24 127	74 522	126 439	14 392	18 704
4th "	94 948	19 424	17 374	38 480	50 902	64 843	56 504	43 466	10 708	22 208	31 258	10 283	22 554
5th "	14 734	14 338	8 054	14 978	10 926	13 924	1 665	12 098	5·052	9 554	21 379	15 433	0·000
6th "	49 958	30 181	64 664	111 087	18 894	25 834	77 967	30 488	43 782	44 923	27 337	15·034	57 819
7th "	103 117	37 567	17 695	38 733	140 932	221 409	64 623	63 580	11 436	38 406	60 495	56 850	37 678
8th "	27 592	21 266	51 978	81 257	55 343	163 152	13 866	55 289	62 003	39 771	60 408	56 068	56 448
9th "	57 361	40 140	39 891	20 698	5 020	20 390	52 565	27 247	9 906	39 858	27 211	25 372	53 231
10th "	12 088	31 926	32 221	8 047	30 424	15 881	4 904	2 875	14 832	11 655	16 841	25 108	21 234
11th "	7 087	6 428	55 697	43 308	60 588	75 824	67 852	52 912	17 095	30 644	116 304	50 840	17 692
Sums	471 532	330 744	449·050	470 065	453·020	797 362	639 409	546 959	295 080	550 649	574 119	367 202	388 687
Means	42 867	30 068	40 823	42 733	41 184	72 487	58 128	49 724	26 825	50·059	52 193	33 382	35 335
Variations	−0 313	−13 112	−2 357	−0 447	−1 996	+29 307	+14 948	+6 544	−16 355	+6 879	+9·013	−9 798	−7 845

* Commencing with March 1847

Easterly Disturbance of Declination in the Sidereal Period of Mercury.

13	14	15	16	17	18	19	20	21	22	23	24	25	{ Successive Months
13	22	5	14	23	6	15	24	7	16	25	8	17	{ Twenty-sixths of the sidereal period of Mercury
−1 000	+ 568	− 355	− 971	+ 749	+ 120	− 885	+ 885	− 120	− 749	+ 971	− 355	− 568	Cosine
0 000	− 823	+ 935	− 239	− 663	+ 993	− 465	− 465	+ 993	− 663	− 239	+ 935	− 823	Sine

of Disturbance.

46 396	25 086	87 569	87 888	44 773	15·669	81 189	233 638	27 256	66 268	94 313	55 685	30 832	1st 9 period*
16 777	42 915	21 585	15 093	8 540	5 443	82 517	74 813	21 741	13 837	20 075	20 990	22 818	2nd „
34·921	32 894	16 508	28 006	36 853	32 650	24 427	42 940	25 270	67 160	39 312	27 949	36 882	3rd „
44 416	18 769	22 813	13 100	7 185	36 425	19 414	8 449	14 416	13 078	10 702	16 942	14 624	4th „
6 893	8 118	6 752	20 732	8 924	10 489	13 311	33 118	12 057	25 447	20 650	40 744	96 261	5th „
69 377	48 189	97 001	52 618	38 394	57 820	114 162	169 905	198 777	15 435	49 717	23 984	70 715	6th „
33 454	19 002	41 033	39 174	32 787	47 383	40 953	35 443	69 321	30 153	26 618	179 782	84 432	7th „
10 299	27 583	32 972	27 227	22 576	12 177	0 000	17 328	32 728	26 261	27 672	6 541	80 336	8th „
70 072	30 130	41 462	50 878	13 940	30 095	81 696	16 635	3 045	17 569	13 226	3 059	27 330	9th „
40 817	27 571	0 000	6 695	4 370	16 389	110 398	5 117	18 591	13 281	38 992	55 874	85 009	10th „
59 819	64 717	57 199	70 171	101 645	77 381	64 724	33 539	62 790	79 164	153 136	51 690	45 709	11th „
442 241	344 974	424 891	411 582	319 987	341 921	632 791	670·925	485 992	367 656	494 413	483 240	594 951	Sums
40 204	31 361	38 627	37 417	29 090	31 084	57 526	60 993	44 181	33 423	44 947	43 931	54 086	Means
−2·976	−11 819	−4 553	−5 763	−14 090	−12·096	+14 346	+17 813	+1 001	−9 757	+1 707	+0 751	+10 906	Variations.

* Commencing with March 1847

Specimen Calculation (continued).

Symbol of operation	P_1						
+(0 to 6)	−0·313	−13·112	−2·357	−0·447	−1·996	+29·307	+14·948
−(13 to 7)	+2·976	+7·815	+9·798	−9·013	−6·879	+16·355	−6·541
−(14 to 19)		+11·819	+4·553	+5·763	+14·090	+12·096	−14·346
+(25 to 20)		+10·906	+0·751	+1·767	−9·757	+1·001	+17·813
Sums	+2·663	+17·458	+12·745	−1·930	−4·542	+58·759	+11·871
Factors	+1·000	−0·568	−0·355	+0·971	−0·749	−0·120	+0·885
Products	+2·663	−9·916	−4·524	−1·874	+3·402	−7·051	+10·506
Sum of Products	−6·794						
Ditto−13	−0·523	$=P_1$					
	P_2						
+(0 to 6)	−0·313	−13·112	−2·357	−0·447	−1·996	+29·307	+14·948
+(13 to 7)	−2·976	−7·845	−9·798	+9·013	+6·879	−16·355	+6·544
+(14 to 19)		−11·819	−4·553	−5·763	−14·090	−12·096	+14·346
+(25 to 20)		+10·906	+0·751	+1·767	−9·757	+1·001	+17·813
Sums	−3·289	−21·870	−15·957	+4·570	−18·964	+1·857	+53·651
Factors	+1·000	−0·355	−0·749	+0·885	+0·121	−0·971	+0·568
Products	−3·289	+7·764	+11·952	+4·044	−2·295	−1·803	+30·474
Sum of Products	+46·847						
Ditto−13	+3·604	$=P_2$					
	P_3						
Sum*	+2·663	+17·458	+12·745	−1·930	−4·542	+58·759	+11·871
Factors	+1·000	+0·971	+0·885	+0·749	+0·568	+0·355	+0·120
Products	+2·663	+16·952	+11·279	−1·446	−2·580	+20·859	+1·425
Sum of Products	+49·152						
Ditto−13	+3·781	$=P_3$					

Symbol of operation	Q_1						
+(0 to 6)	−0·313	−13·112	−2·357	−0·447	−1·996	+29·307	+14·948
−(13 to 7)	−2·976	−7·815	−9·798	+9·013	+6·879	−16·355	+6·544
−(14 to 19)		+11·819	+4·553	+5·763	+14·090	+12·096	−14·346
−(25 to 20)		−10·906	−0·751	−1·767	+9·757	−1·001	−17·813
Sums	−3·289	−20·044	−9·353	+12·562	+28·730	+24·047	−10·667
Factors	0·000	+0·823	−0·935	+0·239	+0·663	−0·993	+0·465
Products	0·000	−16·498	+7·810	+3·002	+19·048	−23·879	−4·960
Sum of Products	−15·475						
Ditto−13	−1·190	$=Q_1$					
	Q_2						
+(0 to 6)	−0·313	−13·112	−2·357	−0·447	−1·996	+29·307	+14·948
−(13 to 7)	+2·976	+7·845	+9·798	−9·013	−6·879	+16·355	−6·544
+(14 to 19)		−11·819	−4·553	−5·763	−14·090	−12·096	+14·346
−(25 to 20)		−10·906	−0·751	−1·767	+9·757	−1·001	−17·813
Sums	+2·663	−27·992	+2·137	−16·990	−13·208	+33·565	+4·937
Factors	0·000	−0·935	+0·663	+0·465	−0·993	+0·239	+0·823
Products	0·000	+26·173	+1·417	−7·900	+13·116	+7·783	+4·063
Sum of Products	+44·652						
Ditto−13	+3·435	$=Q_2$					
	Q_3						
Sum*	−3·289	−20·044	−8·353	+12·562	+28·730	+24·047	−10·667
Factors	0·000	+0·239	+0·465	+0·663	+0·823	+0·935	+0·993
Products	0·000	−4·791	−3·884	+8·329	+23·645	+22·484	−10·592
Sum of Products	+35·191						
Ditto−13	+2·707	$=Q_3$					

* These sums are the same as those in the 16th line above them

XV. *Reduction of Anemograms taken at the Armagh Observatory in the Years 1857–63.* *By* T. R. Robinson, *D.D.*, *F.R.S.*, *F.A.S.*, *&c.*

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In the beginning of the year 1845 I erected a self-recording anemometer at the Armagh Observatory, and have a series of its records up to the present time, unbroken except by accidents to the apparatus or occasional illness of the observers. I, however, soon found it was impossible for me and my single assistant to reduce continuously the mass of materials which was accumulating, without neglecting the primary objects of the establishment, and I was obliged to content myself with preserving them, in hope that they might be available to future inquirers It was thought, however, by some distinguished members of the Royal Society that it was desirable to ascertain how far such observations are able to develop any definite laws amid the seeming lawlessness of the wind, and a grant was made to me from the Government Grant sufficient to discuss the anemograms for the seven years from 1857 to 1863 The work has been long delayed by the death of one of the computers, the migration of another to India, and my own temporary blindness

The anemograph is that described by me in the 'Transactions of the Royal Irish Academy,' vol xxii It differs in nothing essential from that employed by the Meteorological Committee of the Royal Society the recording-apparatus is different, and the direction is observed by a vane whose excursions are controlled by a peculiar contrivance instead of by a windmill The space-records were read to 0 25 of a mile (statute), and the directions to 0° 5 The S and W components of the hourly velocity were computed for each to two places of decimals

Wind is caused by a difference of pressure in the air over adjacent portions of the earth's surface, but of the agencies which produce this difference we as yet are imperfectly informed Heat is obviously a most important one We see that the action of the sun must produce a current from polar towards equatorial regions, and that when the geographical conditions of districts not too far asunder are such as to make their temperatures unequal, air-currents between them will result. The changes of solar action at a given place depending on the hour of the day and the day of the year, ought to produce definite periodical modifications of the wind, and the currents due to the varying tension of aqueous vapour ought to be similarly periodical Were these the only causes of the wind, there seems no reason why its force and direction at a given time and place might not be predicted as certainly as the sun's altitude. But there are evidently disturbing agencies of great power which entirely mask the regular course of

the phenomena, and of whose nature we can only form vague conjectures. The accumulation of ice in the polar regions forming icebergs may be such an influence; and what we have learned recently of the action of the larger planets on the solar spots, and of the connexion of the development of those spots with the magnetic storms and auroral discharges of our own planet, may suggest the possibility of extra-terrestrial forces playing some part in the question before us. But without following in the track of imagination, this is certain, that however complicated and irregular a phenomenon may be, if we have a sufficient number of observations, it is possible to determine the values and periods of those parts of it which are subject to definite laws. Where any of these periods agree with those of agents whose influence is certain, they may be referred to them with certainty, and their effect eliminated, making it much easier to deal with the residual phenomena.

In the present instance the want of self-recording instruments for pressure, temperature, and vapour-tension compelled me to consider the wind solely in reference to time, as depending on the hour of the day and on the month; and even with this simplification it is not easy to come to precise results. Were we to seek a velocity and direction which might be considered *normal* for each hour of the year, such is the irregularity of the air-currents, that I think it could scarcely be obtained in less than 100 years. Even if we confine ourselves to the west and south components, and take for successive hours the mean of the seven years concerned, it differs so widely from the means of the preceding and following hours, that any existence of law might seem impossible. But if the hour-means be taken for 20 or 30 successive days, their means present a very different aspect. I have taken them for months.

Before dealing with these components, I think it may be instructive to present a Table giving a synoptic view of the winds, which may show their general character at Armagh during the seven years concerned. It gives for each month and for each octant of the horizon (S. to S.W., S.W. to W., &c.) the mean hourly velocity, the mean direction, and the approximate number of hours during which this wind has blown.

At the end of each month is given the maximum hourly velocity for each year, the number of hours when the velocity exceeded 25 miles, and the number of hours during which the anemograph has recorded 0. This does not imply that during this time there was no wind, but that there was not enough to move the instrument. This requires a velocity$=1^{m}.74$.

The direction-vane is much more sensitive (very much more so than the windmill-apparatus now used to record the direction), and therefore the records of direction are more numerous than those of velocity.

TABLE I.—January.

		1857	1858	1859	1860	1861	1862	1863	
S. to S.W.	Vel. Dir. Hours	16'·46 28°·7 181	16'·94 24° 211	18'·17 28° 199	13'·89 21° 188	16'·46 22° 249	12'·97 24° 216	17'·67 27° 275	Mean 16'·07 Mean 25° Sum 1519
S.W. to W.	Vel. Dir Hours	12'·7 66°·67 193	17'·09 58° 168	14'·66 60° 412	9'·97 63° 193	11'·43 60° 88	14'·35 63° 192	16'·20 60° 211	Mean 13'·78 Mean 61° Sum 1457
W. to N.W.	Vel. Dir Hours	12'·24 103° 130	10'·57 112° 53	10'·28 103° 65	10'·75 107° 40	5'·30 120° 27	8'·89 114° 88	17'·49 107° 80	Mean 10'·79 Mean 109° Sum 483
N.W. to N.	Vel. Dir Hours	10'·65 152° 92	5·76 150° 46	5'·32 156° 26	5'·85 146° 40	6'·00 146° 3	4'·70 156° 57	5'·73 163° 38	Mean 6'·29 Mean 153° Sum 302
N. to N.E.	Vel Dir Hours	5'·15 183° 106	0 223° 1	5'·43 202° 8	7'·60 208° 62	6'·63 212° 76	6'·50 206° 2	8'·56 188° 45	Mean 5'·70 Mean 203° Sum 300
N.E. to E.	Vel. Dir Hours	6'·39 259° 31	0 0 0	1'·70 242° 18	6'·90 242° 40	14'·89 247° 54	23'·22 265° 9	11'·23 247° 10	Mean 9'·19 Mean 250° Sum 200
E. to S.E.	Vel. Dir Hours	2·10 294° 5	13'·11 307° 37	5'·67 291° 4	11'·02 293° 88	4·70 300° 77	17'·21 287° 43	7'·95 290° 21	Mean 8'·82 Mean 295° Sum 275
S.E. to S.	Vel. Dir. Hours	15'·04 346° 12	18'·59 335° 228	13'·67 344° 12	15'·97 338° 93	13'·62 337° 170	20'·60 339° 135	10'·15 322° 45	Mean 15'·38 Mean 337° Sum 695
Maximum Hours > 25' Hours of 0		44'·2 23 0	46' 92·3 3	68' 98 7	60' 68 21	37' 40 12	48' 95 6	47' 146 15	 Sum 562 Sum 64

TABLE I.—February.

S. to S.W.	Vel. Dir. Hours	17'·48 19°·0 177	16·36 15° 56	16'·19 21° 184	16'·01 21° 110	4'·63 23° 118	12'·30 26° 155	15'·19 28° 183	Mean 13·55 Mean 22° Sum 1003
S.W. to W.	Vel. Dir Hours	11'·18 57° 149	11'·52 60° 40	14'·53 66° 159	13'·32 60° 167	12'·17 65° 110	11'·85 66° 112	14·24 61° 192	Mean 13'·22 Mean 62° Sum 935
W. to N.W.	Vel. Dir. Hours	7'·59 100° 61	5'·81 118° 26	8'·60 95° 140	9'·62 108° 156	9·69 112° 48	7'·00 93° 1	12'·01 99° 72	Mean 9'·24 Mean 103° Sum 504
N.W. to N.	Vel Dir. Hours	2·31 149° 8	4'·78 162° 18	0 0 0	6'·99 157° 129	7'·30 175° 20	3'·13 76° 8	2'·50 172° 11	Mean 6·26 Mean 165° Sum 194
N. to N.E.	Vel Dir Hours	5'·50 215° 11	6'·49 201° 48	2'·50 215° 6	5'·75 188° 77	13·28 200° 50	7'·48 201° 61	17'·05 205° 19	Mean 8'·38 Mean 203° Sum 272
N.E. to E.	Vel. Dir Hours	5'·83 247° 63	7'·41 248° 122	2'·27 233° 15	4'·65 237° 51	8'·73 233° 33	8'·63 265° 97	2'·35 247° 4	Mean 6'·96 Mean 244° Sum 385
E. to S.E.	Vel Dir Hours	5'·20 302° 5	15·57 292° 45	6'·32 301° 25	1'·00 295° 4	14'·49 294° 68	10'·51 292° 102	16'·78 29 ° 14	Mean 13'·61 Mean 295° Sum 263
S.E. to S.	Vel. Dir. Hours	14'·60 348° 98	17'·66 338° 111	19'·23 342° 43	1'·00 337° 2	7'·17 313° 104	18'·23 358° 36	12'·96 340° 71	Mean 14'·87 Mean 339° Sum 465
Maximum .. Hours > 25' .. Hours of 0		44'·5 43 3	42' 91 1	40' 63 6	37' 25 3	45' 36 5	46' 54 4	56' 85 4	 Sum 397 Sum 26

TABLE I.—March.

		1857	1858	1859	1860	1861	1862	1863	
S. to S.W.	Vel.	16'·50	11'·15	20'·35	13'·57	17'·70	11'·85	15'·40	Mean 15'·19
	Dir.	20°	27°	32°	27°	27°	30°	21°	Mean 26°
	Hours	107	115	98	131	276	92	209	Sum 1028
S.W. to W.	Vel.	14·90	9'·31	15'·70	16'·27	14'·25	13'·18	13'·00	Mean 13'·80
	Dir.	73°	68°	62°	67°	67°	69°	65°	Mean 67°
	Hours	173	199	368	193	292	61	209	Sum 1495
W. to N.W.	Vel.	9'·77	9'·59	10'·90	10'·67	13'·13	10'·16	14'·97	Mean 11'·31
	Dir.	117°	110°	117°	113°	107°	114°	107°	Mean 112°
	Hours	62	222	184	279	89	19	103	Sum 958
N.W. to N.	Vel.	6'·45	7·48	9'·30	9'·29	8'·38	5'·86	6'·34	Mean 7·59
	Dir.	172°	161°	147°	151°	149°	153°	156°	Mean 156°
	Hours	69	65	67	70	8	42	38	Sum 359
N. to N.E.	Vel.	8'·58	13·86	12·50	7'·78	1'·50	8'·96	6'·19	Mean 8'·48
	Dir.	241°	209°	197°	202°	195°	206	200°	Mean 207°
	Hours	80	90	14	18	2	96	37	Sum 337
N.E. to E.	Vel.	6'·55	19'·75	11'·00	5'·00	8'·68	11'·30	6'·61	Mean 9'·84
	Dir.	264°	233°	255°	245°	249°	236°	243°	Mean 246°
	Hours	98	28	7	23	19	248	18	Sum 441
E. to S.E.	Vel.	11'·58	4'·80	24'·50	7·50	20'·50	11'·31	12'·39	Mean 13'·23
	Dir.	274°	279°	282°	272°	285°	304°	289	Mean 284°
	Hours	64	5	6	4	6	74	36	Sum 195
S.E. to S.	Vel.	13·71	12'·00	0·00	18·38	13'·88	11'·53	15·94	Mean 12'·21
	Dir.	350°	352°	.	341°	346°	334°	351	Mean 346°
	Hours	91	20		26	50	112	97	Sum 396
Maximum		49'·5	50'	58'	54'	43'·2	57'	40'	
Hours > 25'		45	39	49	35	18	9	55	Sum 250
Hours of 0		3	5	1	4	12	4	1	Sum 30

TABLE I.—April.

S. to S.W.	Vel.	12·38	9'·88	14'·11	11'·23	8'·67	13'·37	14'·90	Mean 12'·08
	Dir.	36	42°	24°	19	23	25°	28°	Mean 28°
	Hours	67	125	116	92	61	181	203	Sum 845
S.W. to W.	Vel.	9·86	7'·50	11·72	14'·06	5'·49	12'·45	12·13	Mean 10'·39
	Dir.	74	61°	62	64	61	63°	66	Mean 64°
	Hours	65	76	150	50	33	131	183	Sum 688
W. to N.W.	Vel.	8·73	7·38	7'·29	8'·86	5'·33	9·38	10'·70	Mean 8'·15
	Dir.	114	117	116°	103°	115°	110	112	Mean 112
	Hours	115	65	54	111	126	42	98	Sum 611
N.W. to N.	Vel.	6·79	6·00	6·20	8'·64	5'·27	6'·53	9'·25	Mean 6·93
	Dir.	154	164	157°	163°	155	136°	157°	Mean 155°
	Hours	60	73	93	119	78	69	55	Sum 547
N. to N.E.	Vel.	5'·29	5·92	6·68	13·28	6'·30	6'·30	7'·50	Mean 7'·18
	Dir.	201°	197°	206°	200°	202°	199°	184°	Mean 198°
	Hours	49	59	89	46	175	83	2	Sum 503
N.E. to E.	Vel.	8·20	13·98	15'·28	6·47	6'·79	6'·36	0'·00	Mean 8'·13
	Dir.	251°	255	250	247°	248°	241°		Mean 213°
	Hours	126	84	153	57	107	22	...	Sum 549
E. to S.E.	Vel.	12·47	16'·78	16·04	8'·73	9'·52	3'·55	8'·64	Mean 10'·82
	Dir.	286	288°	284	290°	284°	292°	302°	Mean 289°·5
	Hours	113	153	49	100	98	45	28	Sum 586
S.E. to S.	Vel.	14'·73	16·43	14'·60	10·40	13·58	13'·01	13'·05	Mean 13'·55
	Dir.	335	335	346°	338	333°	339°	343°	Mean 338°
	Hours	102	81	15	75	40	88	78	Sum 479
Maximum		45·5	57'	46'	34'	23·5	36'	58'	
Hours > 25'		17	27	43	5	0	24	40	Sum 156
Hours of 0		26	108	10	24	5	21	6	Sum 200

TABLE I.—May.

		1857	1858	1859	1860	1861	1862	1863	
S. to S W.	Vel.	15'·31	11'·09	6' 55	8'·21	7' 07	9' 60	12'·39	Mean 9'·95
	Dir.	27°	27°	21°	17°	27°	24°	27°	Mean 24°
	Hours	134	165	86	204	120	158	149	Sum 1016
S.W. to W.	Vel.	6' 87	10'·18	4' 32	5'·98	5'·55	9'·84	8'·67	Mean 7' 34
	Dir.	70°	71°	62°	69°	65°	64°	65°	Mean 67°
	Hours	54	138	50	96	226	174	218	Sum 956
W to N.W.	Vel.	8' 19	4'·72	2' 89	7·24	4' 03	3'·55	6' 20	Mean 5'·23
	Dir	113°	111°	118°	108°	113°	121°	105°	Mean 113°
	Hours	21	59	18	130	159	105	120	Sum 612
N.W. to N	Vel.	3'·85	6' 00	4' 90	3' 93	3'·73	3' 24	5' 02	Mean 4'·26
	Dir	165°	159°	167	156°	162°	157°	162	Mean 161°
	Hours	20	114	70	15	64	54	43	Sum 380
N to N.E	Vel.	5' 15	3'·94	4' 41	3 33	5' 55	3'·24	6'·03	Mean 4'·52
	Dir.	211°	201°	198	205°	197°	198°	207°	Mean 202°
	Hours	57	86	184	36	89	33	67	Sum 552
N E to E	Vel	7'·91	4'·01	6' 36	8' 35	5' 55	4'·44	10' 34	Mean 6 71
	Dir	254	250°	246°	254°	247	248	235°	Mean 248°
	Hours	171	56	140	101	46	27	97	Sum 638
E to S E	Vel	8'·75	5' 62	6 87	8' 10	4' 69	1'·07	9 25	Mean 6 32
	Dir	290°	295°	293	299	300	294	305°	Mean 297
	Hours	183	62	86	75	25	93	20	Sum 544
S E to S	Vel	13·49	10' 77	11' 93	9' 76	11' 90	10' 31	14·27	Mean 11 66
	Dir	315°	337	335	337	330°	336	342	Mean 332
	Hours	97	36	106	86	14	98	18	Sum 455
Maximum		26'	38	42'	28'	21'·2	30'	33'	
Hours > 25'		4	1	7	1	0	7	8	Sum 28
Hours of 0		21	19	49	23	3	52	21	Sum 188

TABLE I.—June

S. to S W	Vel.	8'·19	9·47	12 32	7'·21	5' 88	7' 73	8'·51	Mean 8' 47
	Dir	17	24	20°	26°	22	35	23°	Mean 24
	Hours	131	129	49	131	79	142	182	Sum 843
S.W. to W	Vel	5'·06	6·00	7·67	7'·82	4' 14	6'·96	8' 31	Mean 6' 56
	Dir	65	69	64	61°	66°	65°	65	Mean 65°
	Hours	79	190	92	165	56	193	235	Sum 1010
W. to N.W.	Vel	5' 87	4' 23	4 75	5'·07	5' 11	5'·76	4'·54	Mean 5 05
	Dir	110°	107°	111°	115°	106	111°	112°	Mean 110
	Hours	31	71	135	97	93	243	71	Sum 741
N.W. to N.	Vel	3'·76	5' 02	3' 55	3'·47	3' 94	11·43	4' 91	Mean 5 75
	Dir.	159°	155°	162°	154°	161°	152	161°	Mean 158°
	Hours	58	24	70	88	115	66	62	Sum 483
N. to N.E.	Vel.	4'·76	2' 86	5' 72	5' 15	4'·60	4 00	3'·48	Mean 4' 29
	Dir.	199°	198°	206°	201°	193°	190°	194	Mean 197°
	Hours	65	66	136	33	75	4	88	Sum 467
N E. to E.	Vel	7' 94	4' 18	9' 05	8'·12	9 36	13' 79	4' 24	Mean 6 67
	Dir.	251°	245	243°	249	247°	265	245	Mean 249°
	Hours	125	27	121	56	103	19	25	Sum 476
E. to S.E.	Vel.	5'·36	8'·80	5'·43	8'·03	6' 05	13' 52	6'·61	Mean 7 58
	Dir	291°	300°	296°	267°	297°	297°	303°	Mean 296°
	Hours	137	35	30	60	37	23	13	Sum 335
S.E. to S.	Vel	9' 65	12'·70	10' 74	10'·06	8·57	9·80	11 15	Mean 10'·28
	Dir.	315°	335°	335°	333°	339°	339°	346°	Mean 335°
	Hours	88	173	82	85	129	30	33	Sum 620
Maximum		25'	32'	37'	29'	22'	48'	38'	
Hours > 25'		2	10	4	2	0	18	8	Sum 44
Hours of 0		24	10	15	55	36	10	50	Sum 200

Table I.—July.

		1857	1858	1859	1860	1861	1862	1863	
S.	Vel . .	8′·67	6′·56	7′·79	4′·89	8′·62	10′·53	6′·00	Mean 7′·58
to	Dir	33	24°	27°	29°	31°	27°	19°	Mean 27°
S.W.	Hours	162	119	135	55	109	122	39	Sum 741
S.W.	Vel. . .	7′·83	6′·48	6′·94	4′·59	6′·92	8′·31	5′·19	Mean 6′·61
to	Dir	67°	73°	64°	66°	65°	64°	68°	Mean 67°
W.	Hours	294	191	218	110	213	254	137	Sum 1397
W.	Vel	4′·80	5′·05	4′·18	3′·80	4′·29	6′·53	3′·34	Mean 4′·36
to	Dir	111°	113°	107°	116°	103°	106°	115°	Mean 110°
N.W.	Hours	179	120	99	230	99	141	154	Sum 1022
N.W.	Vel	5′·05	3′·35	2′·33	2′·85	2′·70	2′·06	3′·51	Mean 3′·03
to	Dir .	147°	153	152	159°	155°	158°	157°	Mean 154°
N.	Hours	85	139	12	140	17	49	188	Sum 630
N.	Vel	4′·45	3′·14	5′·51	3′·07	4′·52	1′·50	4′·24	Mean 3′·71
to	Dir .	214°	195°	196°	200°	197°	181°	197°	Mean 197°
N.E.	Hours	33	71	114	78	125	2	112	Sum 535
N.E.	Vel	0′	3′·46	7′·19	7′·50	3′·81	18′·5	6′·00	Mean 6′·51
to	Dir		245°	251°	237	246°	267°	232°	Mean 246°
E.	Hours		28	108	12	43	2	25	Sum 218
E.	Vel	0′	12′·30	4′·69	6′·87	8′·85	11′·95	6′·00	Mean 7′·20
to	Dir		305	288	302	284	293	298°	Mean 295°
S.E.	Hours		23	33	32	67	21	28	Sum 204
S.E.	Vel	10′·90	10′·57	12′·91	5′·59	14′·31	11′·10	13′·90	Mean 11′·08
to	Dir	356	341°	336	332°	252°	341°	344	Mean 343°
S.	Hours	5	54	23	93	67	118	33	Sum 393
Maximum		28	22	32′	19′	32′	41′	22′	
Hours > 25′		2	0	2	0	8	20	0	Sum 32
Hours of 0		5	30	15	83	20	33	47	Sum 233

Table I.—August.

		1857	1858	1859	1860	1861	1862	1863	
S.	Vel	8′·26	7′·11	9′·85	8′·36	9′·54	7′·14	8′·78	Mean 8′·43
to	Dir	24	17	28	30°	26	28	28	Mean 26°
S.W.	Hours	92	75	174	119	377	202	201	Sum 1240
S.W.	Vel	4′·36	6′·01	7′·77	7′·29	7′·50	5′·66	8′·82	Mean 6′·71
to	Dir	72	68	61	70	61°	64	60	Mean 65°
W.	Hours	90	164	242	223	195	195	202	Sum 1311
W.	Vel	3′·09	4′·68	5′·83	4′·75	6′·66	3′·21	6′·14	Mean 4′·91
to	Dir	119	115	104	108°	101	116°	115°	Mean 111°
N.W.	Hours	86	159	153	198	30	93	134	Sum 853
N.W.	Vel	3′·52	3′·82	1′·80	3′·39	9′·94	2′·00	6′·26	Mean 4′·29
to	Dir .	159	151°	155	151°	159°	157°	155	Mean 155
N.	Hours	150	52	56	84	18	65	42	Sum 467
N.	Vel	4′·71	2′·83	4′·14	5′·54	7′·20	4′·22	7′·65	Mean 5′·16
to	Dir	200	211	200	207	218	203°	203°	Mean 206
N.E.	Hours	124	59	88	31	5	45	32	Sum 384
N.E.	Vel	6′·55	3′·78	6′·80	7′·42	9′·77	8′·84	6′·91	Mean 6′·48
to	Dir	246	246°	237	252	255	251°	280°	Mean 252°
E.	Hours	84	14	5	7	22	13	34	Sum 179
E.	Vel	6′·79	6′·55	2′·00	6′·39	12′·14	6′·68	6′·50	Mean 5′·23
to	Dir	289°	298	301	296	303°	289°	306°	Mean 297°
S.E.	Hours	52	70	3	33	14	44	24	Sum 240
S.E.	Vel	9′·48	11′·47	8′·80	8′·77	15′·13	10′·93	15′·72	Mean 11′·37
to	Dir	337	333	344	341	340°	335°	334°	Mean 338°
S.	Hours	46	102	15	40	82	59	36	Sum 380
Maximum		26′	28′	33	42′	28′	28′	31′	
Hours > 25′		1	5	1	2	9	6	14	Sum 38
Hours of 0		20	25	28	24	10	39	10	Sum 156

TABLE I.—September

		1857	1858	1859	1860	1861	1862	1863	
S.	Vel. ...	9'·49	12'·96	10'·64	11'·00	9'·58	9'·05	12'·47	Mean 10'·74
to	Dir. ...	22°	18°	21°	21°	26°	25°	31°	Mean 23°
S.W.	Hours .	193	134	204	118	194	190	165	Sum 1198
S.W.	Vel. . .	7'·42	8'·12	8'·99	5'·35	6'·50	5'·26	1'·13	Mean 6'·04
to	Dir.	73°	63	70°	75°	60°	62°	65°	Mean 67°
W.	Hours	70	296	175	123	66	116	287	Sum 1133
W.	Vel.	4'·27	5'·12	6'·09	3'·29	4'·67	4'·46	8'·46	Mean 5'·19
to	Dir . .	109°	101°	106°	109°	109°	114°	115°	Mean 108°
N.W.	Hours.	83	59	88	112	142	101	88	Sum 673
N.W.	Vel...	4'·34	7'·12	4'·00	2'·82	2'·88	3'·77	9'·64	Mean 4'·94
to	Dir	164°	160°	155°	156°	163°	157°	159°	Mean 159°
N	Hours	75	31	30	88	34	58	54	Sum 370
N.	Vel	3'·47	6'·69	5'·17	4'·06	2'·70	3'·88	4'·35	Mean 4'·24
to	Dir.	200°	199°	204°	205°	170°	193°	193°	Mean 195°
N.E	Hours	79	13	62	128	10	78	54	Sum 424
N E.	Vel. . .	3'·30	4'·72	5'·50	4'·85	4·50	5·77	5'·00	Mean 4'·64
to	Dir	252°	253°	243°	238°	239	253°	226°	Mean 243°
E	Hours	42	102	26	13	2	71	1	Sum 287
E.	Vel.	23·55	7·70	6'·09	4'·53	14·07	8·91	13'·50	Mean 11'·05
to	Dir	296°	283°	281°	301°	269°	290°	291°	Mean 293°
S E	Hours	26	35	21	29	27	23	8	Sum 169
S.E.	Vel	11'·58	13'·75	14·20	11'·48	12·61	11'·05	10'·28	Mean 12·11
to	Dir	335°	339°	343°	349°	334°	359°	344°	Mean 343°
S	Hours	116	36	82	26	244	66	38	Sum 608
Maximum		34'	32	36'	36'	42'	29'	39'	
Hours > 25'		5	6	10	7	27	5	7	Sum 67
Hours of 0		32	6	12	27	10	24	7	Sum 118

TABLE I.—October

S.	Vel.	8·02	12·41	9·11	12·40	7'·54	13·73	9·50	Mean 10·33
to	Dir	22°	29°	20°	27°	17°	28	21°	Mean 24°
S W	Hours	130	131	129	166	233	260	136	Sum 1185
S W	Vel	7·90	9·35	5·12	10·62	2·07	13·60	8·80	Mean 7·91
to	Dir.	59	66	73	71	67°	61°	66°	Mean 66°
W	Hours	149	199	94	263	77	252	298	Sum 1332
W.	Vel.	6·17	5·43	4·84	6·46	3·84	8·43	5·76	Mean 5·85
to	Dir	109°	124°	101°	107°	115	93	111°	Mean 109
N.W	Hours	100	115	90	115	19	60	37	Sum 536
N.W.	Vel	4'·28	2·05	5'·37	4·30	1·45	4·88	9·05	Mean 4·44
to	Dir	165°	161°	153°	151°	140	160°	146°	Mean 154
N.	Hours	7	19	46	27	24	9	8	Sum 140
N.	Vel.	4'·13	5'·25	4'·01	8·59	2·65	4'·25	2·48	Mean 4·48
to	Dir	205°	211°	199°	201°	214°	213°	209	Mean 207°
N E	Hours	71	167	111	32	43	12	9	Sum 445
N.E.	Vel.	10'·71	3'·58	5'·45	4·27	6·45	3·65	10·42	Mean 6·35
to	Dir. .	253°	240°	245°	235°	238°	250°	252°	Mean 244°
E	Hours	82	86	120	27	37	26	80	Sum 458
E.	Vel.	9'·00	3'·50	4'·81	9'·42	9'·68	5·16	8'·40	Mean 7·02
to	Dir .	281°	286°	290°	314°	299°	280°	290°	Mean 292°
S.E.	Hours	38	6	95	4	123	25	71	Sum 362
S.E.	Vel.	11'·43	4'·80	12'·49	14'·70	11'·69	14·37	13·12	Mean 11·61
to	Dir.	335°	322°	349°	341°	340°	338°	333°	Mean 338°
S.	Hours .	46	20	59	37	165	86	102	Sum 515
Maximum .		26'	31'	31'	38'	43'	54'	40'·5	
Hours > 25' .		3	9	5	21	39	60	45	Sum 183
Hours of 0 . .		8	47	27	2	79	8	13	Sum 184

TABLE I.—November.

		1857	1858	1859	1860	1861	1862	1863	
S.	Vel	7'·46	6' 33	12·81	12' 20	17'·47	8' 73	13' 41	Mean 11' 25
to	Dir	19°	27	24°	23°	33°	24°	21°	Mean 24°
S.W.	Hours	200	92	167	83	289	302	242	Sum 1375
S.W	Vel.	5' 03	5 49	9 22	7 52	14'·24	9' 12	12' 16	Mean 8'·97
to	Dir	72°	65°	66°	66°	68°	60°	65°	Mean 66°
W	Hours	35	63	140	79	154	122	294	Sum 887
W.	Vel.	4' 20	7' 49	5·91	5' 81	4'·25	3' 62	9·05	Mean 5' 73
to	Dir.	102°	96°	115°	106°	109°	111°	109°	Mean 107°
N.W.	Hours	41	63	87	11	44	35	81	Sum 362
N W	Vel	3 56	2 32	4 69	4' 15	4 77	1' 87	11' 66	Mean 4' 72
to	Dir	143	151°	153°	154	159°	159°	157°	Mean 154°
N	Hours	38	73	42	2	119	16	9	Sum 299
N	Vel	4' 46	6 32	5 44	7 84	3 54	3 25	8 52	Mean 5 62
to	Dir	199	215	212	214°	187	204°	202°	Mean 205°
N E	Hours	48	28	38	87	46	31	21	Sum 299
N E	Vel	6 26	8 45	4 38	8 78	11 25	4 48	13 22	Mean 8 12
to	Dir	257	248	255	246°	242°	246°	258°	Mean 250°
E.	Hours	107	220	31	231	4	27	9	Sum 629
E	Vel.	6 48	8 48	12 31	10 93	9·66	6 51	5 20	Mean 8' 48
to	Dir	299	283	277°	292	288	293°	294	Mean 289
S E	Hours	87	131	79	130	6	47	10	Sum 490
S.E	Vel	6 86	15 05	14 41	7'·93	6 04	10 48	13 58	Mean 10 62
to	Dir	329	339°	337°	344	335°	342°	340	Mean 338°
S	Hours	44	39	99	46	23	127	51	Sum 429
Maximum		26'	36'	37	29'	75'	37	34	
Hours > 25'		6	9	19	5	90	19	30	Sum 178
Hours of 0		43	80	37	9	13	54	1	Sum 237

TABLE I.—December.

		1857	1858	1859	1860	1861	1862	1863	
S	Vel	26 11	16 68	13 35	6 78	18' 18	21 98	16 52	Mean 17' 08
to	Dir	25°	26	28	15	20	27	37°	Mean 25°
S W	Hours	210	230	258	116	121	254	153	Sum 1342
S W	Vel	11 04	11 91	8 41	3 06	8 84	14 13	14 40	Mean 10' 20
to	Dir	69	69	64	60°	64	62	62°	Mean 64°
W.	Hours	175	233	167	78	182	177	335	Sum 1367
W	Vel	35 70	10 98	2 92	2 20	5 44	16 33	12 71	Mean 12 21
to	Dir	95	104	113	114	107°	109°	105°	Mean 107°
N W	Hours	12	56	26	128	43	90	135	Sum 490
N W	Vel	0	2 66	3 38	3' 20	4 04	13' 96	8 59	Mean 5 09
to	Dir		151°	157°	159	154°	154°	155	Mean 155°
N	Hours		3	116	38	43	62	32	Sum 294
N	Vel	0	0	2·11	2 37	4' 38	0	2 00	Mean 1 55
to	Dir			203	223	212'		212°	Mean 212°
N E	Hours			36	59	31		1	Sum 127
N E	Vel	12' 04	0'	4 29	5 40	3 67	12' 55	3' 50	Mean 5 80
to	Dir	254		251°	255°	258	261°	242°	Mean 254°
E	Hours	3		17	62	56	18	2	Sum 158
E	Vel	0'	17 20	14 97	10 15	6 86	14' 26	12 63	Mean 10 64
to	Dir		303	295°	294	294°	290°	295°	Mean 295°
S E	Hours		5	37	109	61	64	36	Sum 312
S E	Vel	22 26	21 95	14' 54	12' 87	14 24	21' 68	18' 96	Mean 18' 07
to	Dir	312°	332°	334	326°	344°	340°	343°	Mean 337°
S	Hours	82	205	84	127	177	57	28	Sum 760
Maximum		42	48'	45'	38 8	40'	45'	50'	..
Hours > 25'		81	102	40	15	56	106	94	Sum 494
Hours of 0		1	2	60	56	41	0	0	Sum 160

The first thing which strikes one in this Table is the irregularity of the wind. It varies in each octant, in each octant it varies with the month, and in each octant and month it varies with the year. As to the first of these variations, both the velocity of the wind and the number of hours during which it blows are, in general, a maximum in the first octant (S to S.W.), they decrease from this to a minimum at octants N to N.E., and increase to octant 1. The products of the velocity and time at the maximum and minimum are as 6 : 1 The predominance of south-westerly winds is what might be expected from the combination of an equatorial current with the earth's rotation; but it is not obvious why it is not absolute Probably much of the change of direction arises from circumstances local to the place of observation. For instance, the direction of the west coast of Ireland, which runs nearly N and S, may occasionally turn the S W currents northward, and the mountainous ground of Antrim may divert it here towards the east. It must also be remembered that our anemographs give only measure of the wind at the earth's surface, where it is at once retarded and thrown into gigantic eddies and vortices by the effects of friction

The experience of aeronauts shows that at a few thousand feet elevation the velocity is often far greater than it is below, and that the direction is much more uniform But I do not see how this error is to be remedied The summit of a mountain is not exempt from it, and though a small and lofty island, like St Kilda, far from any extensive land, would be better, yet even here the friction of the sea's surface will destroy velocity It is possible that an anemograph at the top of a tall and slender "*stack*" would give a much larger velocity than one at its base, the record could be easily effected below by telegraphy We must remember that a current of air comports itself like one of water, and shall be assisted in comprehending the nature of a gale by watching the irregular movements of a river in flood There must also be eddies in a vertical plane On the action of these see a valuable paper by Prof HENNESSEY in Phil Trans 1860 An anemograph for vertical currents might be made by a set of windmill-vanes placed horizontal.

Secondly, in *each* octant the amount of wind varies with the month It is a maximum in January, decreases from this to July, the ratio being $2\frac{1}{4}$: 1. From this it increases to the end of the year There is an exception to this in March, where the daily amount is greater than in February in the ratio of 1.13 : 1 This might seem to countenance the vulgar notion of stormy weather prevailing near the equinoxes, but there is no such excess in September above October, and in March, though the yearly maxima are higher than in February, yet the number of hours when the velocity exceeds 25 miles is considerably less. This monthly change is an obvious consequence of the change of the sun's declination, for the zone where the easterly winds of low latitudes confine with the westerly ones of more northern regions must shift with that to which the sun is vertical

For the third of these irregularities, that which prevails from year to year, there can, in the present state of our knowledge, be no certain cause assigned It will be seen that in the same octants the variation is very different in each month, and that the

maxima in each octant do not belong to the same years; while the amount of discordance is so great as to almost exclude the idea of any law. I looked for one in the direction already noticed In 1860 the sun-spots were at a maximum, in 1856 at a minimum, and if they exert any influence it must have been considerably less in 1857 than in 1860 The products of velocity and time were accordingly examined in these years, and that for 1860 is 4167 greater than in the other But this result is reversed by 1863, which exceeds 1860 by a still greater amount, 5223, and evidently many decennial periods must be examined before any reliable conclusion can be attained as to this influence

The same lawless irregularity may be observed in the maximum velocities of separate years. The highest in the period before us is 71 miles in November 1861, the lowest 19, in July 1860. Far higher velocities than these are sometimes attained, but only for a few minutes It holds also as to the number of hours when the velocity exceeds twenty-five miles As instances in January 1863 this number is 146, in 1857 it is 23, in April 1859 it is 40, in 1861 it is 0, in November 1861 it is 90, in 1860 it is 5. It occurs also, though not intensely, in the hours of calm It may have some interest to give the mean velocity for each month irrespective of the direction.

TABLE II

Month	Velocity	Total miles
January	13 51	70336
February	12·82	60422
March	13·00	67691
April	11·62	51587
May	7 78	39664
June	4·24	35353
July	6·59	34343
August	7·29	35986
September	8·02	39513
October	9·12	45568
November	9 97	47671
December	12·98	166498

Here also there seems little indication of equinoctial gales March is a trifle more windy than February, but September less so than October The yearly sums also do not show any special relation to the solar spots, the total in 1857=79865; in 1860=73067, but in 1863=95583 The total miles in the seven years=590672, and the mean velocity during that time is 9 729.

11 The most obvious way of dealing with the west and south components of V is to derive from them interpolation formulæ for each year involving periodic functions of the time, and deduce from the coefficients of these formulæ in successive years some general law This, however, seems impracticable, for the components differ so widely in successive years as to preclude any hope of reconciling them As a specimen of this discordance I give the values for the first hour of the series for January 1:—

1857 .	W=12·59	S= 9·81
1858 .	−6·27	22·14
1859 . .	7·78	6·31
1860 . . .	6·80	20·91
1861 . .	−3·91	−1·21
1862 . . .	1·29	−2·70
1863 . .	14·62	22·65

It is evident that here there is no regular succession, and equally so that little dependence can be put on even the mean of the seven as representing the hour 0 for that day. But if, as is probable, these discordances are casual, we may expect they will disappear from the mean of a large number of observations—how large may be estimated from the Probable Error of these observations, though, on account of the magnitude of their discordances, this cannot be determined with great precision. There is also this difficulty in the process of finding the Probable Error, that the coordinates undergo daily and monthly variations, which must not be confounded with the casual errors. It is therefore necessary to confine ourselves to the observations of *each individual* hour during the seven years, and combine any number of these groups of seven. This is effected by the simple means of using as the divisor $n-m$ instead of $n-1$, n being the number of terms in the entire set, and m the number of groups. I have only thought it necessary to make the computation for W in January and June, and I find

P E of a single observation .	±5·901	±3·913
P E of mean of seven	±2·230	±1·479
P E of W in Table, mean of 217	±0·401	±0·266
P E of mean of month . .	±0·082	±0·054

The discordancy in summer is only two thirds of that in winter, and in both is so great that the mean of seven is not to be relied on, and even the numbers of Table III are not sufficiently certain. Perhaps these seven years may have been exceptionally irregular. The discordancy of S is still greater than that of W. Evidently single hours were out of the question, I therefore took for each hour the mean of the month in the first instance, I then grouped these means for every ten days, but ultimately adopted the entire month as the group.

Before discussing these means individually, it may be useful to give their means for the entire period of seven years. Supposing the winter from October to March inclusive, the summer from April to September, the day hours from 7 A.M. to 6 P.M., the night from 7 P.M. to 6 A.M., we find —

Winter Day.

Sum W=7899^{m}·315, Sum S=11239·92, Ann. Translation=13738, D=35° 6′

Winter Night.

Sum W=7264^{m}·43, Sum S=11527·75, Ann. Translation=13812; D=33° 25′.

Summer Day.

Sum W=3519^m·23, Sum S=5454·50, Ann. Translation=6491; D=32° 50′.

Summer Night.

Sum W=2831^m·43; Sum S=5081·65, Ann Translation=5817, D=29° 8′.

Both components are more than twice as great in winter as in summer, the day components are greater than the night ones, except the winter S.

The sums of all are Sum W=21514^m·40, Sum S=33303^m; Ann Trans.=39648^m; D=32° 51′

On examining the records of the components, I find that 630 hours were missed by various accidents, so that the total number of hours is 60714, and the above sums, $\times 7^y \div 60714^h$, will give for the mean hourly values $\overset{1}{W}=2^m\cdot 4805$, $\overset{1}{S}=3^m\cdot 8398$; $\overset{1}{V}=4^m\cdot 5713$, $\overset{1}{D}=32° 54′ 44″$. The value of $\overset{1}{V}$ shows that the wind in the first quadrant is nearly half the total amount

The monthly means of the components are given in the following Table (p 415)

On examining this Table we observe, *First*, that all the values both of W and S are positive, in other words, that in a considerable number of observations the aerial currents from west and south have at this station a decided predominance over all the others *Secondly*, that, as was anticipated, however discordant the results of individual hours or days may be, yet the means of from 196 to 217 present a notable agreement, and the differences which they exhibit are evidently subject to law If we look down the vertical columns (which give approximate values for each hour of the middle day of each month) we find in each a decided maximum and minimum, and another, or even more than one of each, less in amount The hours of these phases vary with the months, that of the principal maximum occurs in the winter months from noon to 3 P.M. for W, in the summer from 9 A.M. to noon, for S it varies less, being a little before noon

The principal minimum occurs in the evening, from 6 P.M. to 10 P.M., both for W and S The extreme diurnal ranges are greatest in March, being for W 2^m·14, for S 2^m·40, they are least in November, being 0·74 and 0·79

It deserves notice that during the winter months the horary values of W for the four afternoon hours exceed those for the four that precede, the sum of the differences being 9^m·95 In the summer months the reverse is the case, but the − differences are only 7^m·02.

Does this arise from the great extent of land to the east of Ireland as contrasted with the ocean to its west, and the greater evaporation from the latter in summer?

If we examine the horizontal columns (which show the monthly variations) the dominion of Law is still more manifest W has a maximum in January, a minimum in February, the greatest maximum is in March, the least minimum in April: these abrupt changes are remarkable; but it is possible that the great value of W in March is abnormal, and may not occur in subsequent years. It then increases with a slight

TABLE III

Hours.	January	February	March	April	May	June	July	August	September	October	November	December	Hours.
0	W 4 563 S 6 70	W 2 76 S 5 97	W 4 57 S 3·09	W 0 24 S 2 69	W 0·90 S 2 21	W 0 85 S 1 87	W 1 680 S 1 442	W 1 95 S 2 695	W 2 11 S 3 67	W 1 87 S 3 86	W 1 70 S 3 97	W 4·08 S 8·05	0
1	W 4 73 S 6 87	W 3 01 S 5 87	W 4 55 S 3 49	W 0·26 S 3 11	W 0 83 S 2 22	W 0 93 S 1 81	W 1 669 S 1 70	W 1 82 S 2 70	W 2 39 S 3 69	W 1·65 S 4·09	W 2·90 S 4·01	W 4·00 S 7 52	1
2	W 4 59 S 6 89	W 2 81 S 5 84	W 4 81 S 3 61	W 0 40 S 2 865	W 1·06 S 2 22	W 0 91 S 1 93	W 1 912 S 1 581	W 2 00 S 2 66	W 2 23 S 3 48	W 1 85 S 4 12	W 1 81 S 3 76	W 3 99 S 7 615	2
3	W 4 54 S 7 18	W 2 77 S 5·94	W 4 90 S 3 62	W 0 60 S 2 88	W 1 13 S 2 18	W 1·06 S 1 92	W 1 565 S 1 72	W 2 12 S 2 71	W 2 44 S 3 52	W 2·015 S 4 23	W 1 70 S 3 23	W 4 30 S 7 90	3
4	W 4 72 S 7 07	W 2 76 S 6·00	W 4 99 S 3 65	W 0 41 S 2 7	W 1 20 S 2 18	W 1 08 S 2 01	W 1 80 S 1 73	W 2 14 S 2 67	W 2 39 S 3 45	W 2 245 S 4 35	W 1·835 S 3 63	W 4 24 S 8 285	4
5	W 4 47 S 6 66	W 2 59 S 5 88	W 5 00 S 3 43	W 0·47 S 2 79	W 1 12 S 2 285	W 0·99 S 2·01	W 2 18 S 1 81	W 1 93 S 2 90	W 2 55 S 3 53	W 2 13 S 4 36	W 1 84 S 4 11	W 4 32 S 7 78	5
6	W 4 36 S 6 91	W 2 65 S 5 72	W 4 90 S 3 59	W 0·48 S 2 76	W 1 10 S 2 13	W 1 04 S 1 70	W 2 14 S 1 88	W 2 20 S 2 95	W 2 58 S 3 44	W 2 05 S 3 95	W 1 665 S 4 29	W 4 32 S 7 91	6
7	W 3 71 S 6 25	W 2 30 S 6 00	W 4 94 S 3 75	W 0 41 S 2 75	W 0 92 S 2 49	W 1 01 S 1 95	W 2 30 S 1 635	W 2 25 S 2 86	W 2 66 S 3 56	W 1 86 S 3 80	W 1 70 S 4 02	W 4 16 S 7 48	7
8	W 3 88 S 6 53	W 2 52 S 5 94	W 5 17 S 3 73	W 0·70 S 2 94	W 0 79 S 2 30	W 1 02 S 1 88	W 2 65 S 1 63	W 2 25 S 2 72	W 2 565 S 3 79	W 1 70 S 3 42	W 1 72 S 3 97	W 4 00 S 7 44	8
9	W 4 62 S 6 91	W 2 59 S 6 19	W 5 73 S 3 48	W 0 42 S 3·01	W 0 70 S 2 81	W 0 77 S 2 05	W 3 405 S 1 63	W 2 72 S 2 63	W 2 72 S 4 20	W 2 21 S 3 67	W 1 91 S 4 08	W 4 31 S 7 84	9
10	W 4 27 S 7 33	W 3 48 S 6 69	W 6 20 S 3 76	W 0 39 S 3 075	W 0·92 S 2 75	W 1 21 S 2 30	W 3 14 S 1 63	W 3 34 S 3 11	W 2 29 S 4 03	W 2 47 S 3 76	W 1 76 S 4·21	W 3 55 S 7 28	10
11	W 4 51 S 7 22	W 3 94 S 6 53	W 5·05 S 3 62	W 0 52 S 2 26	W 0 55 S 2 57	W 1 05 S 1 99	W 2 50 S 1 88	W 2 98 S 3 19	W 2·29 S 4 32	W 2 52 S 3 97	W 1 495 S 4·27	W 3·92 S 8 14	11
12	W 4 78 S 7 46	W 3 75 S 6 93	W 5 37 S 3 34	W 0 51 S 2 60	W 0 01 S 2 33	W 0 93 S 2 24	W 2 575 S 1 89	W 3 07 S 3 09	W 2 30 S 4 01	W 2 52 S 3 76	W 1 80 S 3 73	W 4 18 S 7 76	12
13	W 4 88 S 7 685	W 4 11 S 6 64	W 5 63 S 2 89	W 0 85 S 2 72	W 0·30 S 2 52	W 1 12 S 2 12	W 2 58 S 1 84	W 3 10 S 2 90	W 2 59 S 3 79	W 2 63 S 3 61	W 2 01 S 4·05	W 4 01 S 7 66	13
14	W 4 91 S 7 02	W 3 87 S 6 11	W 5 62 S 2 62	W 0 87 S 2 67	W 0 10 S 2 32	W 0 82 S 1 97	W 2 59 S 1 63	W 3 15 S 2 85	W 2 645 S 3 83	W 2 42 S 3·67	W 2 16 S 4 12	W 4 34 S 7 44	14
15	W 5 09 S 7 14	W 3 28 S 6 24	W 6 56 S 2 095	W 0 785 S 2 3·5	W 0 00 S 2 11	W 0 56 S 1 74	W 2 42 S 1 50	W 3 06 S 2 52	W 2 37 S 3 59	W 2 44 S 3 29	W 1 90 S 3 86	W 4 17 S 7 36	15
16	W 4 62 S 6 81	W 3 28 S 5 65	W 6 22 S 1 94	W 0 64 S 1 88	W 0 13 S 2 345	W 0 51 S 1 86	W 2 29 S 1 32	W 2 80 S 2 14	W 2 295 S 3 37	W 2 20 S 3 24	W 1 86 S 3 84	W 4·07 S 7 17	16
17	W 4 39 S 6 87	W 2 38 S 5 27	W 5 85 S 1 79	W 0·76 S 1 65	W 0 39 S 2 15	W 0 53 S 1 32	W 2 37 S 1 43	W 2 35 S 2 235	W 2 12 S 3 07	W 1 83 S 3 47	W 1 58 S 3 72	W 4 11 S 7 29	17
18	W 4 39 S 7 14	W 2 34 S 5 31	W 5 14 S 1 80	W 0 48 S 1 41	W 0 05 S 1 91	W 0 56 S 1 17	W 1 99 S 1 24	W 2 205 S 2 01	W 1 89 S 3 45	W 1 90 S 3 40	W 1 48 S 3 79	W 3 99 S 7 33	18
19	W 4 30 S 6 97	W 2 53 S 5 65	W 4 72 S 2 19	W 0·29 S 1 71	W 0 12 S 1 56	W 0 58 S 1 08	W 1 56 S 0 82	W 1 87 S 1 91	W 1 59 S 3 34	W 1 75 S 3·60	W 1 57 S 4·03	W 4 06 S 7 64	19
20	W 4 62 S 7 44	W 2 39 S 5 53	W 4 48 S 2 51	W 0 04 S 2 03	W 0 20 S 1 62	W 0 73 S 0 89	W 1 38 S 0 85	W 1·64 S 2 03	W 1 75 S 3 68	W 1 885 S 3 835	W 1 42 S 3 95	W 3·86 S 7 83	20
21	W 4 69 S 6 88	W 2 58 S 5 93	W 4 42 S 2 41	W 0 06 S 1 91	W 0·24 S 1 73	W 0 59 S 1 26	W 1 29 S 1 265	W 1 88 S 2 26	W 1 71 S 1 01	W 1 95 S 4 10	W 1 73 S 4 40	W 4·01 S 8 10	21
22	W 5 10 S 6 79	W 2 68 S 6 12	W 4 50 S 2 53	W 0 17 S 2 24	W 0 495 S 1 91	W 0 65 S 1 315	W 1 54 S 1 405	W 1 995 S 2 37	W 1 65 S 4 05	W 2 01 S 4 39	W 1 77 S 4 00	W 3 96 S 8 095	22
23	W 4 81 S 6 99	W 2 34 S 6 46	W 4 65 S 3 20	W 0 34 S 2 24	W 0 81 S 2 12	W 0 685 S 1 58	W 1 625 S 1 29	W 2 00 S 2 72	W 1 93 S 4 12	W 2 04 S 4 07	W 1 69 S 4 11	W 3 77 S 7 80	23

maximum in August and a slight minimum in November. The variation is greater here than in the horary columns, being for hour $15=6^{m}\cdot56$. The largest W is at March $15^{h}=6^{m}\cdot56$, the least at May $15^{h}=0\cdot00$.

The law of S is simpler, it has one maximum in December and one minimum in July; its range, too, is something greater, being in hour $20=6^{m}\cdot98$. The greatest magnitude $=8^{m}\,285$ at December 4^{h}, its least $=0^{m}\cdot82$, July 19^{h}. There is a general agreement in the change of the two components, with one striking exception, the maximum and minimum which W has in March and April Such a general agreement might be expected, for any air coming from the south must have a westward motion due to the greater velocity of the earth's rotation in a southern parallel. This anomaly, if real, may be caused by the geographical conditions to which I have already alluded. To them also may be referred the fact that at May 15^{h} $W=0$, though $S=2^{m}\,41$, from which a sensible magnitude of the other might be expected. It must, however, be observed that some of the changes exhibited in this Table can scarcely be regarded as periodical. I have already pointed out that from the very great discordance of individual observations it is evident that a much greater number of them than is afforded by a period of seven years is required to eliminate the barometric and hygrometric influences Yet these disturbances might be expected to be distributed with some uniformity through the day, while the changes from hour to hour are sometimes considerable Thus in February 9^{h} to 10^{h} $\Delta W=0\cdot89$, 16^{h} to 17^{h} $\Delta W=-0\,90$, March 10^{h} to 11^{h} $\Delta W=-1\,15$, 14^{h} to 15^{h} $\Delta W=0\,94$, April 10^{h} to 11^{h} $\Delta S=-0\,71$, December 10^{h} to 11^{h} $\Delta S=0\,86$. These are the largest, and it deserves notice that they occur in winter months, in summer there is much less abruptness of change.

It occurred to me that some of these irregularities might be due to errors in the records of velocity, but this seems quite improbable. Such errors could only arise from three possible causes

1 Referring to my description of the anemograph in the 'Transactions of the Royal Irish Academy,' vol xxii, it will be obvious that the track of the recording pencil might be excentric to the brass disk which carries the paper. It was carefully adjusted whenever the clock was cleaned, but was liable to derangement from rough handling The error which would thus arise was avoided by an easy adjustment, which made the edge of the reading alidad coincide with the right line drawn by the pencil when the clock was wound up. It will easily appear that the readings so made are true

2 The paper may be excentric to the centre of rotation Let e be its excentricity, ϵ that of the pencil, θ the reading of any distance from the winding line, ψ the angle between that line and the line of the two centres, the correction for θ

$$=\frac{-\epsilon(\sin\psi-1)-(e\sin\psi(-\sin\psi-\theta))}{}$$

and calling V the change of θ in the following hour,

$$\text{correction of } V=\frac{2e}{r}\cos(\psi-\theta-\tfrac{1}{2}V)\times\sin\tfrac{1}{2}V,$$

supposing $e=0\cdot05$ (and such an error is not probable) the maximum error would be $0^{m}\cdot27$. This, therefore, cannot do much harm.

3. A much more serious error may be caused by the rate of the clock which moves the pencils of the instrument. Suppose it a gaining one, the hour-circles on the paper are less than hours, and the recorded Vs belong, not to the times to which they are ascribed, but to periods a little in advance. The error is negligible, except for the hour of winding-up. There the space-curve goes beyond the last hour-circle to a distance equal to the rate in 24^{h}, and the measured V is proportionally too large. If the velocity were uniform, this would be corrected by multiplying $\underset{n}{V'}$ by $\frac{H+x}{H+nx}$, where H is the hour-space, x its hourly increase, but as this seldom is the case, the change must be allowed for by interpolation. In all cases but the last we thus obtain

$$\underset{n}{V}=\frac{H+x}{H}\left\{\underset{n}{V'}+\left(\underset{n+1}{V'}-\underset{n}{V'}\right)\left(\frac{2n-1}{2H^{2}}\right)x\right\}.$$

As I never have found $\frac{x}{H}$ greater than $\frac{1}{45}$, the second term may be neglected, and the coefficient scarcely differs from unity. In the last we have

$$\underset{n}{V}=\frac{H+x}{2H+nx}\left\{\underset{n}{V'}\times\frac{2H+x}{H+nx}+\underset{n+1}{V'}\times\frac{(n-1)x}{H}\right\},$$

which may be considerably less than $\underset{n}{\overset{\prime}{V}}$. The projection of the space-curve beyond the last hour-circle gives $24x$. This excess occurred most frequently in gales from S.W., and was, I think, often caused by the vibration of the lofty structure which supports the instrument. I have not applied these corrections except in a few cases when the error was glaring. The winding-hour was at 9 A.M. in 1857 and 1858, at 10 A.M. in the other years, and at these hours this influence might be expected, but on comparing their values in Table III. with the formulæ of Tables V. and VI., they seem as well represented as any of the others*.

The discordances of these quantities would have been less striking had they been grouped as three-hourly means, and this was my original intention, which I abandoned on account of a difficulty in respect of interpolation to which I will refer presently.

It is, however, necessary to remark that the numbers of Table III. are merely *probable* values. A sensible proportion of the individual values is invariably negative for each hour, and my first idea was to keep the positive and *negative* means separate. I tried it for January and June as extreme cases, and came to the conclusion that this separation would be useless. The negative values occur so constantly, that they can

* I have given these details as they will be useful in case it be ever thought desirable to reduce the entire series of their anemograms, which extends from 1847 to 1870.

scarcely be deemed casual In the 744 septimates of January there are only seven in which all the W and S are positive, in the 720 of June there are none.

It might be expected, from the mechanism of the polar and equatorial currents, that both components would change signs simultaneously; but it is not so. I find that the proportion of the combinations is —

In January .	$\begin{matrix}W+\\S+\end{matrix}$ 0 589,	$\begin{matrix}W+\\S-\end{matrix}$ 0 141,	$\begin{matrix}W-\\S+\end{matrix}$ 0 184;	$\begin{matrix}W-\\S-\end{matrix}$ 0 086.
In June . .	„ 0 406,	„ 0 220,	„ 0 198,	„ 0 176.

The combination of +W with −S may arise from the influence of a continent to the east of Ireland, and that of −W and +S from a north-east current whose north component has been destroyed by friction; but I looked for a greater frequency of −W and −S If we confine ourselves to consider +W, −W, +S, and −S separately, we find —

For January	Sum (+W)=32121,	Sum (−W)=−8315;
	Sum (+S) =42733,	Sum (−S) =−6372.
For June . .	Sum (+W)=13298,	Sum (−W)=−9002,
	Sum (+S) =15978,	Sum (−S) =−7163

The amount of negative components does not differ very much in the two months, but that of the positive is nearly triple in January what it is in June. Were we to attempt to develop separately these + and − values, we should be embarrassed by the different numbers of them belonging to each hour. Thus in January the number for −W is 47 at 2^h, 69 at 11^h, for −S is 41 at 2^h, 56 at 1^h. In June, for −W it is 63 at 3^h, 91 at 15^h, for −S it is 65 at 2^h, 98 at 9^h Supposing them developed in terms of the time, we should still be unable to obtain any *absolute* values of the components at a given epoch unless we knew the causes which produce these negative values and the laws of their action It is evident that the equatorial current predominates here, but that there coexists with it a polar one, probably above, possibly collateral, which is occasionally mixed with the other by some disturbing force—probably barometric. It seems also that the monthly variation of the components is in a great measure limited to the positive values. For these reasons I have confined myself to the simple means of the entire set But I think it might be well, in a series extending to several periods of five or seven years, to keep them so far separate as to be able to examine whether the occurrence of the negative values has any relation to time.

A Table like this, whose data refer to dates separated by considerable intervals, will not suffice to give the components generally without some process of interpolation; and we proceed to consider this. The form universally adopted where the quantities con-

cerned are periodic functions of the time-angle is that given by BESSEL, in which, calling the quantity u and the angle θ, we have

$$u = K + A\cos\theta + B\cos 2\theta + C\cos 3\theta + D\cos 4\theta + \&c.$$
$$+ O\sin\theta + P\sin 2\theta + R\sin 3\theta + S\sin 4\theta + \&c$$

But as the monthly variations must be represented as well as the horary, a formula of this nature including two variables would be very complicated; and it seems best to obtain, as proposed by BESSEL, the horary formula for each month, and to regard the constants of this formula as themselves periodic functions of the monthly time, and develop them in similar formulas of the month-angle, ϕ. Stopping at terms of the fourth order, we should have nine of these for each component; and for a given day of the year and hour of the day we must compute the constants for the ϕ of the day, and multiply each of the last eight by the cosine or sign of the corresponding multiple of θ The calculation of the horary constants is shortened by observing that for the angles θ, $180+\theta$, $180-\theta$, and $360-\theta$ the sines and cosines have the same numerical value, and hence the calculation need only be made for the first quadrant

Supposing the circle divided into $2n$ equal parts, and that θ contains m of these, the u corresponding to any θ may be characterized as u_m, that corresponding to $\theta+180$ as u_{n+m}, and the sum or difference of these two as s_m, d_m

As the cosines and sines of odd multiples of θ and $180+\theta$ differ in sign, but those of even multiples agree, the expressions of A, O, C, and R will contain only d, those of the others only s The signs of s_m and d_m are easily determined in each case. Thus for the first multiples of θ the cosine and sine are + for m through the entire quadrant, they are − and + for $n-m$ For the second multiples the sine is + through the quadrant, the cosine is + up to 45°, − through the rest, for $n-m$ the cosine is the same as for m, the sine different I take, as in the first instance, the horary division in which $n=12$, and BESSEL's formulæ become

$$K = \tfrac{1}{24}\left\{s_0 + s_1 + s_2 \quad . + s_{11}\right\},$$

$$A = \tfrac{1}{12}\left\{d_0 + \left(d_1 - d_{11}\right)\cos 15^\circ + \left(d_2 - d_{10}\right)\cos 30^\circ + \left(d_3 - d_9\right)\cos 45^\circ + \tfrac{1}{2}\left(d_4 - d_8\right) + \left(d_5 - d_7\right)\cos 75^\circ\right\},$$

$$B = \tfrac{1}{12}\left\{s_0 - s_6 + \tfrac{1}{2}\left(s_2 + s_{10} - s_4 - s_8\right) + \left(s_1 + s_{11} - s_5 - s_7\right)\cos 30^\circ\right\},$$

$$C = \tfrac{1}{12}\left\{d_0 - \tfrac{1}{2}\left(d_4 - d_8\right) + \left[d_1 - d_{11} - \left(d_3 - d_9 + d_5 - d_7\right)\right]\sin 45^\circ\right\},$$

$$D = \tfrac{1}{12}\left\{s_0 + s_6 + \tfrac{1}{2}\left(s_1 + s_{11}\right) - \tfrac{1}{2}\left(s_2 + s_{10}\right) - \left(s_3 + s_9\right) - \tfrac{1}{2}\left(s_4 + s_8\right) + \tfrac{1}{2}\left(s_5 + s_7\right)\right\},$$

$$O = \tfrac{1}{12}\left\{\left(d_1 + d_{11}\right)\sin 15^\circ + \tfrac{1}{2}\left(d_2 + d_{10}\right) + \left(d_3 + d_9\right)\sin 45^\circ + \left(d_4 + d_8\right)\sin 60^\circ + \left(d_5 + d_7\right)\sin 75^\circ + d_6\right\},$$

$$P = \tfrac{1}{12}\left\{\tfrac{1}{2}\left(s_1 - s_{11} + s_5 - s_7\right) + s_3 - s_9 + \left(s_2 - s_{10} + s_4 - s_8\right)\cos 30^\circ\right\},$$

$$R=\tfrac{1}{12}\left\{d_{2}+d_{10}-d_{6}+\left(d_{1}+d_{11}+d_{3}+d_{9}-d_{5}-d_{7}\right)\cos 45^\circ\right\},$$

$$S=\tfrac{1}{12}\left\{s_{1}-s_{11}+s_{2}-s_{10}-\left(s_{4}-s_{8}+s_{5}-s_{7}\right)\cos 30^\circ\right\},$$

and so on.

These are all combinations of the groups $s_{m}\pm s_{n-m}$, $d_{m}\pm d_{n-m}$, and by forming these groups the computation is evidently much simplified.

This simplification is, however, only possible when n is an integer, and a the first arc of the series$=\frac{\pi}{2n}$ or$=0$.

Whatever be the value of a, BESSEL'S formula fails generally to give G and U the cosine- and sine-coefficients of the nth order The θ_{m} correspond to $u_{m}=a+(m-1)\frac{\pi}{n}$, and this for the order p becomes $pa+p(m-1)\frac{\pi}{n}$ Then cosine $\theta_{m}=\cos(na)$, sin $\theta_{m}=$sine na, both + for odd values of m, − for even ones Thence the nth coefficient—

$$\begin{aligned} u_{1}\cos na &= \ \ K\cos na+\&c+G\cos^2 na+U\sin na\cos na, \\ -u_{2}\cos na &= -K\cos na-\&c.+G\cos^2 na+U\sin na\cos na \end{aligned}$$

Then summing from $m=1$ to $m=2n$, we get

$$\cos na\, S\left(u_{1}-u_{2}\right)=(\cos^2 na\, G+\sin na\cos na\, U)\times 2n,$$

$$S\left(u_{1}-u_{2}\right)=2n\,(G\cos na+U\sin na).$$

Here the divisor of S $\left(u_{1}-u_{2}\right)$ is $2n$ instead of n, and these coefficients cannot be obtained separately unless $a=0$ or $\frac{\pi}{2n}$, in which case the cosine or sine $=0$.

How far the series is to be continued depends on the periodic fluctuations of the us, and may be found by trial, or by BESSEL'S expression for the squares of the residual errors In any case it should not be carried further than the order $\frac{\pi}{2a}$, as after that the coefficients coalesce. BESSEL has shown this for $a=0$, and it can easily be proved to hold good when a is a submultiple of $\frac{\pi}{2}$ and b a multiple of a

For the horary groups I find the fourth order sufficient. These horary groups might be combined in triple sets, but, as I have said, there is a difficulty in the interpolation due to the fact that while u'_{θ}, the mean of any three, is multiplied by cosine or sine of θ, the first and third components of it should be multiplied by the same functions of $\theta-b$ and $\theta+b$ This, however, may easily be corrected. Take the case of A. the effect of three components to determine this is —

$$\underset{-b}{u}\times\cos(\theta-b)+\underset{\theta}{u}+\underset{\theta+b}{u}\times\cos(\theta+b)=\underset{\theta-b}{u}\cos\theta\cos b+\underset{\theta}{u}+\underset{\theta+b}{u}\cos\theta\cos b-\left(\underset{\theta+b}{u}-\underset{\theta-b}{u}\right)\sin\theta\sin b$$

$$=\left(\underset{\theta-b}{u}+\underset{\theta}{u}+\underset{\theta+b}{u}\right)\cos\theta-\left(\underset{\theta+b}{u}+\underset{\theta-b}{u}\right)\cos\theta\text{ versine } b-\left(\underset{\theta+b}{u}-\underset{\theta-b}{u}\right)\sin\theta\sin b.$$

Developing the sum and difference of the *u*s, which gives

$$\underset{\theta+b}{u}+\underset{\theta-b}{u}=2K+2A\cos\theta\cos b+2O\sin\theta\cos b+2B\cos 2\theta\cos 2b+\&c.,$$

$$\underset{\theta+b}{u}-\underset{\theta-b}{u}=-2A\sin\theta\sin b+2O\cos\theta\sin b-2B\sin 2\theta\sin 2b\ \&c.,$$

we obtain the term

$$=3\acute{u}\cos\theta-2\text{ versine } b\{K\cos\theta+A\cos^2\theta+O\sin\theta\cos\theta+\&c.\}$$
$$+2\sin^2 b\{A\sin^2\theta+O\sin\theta\cos\theta+\&c.\}.$$

Summing round the circle, calling $Su'\cos\theta=F$, and remembering that all except $S\cos^2\theta$ and $S\sin^2\theta$ vanish, that each of these $=4$, and $12A=3F$ in ordinary cases, we have

$$12A=3F-8A\cos\text{ versine } b+8A\sin^2 b,$$

and ultimately

$$A\times 4\left(1-\tfrac{2}{3}\text{ versine } b\right)=F.$$

O is given by the same formula, changing the cosines for sines in F. For higher orders, P, it is only necessary to use $p\theta$ and pb. In the case of D, however, the formula must be modified; for in this instance $S\cos^2=8$, $S\sin^2=0$, and the expression is $D\left(4+\frac{16}{3}\cos\text{ versine } 4b\right)=F$. The values of the constants are:—

$$A\ (3\cdot9091)=\underset{0}{d}+\left(\underset{3}{d}-\underset{9}{d}\right)\sin 45^\circ.$$

$$B\ (3\cdot6428)=\underset{0}{s}-\underset{6}{s}.$$

$$C\ (3\ 2190)=\underset{0}{d}-\left(\underset{3}{d}-\underset{9}{d}\right)\sin 45^\circ$$

$$D\ (5\cdot333)=\underset{0}{s}+\underset{6}{s}-\left(\underset{3}{s}+\underset{9}{s}\right).$$

$$O\ (3\ 9091)=\underset{6}{d}+\left(\underset{3}{d}+\underset{9}{d}\right)\sin 45^\circ.$$

$$P\ (3\ 6428)=\underset{3}{s}-\underset{9}{s}.$$

$$R\ (3\ 2190)=-\underset{6}{d}+\left(\underset{3}{d}+\underset{9}{d}\right)\sin 45^\circ$$

$$S\qquad=0$$

The suffixes here are the same as in the preceding formulæ. Thus $\underset{3}{\theta}$ is 45°, $\underset{6}{\theta}$ is 90°.

I have compared this formula with the observations of February and March, the most irregular of the whole set, and the results, along with those of the preceding one, are given in the following Table. The numbers are the observed — the calculated values.

TABLE IV.

Hours	February		March	
	Normal	Triplet	Normal	Triplet.
	m	m	m	m
0	0·07	0 02	0 02	0 00
1	0·22	0 26	0 02	−0·20
2	−0 03	0 04	−0·07	−0 12
3	−0 11	−0·04	−0 07	−0·06
4	−0 02	0 08	−0 11	0·16
5	−0 03	0 02	−0 18	0·29
6	0 21	0·17	−0 07	0 04
7	−0 17	−0 18	0·01	−0·30
8	0 00	0 08	−0·30	−0 52
9	−0 24	−0·23	0·01	−0 15
10	0 11	0·25	0·38	0 57
11	0 21	0 28	−0·48	−0 29
12	−0 28	−0 26	−0·02	0·15
13	0 04	0·00	0 21	0·15
14	0·01	−0·08	−0 17	−0 32
15	−0·12	−0·25	0 28	0·31
16	0·34	0 38	−0 20	−0·01
17	−0·23	−0 13	0 05	0·01
18	−0·11	0·11	−0·05	−0 05
19	0 07	0·15	0·16	−0·03
20	0·07	0·19	−0·05	−0·06
21	0·00	−0 06	−0 09	−0·08
22	0 11	−0·03	−0 07	−0 17
23	−0·25	−0·26	0 06	−0 22
PE	±0·110	±0·123	±0·123	±0 161

The triplet combinations are not much inferior to the others, and might possibly be sufficient, but I prefer the latter Even in the extreme cases of February 16 and March 8, 10^h and 11^h, the discordance is not as great as I anticipated from the absence of the constant S. I tried them, omitting the terms of the fourth order, but the results were decidedly inferior.

In considering the magnitude of some of these errors, it must be remembered that the formula expresses only that part of the coordinates which is periodic; and they are the residues of other effects which do not depend on the time θ, and which disappear from a larger series of observations, for the other hours the errors are much smaller. I thought of grouping the hours in pairs, which would probably have given a better result than the triple combination, but on deducing the formula, I found it would require more logarithmic work than the complete process. In it the coefficient of a constant of the p order has the coefficient $= 6 \cos p \times 15°$, instead of 6, as is evident from what precedes.

The horary constants of W for the twelve months are given in

TABLE V.

Month	K	A	B	C	D	O	P	R	S
January	4·571	0·058	0·234	−0·104	−0·115	−0·211	0·133	−0·126	−0·080
February	2·913	−0·474	0·457	−0·199	−0·010	−0·048	0·219	−0·046	0·021
March	5·165	−0·645	−0·043	0·244	−0·202	−0·108	0·272	0·072	−0·183
April	0·462	−0·212	−0·016	0·089	−0·000	0·008	0·081	−0·024	0·086
May	0·594	0·161	0·044	0·098	0·021	0·458	0·127	0·055	−0·066
June	0·837	−0·052	0·080	−0·090	0·004	0·252	0·021	0·026	0·037
July	2·104	−0·651	0·052	0·197	−0·046	0·225	−0·033	0·104	−0·043
August	2·367	−0·630	0·228	0·031	−0·114	−0·012	0·115	−0·005	−0·105
September	2·252	−0·236	0·011	0·105	−0·018	0·203	0·122	−0·053	0·067
October	2·090	−0·274	0·159	−0·131	−0·040	0·049	0·074	−0·085	−0·123
November	1·754	−0·084	0·094	0·044	−0·068	0·046	0·112	−0·082	0·040
December	4·071	−0·023	−0·083	−0·015	−0·003	0·103	0·127	0·006	0·032

The similar constants of S are given in

TABLE VI.

Month	K	A′	B′	C′	D′	O′	P′	R′	S′
January	6·982	−0·131	0·170	−0·252	−0·002	−0·143	0·049	0·089	−0·007
February	6·017	−0·239	0·419	−0·108	−0·012	0·160	−0·085	−0·109	−0·085
March	2·976	0·012	0·311	−0·080	−0·012	0·944	−0·155	−0·009	0·058
April	2·472	−0·026	0·248	0·053	−0·112	0·561	0·070	−0·037	0·137
May	2·221	−0·252	0·142	0·142	−0·054	0·226	0·054	−0·042	−0·077
June	1·749	−0·237	0·139	0·043	−0·015	0·379	0·155	−0·006	−0·054
July	1·539	−0·150	0·120	−0·015	0·080	0·287	0·124	−0·048	−0·007
August	2·621	−0·176	0·241	−0·033	0·089	0·354	0·116	−0·078	−0·123
September	3·715	−0·093	0·284	−0·037	−0·086	0·071	−0·259	−0·030	−0·057
October	3·834	0·289	0·104	−0·202	−0·018	0·217	−0·033	−0·072	−0·081
November	3·990	−0·041	0·038	−0·013	0·006	0·081	−0·145	−0·060	0·001
December	7·696	0·181	0·102	−0·214	−0·006	0·121	−0·100	−0·093	−0·087

The degree of precision with which these constants represent the observations will appear from the number of errors between certain limits. W has from 0·0 to 0·10 inclusive, 177; from 0·11 to 0·20, 72; from 0·21 to 0·30, 33; from 0·31 to 0·40, 3; above 0·40, 3. S has from 0·00 to 0·10, 172; from 0·11 to 0·20, 85; from 0·21 to 0·30, 27; from 0·31 to 0·40, 3; above 0·40, 1.

We now proceed to develop these constants in terms of ϕ, but as four orders do not give K and K′ with sufficient exactness, I have carried the formula to the sixth order, its utmost extent.

Formula where $b=30°$ and $a=15°$.

$$6A=\left(d_1-d_6\right)\cos 15+\left(d_2-d_5\right)\cos 45+\left(d_3-d_4\right)\cos 75.$$

$$6O=\left(d_1+d_6\right)\sin 15+\left(d_2+d_5\right)\sin 45+\left(d_3+d_4\right)\sin 75.$$

$$6B=\left\{s_1+s_6-\left(s_3+s_4\right)\right\}\cos 30.$$

$$6P=\tfrac{1}{2}\left\{s_1-s_6+s_3-s_4\right\}+s_2-s_5.$$

$$6C=\left\{d_1-d_6-\left(d_2-d_5+d_3-d_4\right)\right\}\cos 45$$

$$6R=\left\{d_1+d_6+d_2+d_5-\left(d_3+d_4\right)\right\}\sin 45.$$

$$6D=\tfrac{1}{2}\left\{s_1+s_6+s_3+s_4\right\}-\left(s_2+s_5\right).$$

$$6S=\left\{s_1-s_6-\left(s_3-s_4\right)\right\}\cos 30°.$$

$$6E=\left(d_1-d_6\right)\sin 15°-\left(d_2-d_5\right)\cos 45°+\left(d_3-d_4\right)\cos 15°.$$

$$6T=\left(d_1+d_6\right)\cos 15°-\left(d_2+d_5\right)\cos 45+\left(d_3+d_4\right)\sin 15$$

$$12U=s_1-s_6-\left(s_2-s_5\right)+s_3-s_4$$

G, for reasons already given, cannot be determined

It is, however, necessary to obviate two difficulties which interfere in the present instance with the accuracy of this process, but which do not affect the horary interpolation. It supposes that the us employed represent values of the coordinates belonging to dates which correspond with a series of ϕ in arithmetical progression.

This is not the case, for (1) the means of each month do not represent exactly the coordinates belonging to the middle of that month, and (2) the angles representing the distance of the middle of each month from the beginning of each year are not in arithmetical progression, as is evident from the following Table, which gives these angles $=\psi$, and also those belonging to each half month $=\mu$.

TABLE VII.

Month	ψ.	μ	Month	ψ	μ.
January . .	15° 16′·5	15° 16′·8	July	193° 40′ 8	15° 16′·8
February .	44 21·2	13 53·4	August .	224 13·8	15 16·8
March .	73 25·9	15 16 8	September	254 18 0	14 46·6
April ..	103 25·2	14 46 6	October .	284 24·6	15 16·8
May	133 33·6	15 16 8	November	314 25·2	14 46 6
June	163 37·2	14 46 6	December .	344 28·8	15 16·8

Both these difficulties are overcome by a process based on a suggestion of Professor STOKES.

Let the true constants of the formula be denoted by small *italic* letters, so that

$$\underset{1}{u}=\mathrm{K}+a\cos\theta+o\sin\theta+b\cos 2\theta+\&c.,$$

then, as the mean of u through the space $\theta'-\theta=\int_\theta^{\theta'} u\,d\theta$, we have

$$\text{mean } u=\left\{\int \mathrm{K}d\theta+\int a\cos\theta d\theta+\int o\sin\theta d\theta\ \&c.\right\}\frac{1}{\theta'-\theta}.$$

Let $\theta'=\psi+\mu$, $\theta=\psi-\mu$, and as all the pairs of terms are of the same form,

$$\underset{p}{a}\cos p\theta+\underset{p}{o}\sin p\theta,$$

integrating this will do for all The integral is

$$\mathrm{K}\theta+\ldots\quad+\underset{p}{a}\frac{\sin p\theta}{p}-\underset{p}{o}\frac{\cos p\theta}{p},$$

which within the limits

$$=\mathrm{K}2\mu\quad .\quad+\underset{p}{a}\left\{\frac{\sin(p\psi+p\mu)}{p}-\sin\frac{(p\psi-p\mu)}{p}\right\}-\underset{p}{o}\left\{\frac{\cos(p\psi+p\mu)}{p}-\frac{\cos(p\psi-p\mu)}{p}\right\}$$

$$=2\mathrm{K}\mu+\frac{2\underset{p}{a}\cos p\psi\sin p\mu+2\underset{p}{o}\sin p\psi\sin p\mu}{p\mu-\sin p\mu},$$

and dividing by $2\mu=\theta'-\theta$, we obtain, calling $\frac{p\mu}{\sin p\mu}=\underset{p}{r}$,

$$\text{mean } u=\mathrm{K}+a\frac{\cos\psi}{\underset{1}{r}}+\frac{o\sin\psi}{r}+b\frac{\cos 2\psi}{\underset{2}{r}}+\&c$$

Now we might form the n equations for u and treat them by minimum squares, but as in this case none of the terms would vanish on summing, though all (except the one, say $\underset{p}{a}$, whose square appears) are small, the labour of eliminating 12 quantities 13 times over would be truly formidable This might be evaded by substituting in each sum for the true constants those given by the series of ϕ, which differ little from them, and all, except $\underset{p}{\mathrm{A}}$, are multiplied by small coefficients This will give $\Delta\underset{p}{a}$ with close approximation. The process may be repeated with the corrected values, but $\Delta\underset{p}{a}$ alone will have any notable effect. Yet even with this simplification the labour is very great But it may be superseded thus We have the above equation for $\underset{1}{u}$, but we have also $u'=\mathrm{K}+\mathrm{A}\cos\phi+\mathrm{O}\sin\phi+\mathrm{B}\cos 2\phi+\mathrm{P}\sin 2\phi\ \&c.$; and equating the two values,

$$\mathrm{K}+\frac{a\cos\psi}{r}+\frac{o\sin\psi}{r}+\frac{b\cos 2\psi}{\underset{2}{r}}+\frac{p\sin 2\psi}{\underset{2}{r}}=\mathrm{K}+\mathrm{A}\cos\phi+\mathrm{O}\sin\phi+\&c$$

It is evident that if we put $a=\mathrm{A}\frac{r\cos\phi}{\cos\psi}$, $o=\mathrm{O}\frac{r\sin\phi}{\sin\psi}$, and so on, the equation would

be satisfied, if the factors $\frac{r\cos\phi}{\cos\psi}$, $\frac{r\sin\phi}{\sin\psi}$ &c were equal in every month. They, however, differ so little that I have thought it lawful to take their means for the twelve months.

Though this is fairly warranted, yet it seemed advisable to test it by comparing for E the cosine constant, of the fifth order in the series for K, with the minimum square process It gives for E 1 0525, the second approximation, using ΔE alone, gives 1·0305, which would be a little increased by using the corrections of the other constants, so that the agreement is sufficient. As the factors $r_p\frac{\cos p\phi}{\cos p\psi}$, $r_p\frac{\sin p\phi}{\sin p\psi}$ will answer for any year, I give their logarithms

A	O	B	P	C	R
0 00672	0 00346	0·02156	0·02119	0 04411	0·04976
D	S	E.	T	G	U
0 09300	0·08400	0 16622	0·11988	Not determined	0·19731

It does not seem necessary to give the constants A, O, &c., but instead the secondary constants of the formula $u_\theta = K_0 + K_1 \sin(r+\theta_1) + K_2 \sin(r+2\theta_2) + \&c$, deduced from their corrected values, are given as more convenient for computation in Table VIII. (p 427)

I have given the constants for the horary coefficients A, O, &c to the 6th order for symmetry, but in fact I do not think any of them less than 0 05 need be attended to. Even this limit is beyond what can be expected to be available when they are only determined by the observations of seven years, as is evident from what I have already said as to the P E of the quantities from which they are determined. Whether the diurnal variation of the coordinates follows the same law in different septennial periods remains to be determined, probably it does. The constants belonging to K_ϕ and K'_ϕ are larger than the others, and, as derived from larger coefficients, merit more confidence

The effect of the terms of the first and second orders, which are the chief, are similar; but the others present opposite phases, and would probably be modified by more accurate determination It is here that I think changes in successive years will probably be found, and were I to pursue this work further, I would combine the observations rather differently from what I have done in the present case. I would mean the homonymous hours of each month of each year, combine them in pairs, and mean *them* to get the K of each month I would then compute the K constants, retaining the cosine and sine form, and this should be continued through a few periods of the solar spots This would decide the question whether the wind is affected by the conditions which modify those phenomena

At the same time the inspection of the horary means would show whether *their* laws vary with the time Then the final constants could be determined for such intervals as might be considered sufficient. The sine and cosine formula, though requiring more

TABLE VIII.

$K_0=2{\cdot}4307$	$K_1=1{\cdot}3393$ $\kappa_1=85^\circ 37'{\cdot}4$	$K_2=1{\cdot}1203$ $\kappa_2=12^\circ 37'{\cdot}7$	$K_3=0{\cdot}3354$ $\kappa_3=191^\circ 24'{\cdot}2$	$K_4=0{\cdot}8488$ $\kappa_4=130^\circ 9'{\cdot}3$	$K_5=1\ 1377$ $\kappa_5=89^\circ 43'\ 3$	$K_6=0{\cdot}4856$ $\kappa_6=0^\circ 0'\ 0$
$A_0=-0{\cdot}2552$	$A_1=0{\cdot}1601$ $\alpha_1=33^\circ 21'{\cdot}2$	$A_2=0{\cdot}3159$ $\alpha_2=159^\circ 44'{\cdot}6$	$A_3=0\ 1927$ $\alpha_3=60^\circ 9'{\cdot}0$	$A_4=0{\cdot}0215$ $\alpha_4=179^\circ 22'{\cdot}9$	$A_5=0\ 1235$ $\alpha_5=333^\circ 51'{\cdot}8$	$A_6=0{\cdot}0351$ $\alpha_6=0^\circ 0'{\cdot}0$
$B_0=0{\cdot}1014$	$B_1=0{\cdot}0419$ $\beta_1=91^\circ 14'\ 5$	$B_2=0{\cdot}1090$ $\beta_2=13^\circ 56'\ 5$	$B_3=0\ 9878$ $\beta_3=333^\circ 23'{\cdot}7$	$B_4=0{\cdot}1551$ $\beta_4=301^\circ 45'\ 1$	$B_5=0\ 0413$ $\beta_5=330^\circ 51'{\cdot}9$	$B_6=0{\cdot}0569$ $\beta_6=180^\circ 0'\ 0$
$C_0=0{\cdot}0224$	$C_1=0{\cdot}0299$ $\gamma_1=314^\circ 21'{\cdot}1$	$C_2=0{\cdot}0485$ $\gamma_2=266\ 47'{\cdot}5$	$C_3=0{\cdot}1202$ $\gamma_3=171^\circ 26'\ 1$	$C_4=0{\cdot}0427$ $\gamma_4=216^\circ 34'\ 8$	$C_5=0{\cdot}0387$ $\gamma_5=153^\circ 44'{\cdot}8$	$C_6=0{\cdot}1180$ $\gamma_6=0^\circ 0'\ 0$
$D_0=-0{\cdot}0492$	$D_1=0{\cdot}0207$ $\delta_1=261^\circ 18'{\cdot}7$	$D_2=0{\cdot}0628$ $\delta_2=167^\circ 0'{\cdot}0$	$D_3=0{\cdot}0399$ $\delta_3=25^\circ 31'\ 2$	$D_4=0{\cdot}0087$ $\delta_4=291^\circ 22'{\cdot}8$	$D_5=0{\cdot}0830$ $\delta_5=229^\circ 34'{\cdot}7$	$D_6=0\ 0348$ $\delta_6=180^\circ 0'{\cdot}0$
$O_0=0{\cdot}0804$	$O_1=0\ 1630$ $o_1=261^\circ 11'{\cdot}8$	$O_2=0\ 1296$ $o_2=165^\circ 16'\ 3$	$O_3=0\ 0597$ $o_3=16^\circ 34'{\cdot}7$	$O_4=0{\cdot}0776$ $o_4=209^\circ 43'\ 0$	$O_5=0{\cdot}1591$ $o_5=189\ 21'{\cdot}5$	$O_6=0{\cdot}0342$ $o_6=0\ 0'\ 0$
$P_0=0{\cdot}1142$	$P_1=0{\cdot}0650$ $\varpi_1=55^\circ 28'{\cdot}5$	$P_2=0\ 0565$ $\varpi_2=306^\circ 8'\ 2$	$P_3=0{\cdot}0069$ $\varpi_3=48^\circ 33'{\cdot}1$	$P_4=0\ 0626$ $\varpi_4=215^\circ 38'{\cdot}0$	$P_5=0{\cdot}0339$ $\varpi_5=46\ 45'\ 1$	$P_6=0\ 0126$ $\varpi_6=0^\circ 0'{\cdot}0$
$R_0=-0{\cdot}0095$	$R_1=0\ 0209$ $\rho_1=298^\circ 39'{\cdot}9$	$R_2=0{\cdot}0205$ $\rho_2=74^\circ 20'{\cdot}1$	$R_3=0{\cdot}0430$ $\rho_3=204^\circ 6{\cdot}5$	$R_4=0{\cdot}0276$ $\rho_4=152^\circ 52'\ 7$	$R_5=0{\cdot}0621$ $\rho_5=148^\circ 23'{\cdot}7$	$R_6=0\ 0186$ $\rho_6=0^\circ 0'{\cdot}0$
$S_0=-0{\cdot}0264$	$S_1=0{\cdot}0035$ $\sigma_1=166^\circ 20'{\cdot}3$	$S_2=0{\cdot}0339$ $\sigma_2=156^\circ 5'{\cdot}1$	$S_3=0\ 0245$ $\sigma_3=80^\circ 58'\ 6$	$S_4=0\ 0272$ $\sigma_4=225^\circ 20'\ 3$	$S_5=0{\cdot}0152$ $\sigma_5=264^\circ 12'{\cdot}8$	$S_6=0{\cdot}0282$ $\sigma_6=180^\circ 0'\ 0$

TABLE IX.—Secondary Constants for S.

$\overset{1}{K}_{0}=3\cdot8177$	$\overset{1}{K}_{1}=2\cdot5130$ $\overset{1}{\kappa}_{1}=93^\circ 52'\cdot1$	$\overset{1}{K}_{2}=0\cdot8748$ $\overset{1}{\kappa}_{2}=63^\circ 28'\cdot5$	$\overset{1}{K}_{3}=0\cdot9444$ $\overset{1}{\kappa}_{3}=66^\circ 16'\cdot5$	$\overset{1}{K}_{4}=0\cdot2678$ $\overset{1}{\kappa}_{4}=150^\circ 17'\cdot6$	$\overset{1}{K}_{5}=0\cdot5199$ $\overset{1}{\kappa}_{5}=182^\circ 59'\cdot8$	$\overset{1}{K}_{6}=0\cdot3894$ $\overset{1}{\kappa}_{6}=180^\circ 0'\cdot0$
$\overset{1}{A}_{0}=-0\cdot0749$	$\overset{1}{A}_{1}=0\cdot1361$ $\overset{1}{\alpha}_{1}=127^\circ 43'\cdot2$	$\overset{1}{A}_{2}=0\cdot1062$ $\overset{1}{\alpha}_{2}=227^\circ 53'\cdot9$	$\overset{1}{A}_{3}=0\cdot0628$ $\overset{1}{\alpha}_{3}=195^\circ 36'\cdot3$	$\overset{1}{A}_{4}=0\cdot1319$ $\overset{1}{\alpha}_{4}=80^\circ 54'\cdot3$	$\overset{1}{A}_{5}=0\cdot1183$ $\overset{1}{\alpha}_{5}=118^\circ 31'\cdot6$	$\overset{1}{A}_{6}=0\cdot0600$ $\overset{1}{\alpha}_{6}=180^\circ 0'\cdot0$
$\overset{1}{B}_{0}=0\cdot1932$	$\overset{1}{B}_{1}=0\cdot0658$ $\overset{1}{\beta}_{1}=5^\circ 8'\cdot8$	$\overset{1}{B}_{2}=0\cdot1265$ $\overset{1}{\beta}_{2}=330^\circ 3'\cdot5$	$\overset{1}{B}_{3}=0\cdot0272$ $\overset{1}{\beta}_{3}=18^\circ 14'\cdot3$	$\overset{1}{B}_{4}=0\cdot0405$ $\overset{1}{\beta}_{4}=211^\circ 32'\cdot3$	$\overset{1}{B}_{5}=0\cdot0432$ $\overset{1}{\beta}_{5}=248^\circ 27'\cdot0$	$\overset{1}{B}_{6}=0\cdot0247$ $\overset{1}{\beta}_{6}=180^\circ 0'\cdot0$
$\overset{1}{C}_{0}=-0\cdot0597$	$\overset{1}{C}_{1}=0\cdot1294$ $\overset{1}{\gamma}_{1}=290^\circ 14'\cdot5$	$\overset{1}{C}_{2}=0\cdot0590$ $\overset{1}{\gamma}_{2}=206^\circ 16'\cdot4$	$\overset{1}{C}_{3}=0\cdot0153$ $\overset{1}{\gamma}_{3}=163^\circ 19'\cdot2$	$\overset{1}{C}_{4}=0\cdot0737$ $\overset{1}{\gamma}_{4}=252^\circ 17'\cdot4$	$\overset{1}{C}_{5}=0\cdot0642$ $\overset{1}{\gamma}_{5}=273^\circ 32'\cdot0$	$\overset{1}{C}_{6}=0\cdot0270$ $\overset{1}{\gamma}_{6}=180^\circ 0'\cdot0$
$\overset{1}{D}_{0}=-0\cdot0083$	$\overset{1}{D}_{1}=0\cdot0284$ $\overset{1}{\delta}_{1}=199^\circ 28'\cdot2$	$\overset{1}{D}_{2}=0\cdot0455$ $\overset{1}{\delta}_{2}=49^\circ 15'\cdot3$	$\overset{1}{D}_{3}=0\cdot0390$ $\overset{1}{\delta}_{3}=220^\circ 55'\cdot3$	$\overset{1}{D}_{4}=0\cdot0198$ $\overset{1}{\delta}_{4}=257^\circ 13'\cdot3$	$\overset{1}{D}_{5}=0\cdot0429$ $\overset{1}{\delta}_{5}=83^\circ 59'\cdot9$	$\overset{1}{D}_{6}=0\cdot0179$ $\overset{1}{\delta}_{6}=180^\circ 0'\cdot0$
$\overset{1}{O}_{0}=0\cdot2715$	$\overset{1}{O}_{1}=0\cdot2265$ $\overset{1}{o}_{1}=324^\circ 48'\cdot8$	$\overset{1}{O}_{2}=0\cdot1762$ $\overset{1}{o}_{2}=278^\circ 25'\cdot5$	$\overset{1}{O}_{3}=0\cdot2199$ $\overset{1}{o}_{3}=211^\circ 4'\cdot6$	$\overset{1}{O}_{4}=0\cdot1341$ $\overset{1}{o}_{4}=141^\circ 41'\cdot0$	$\overset{1}{O}_{5}=0\cdot1469$ $\overset{1}{o}_{5}=72^\circ 50'\cdot9$	$\overset{1}{O}_{6}=0\cdot0400$ $\overset{1}{o}_{6}=180^\circ 0'\cdot0$
$\overset{1}{P}_{0}-0\cdot0226$	$\overset{1}{P}_{1}=0\cdot1098$ $\varpi_{1}=292^\circ 1'\cdot8$	$\overset{1}{P}_{2}=0\cdot0936$ $\varpi_{2}=101^\circ 28'\cdot8$	$\overset{1}{P}_{3}=0\cdot0097$ $\varpi_{3}=182^\circ 35'\cdot5$	$\overset{1}{P}_{4}=0\cdot0992$ $\varpi_{4}=5^\circ 40'\cdot3$	$\overset{1}{P}_{5}=0\cdot0699$ $\varpi_{5}=43^\circ 57'\cdot6$	$\overset{1}{P}_{6}=0\cdot0327$ $\varpi_{6}=180^\circ 0'\cdot0$
$\overset{1}{R}_{0}=-0\cdot0315$	$\overset{1}{R}_{1}=0\cdot0285$ $\overset{1}{\rho}_{1}=342^\circ 57'\cdot0$	$\overset{1}{R}_{2}=0\cdot0216$ $\overset{1}{\rho}_{2}=40^\circ 45'\cdot9$	$\overset{1}{R}_{3}=0\cdot0407$ $\overset{1}{\rho}_{3}=39^\circ 8'\cdot0$	$\overset{1}{R}_{4}=0\cdot0269$ $\overset{1}{\rho}_{4}=61^\circ 35'\cdot7$	$\overset{1}{R}_{5}=0\cdot0453$ $\overset{1}{\rho}_{5}=30^\circ 13'\cdot3$	$\overset{1}{R}_{6}=0\cdot0343$ $\overset{1}{\rho}_{6}=0^\circ 0'\cdot0$
$\overset{1}{S}_{0}=-0\cdot0487$	$\overset{1}{S}_{1}=0\cdot0508$ $\overset{1}{\sigma}_{1}=44^\circ 29'\cdot7$	$\overset{1}{S}_{2}=0\cdot0023$ $\overset{1}{\sigma}_{2}=280^\circ 53'\cdot1$	$\overset{1}{S}_{3}=0\cdot0452$ $\overset{1}{\sigma}_{3}=186^\circ 38'\cdot1$	$\overset{1}{S}_{4}=0\cdot0283$ $\overset{1}{\sigma}_{4}=95^\circ 19'\cdot5$	$\overset{1}{S}_{5}=0\cdot0565$ $\overset{1}{\sigma}_{5}=304^\circ 41'\cdot1$	$\overset{1}{S}_{6}=0\cdot0526$ $\overset{1}{\sigma}_{6}=0^\circ 0'\cdot0$

work in computing, has this advantage, that it permits the combining the constants obtained at different periods by simple meaning, which the sine formula does not. It also lends itself more easily to an examination of any influence which may be supposed to change the coordinates periodically. Any such may be developed in a similar series, and the sum or difference of the two will give the residual part which is to be accounted for by other causes. If this residue be larger than the original periodic part, the hypothesis must be rejected, and even though it be diminished, this is not sufficient unless there be *à priori* evidence of a *vera causa*. As an example of this may be mentioned one of the elements of the sun's action. Its heating-power on a given day depends, among other things, on the sum of the sines of its altitude during that day. This sum

$$=2\int_{\theta'}^{180-\theta'} d\theta\{\sin \text{lat } \sin \text{declin} - \cos \text{lat } \cos \text{decl } \sin \theta\}$$

$$=2 \sin \text{lat.} \sin \text{decl.} \times \theta' + 2 \cos \text{lat } \cos \text{decl.} \sin \theta',$$

θ' being the value of θ at sunrise. If the value of this integral be computed for 12 values of ϕ, it can be developed in a series $y=k+a\cos\phi+o\sin\phi+b\cos 2\phi+$ &c. This belongs to the midday of each month, and ought in strictness to be summed for the entire month by means of the expression of decl. in terms of ϕ, but it is sufficient for illustration. u is evidently diminished by y, and we have what would be found if the altitude had no effect,

$$x=u+qy=\text{K}+kq+\cos\phi(\text{A}+aq)+\sin\phi(\text{O}+oq)+\cos 2\phi(\text{B}+bq)+\&c$$

If q, the measure of the altitude's effect on the coordinates, were known, no more would be required, but a probable value of it is that which would make the sum of the squares of the periodic parts of the residues or $\text{S}(u_\phi+qy_\phi-\text{K}-k)$ a minimum. This gives

$q(\text{S}y^2-12k^2)=-\text{S}uy+12\text{K}k$

For $\text{K}q=2{\cdot}422$, for $\text{K}'u=4{\cdot}723$. With these I computed the series for x and x', which need not be given, remarking merely that the coefficients of the first order are the only ones much altered. It may suffice to give the variable parts of u, x, u', x'.

January	February	March	April	May	June	July	Aug	Sept	October	Nov	Dec
2·109	0·778	2·758	−1·931	−1·715	−1·576	0·438	0·217	−0·180	−0·361	−0·386	1·658
1·016	0·032	2·630	−1·258	−0·575	−0·299	0·680	0·886	−0·150	−0·929	−1·397	0·467
3·170	2·190	−0·541	−1·345	−1·596	−2·068	−2·278	−1·196	−0·102	0·017	0·173	3·879
1·030	0·778	−1·092	−0·137	0·398	0·420	−0·100	0·117	−0·045	−1·092	−1·800	1·655

It seems from these numbers that the sun's altitude may account for 0·27 of the variation of W, and for 0·53 of that of S.

This discussion suggests the notion that the equatorial current which produces the positive W and S coordinates may possibly be more constant than appears at first sight, and that a part of these variations may be due to a current in the opposite direction

caused by the solar action in the vicinity of the place of observation and varying with the sun's declination. Supposing $q y_{\phi}$ to be that part of it due to the altitude, its mean annual value would be V=2 721, about 0 6 of V′ (page 412), and its D=207° 8′·7. Other periodical causes, such as the length of air traversed by the sun's rays at different altitudes, the difference of the earth's daily and nightly radiations, and the amount of watery vapour in the air, might be similarly taken into account.

I have already stated that I thought it useless to deal with the observations of single days; I, however, tried two experiments in this direction, which may be of some interest, though the first of them was unsuccessful.

1 In many instances, even when the wind is moderate, there are variations in its direction which suggest the notion that they are due to aerial whirlpools on so small a scale that they are not likely to reach any other meteorological station.

I thought it might be possible to determine the constants of such a motion in the following way The curve described on such a supposition by the thread of wind which passes the anemometer at a given station is that which would be traced by a pencil fixed there on a plane revolving with an angular hourly velocity ω' round a centre which is carried in a line inclined at the angle α to the axis of x with the hourly velocity V, ξ and η being the coordinates of that centre at the origin of the time, and ω the angular motion there. It is obvious that we have

$$dx=dt\{\mathrm{V}\cos\alpha(1+\omega' t)+\xi\omega'\},\ dy=dt\{\mathrm{V}\sin\alpha(1+\omega' t)-\eta\omega'\}.$$

Then at successive hours equating $\frac{dy}{dx}$ to tang D, and $\int\sqrt{dx^2+dy^2}$ to s, I would be able to get values of the unknown quantities But against this is my ignorance of the relation between ω' and this distance from the centre of the circle, which is not given in any book to which I can refer NEWTON, in the vortex which he considers, gives it inversely as the distance It is probably nearer the inverse square. Either of these suppositions would make direct integration impossible, so I gave up the project.

2. The other was an attempt to determine from these observations the existence of an atmospheric tidal current As in the case of the ocean, so in the atmosphere, the air must be heaped up in the meridian passing through the moon, or a little to the east of it, and this elevation must be accompanied by a horizontal current.

LAPLACE (Méc Cél ii.) has shown that the maximum air-tidal current is 0·07532 metre in a centesimal second*, which in English measure and time is 0·195 mile in an hour He, however, gives no indication of the phase of this maximum, or in what stratum of the atmosphere it occurs. At the earth's surface, owing to friction and other causes, it must be considerably less than the above value, and the analogy of sea-tides is too slight to give much assistance in the research. It may, however, authorize us to assume that on opposite sides of the lunar meridian the directions of this current will be opposite.

* It is to be regretted that in this noble work LAPLACE used the centesimal division of the quadrant, and the decimal and centesimal divisions of the day Whatever be the fate of the metric system, it is very unlikely that either of the others will be generally adopted.

Having no data to guide me in detecting the most favourable Lunar hours, I began by comparing the Ws for 0^h, 6^h, 12^h, 18^h, and 3^h, 9^h, 15^h, 21^h. I soon, however, found that this involved too much labour, and confined myself to the last hour.

Calling C′ the current, $C' = \frac{1}{2}(W_{21} - W'_{3}), = \frac{1}{2}(W_{9} - W'_{15})$. In this I made no attempt to allow for the sun's elongation from the moon, or for their declination, nor for the horary changes of the coordinates, as the selected lunars are nearly uniformly distributed in each of the 24 common hours

By a Table with the moon's hourly motion in Right Ascension for argument I found the time which should be added to the Greenwich time of its culmination to obtain the common time of the above-named lunar hours at Armagh, and entering the Journal with these I obtained for each day two values of $\frac{1}{2}(W - W')$, belonging to the upper and lower culminations. From the irregularity of these values it might seem hopeless to get any result, but I pursued the inquiry in hopes of ascertaining the limits within which the mean of a considerable number of observations (even though very discordant) might be depended on

I only took the first six months of the year, as the results which they gave were quite satisfactory.

TABLE X

Month	Current	No	PE	Weight	C×W	
January	0·2289	404	±1·939	1·000	0·2289	The mean according to the weight.
February	0·1549	376	±1·809	1·020	0·1580	
March	−0·0414	415	±1·886	1·082	−0·0459	
April	0·0072	396	±1·737	1·190	0·0084	C′=0·0906.
May	0·0844	423	±1·345	2·175	0·1837	Not differing much from the single mean
June	0·1125	404	±1·247	2·416	0·2720	
	0·0911	2418	±1·661	8·883	0·8051	

The weights are proportional, the least, that for January, being taken as unity It will be observed that these probable errors are far less than those given in page 415, but it should be recollected that here the variations can only occur within 6 lunar hours, while in the other case they range through months and years. Even so there are occasionally very great and startling changes when a gale bursts out suddenly or suddenly ceases. There were two values of $W - W'$ above 40, and three above 30. Yet with all this I think the result is very remarkable I do not pretend to assert that this value of C′ really represents the tidal current at these hours, though it is in the right direction and of not improbable amount, for it may be some uncompensated residue of the horary changes. But it is of great importance, as giving what must be a close approximation to the real value of the average air-tidal stream, and as verifying my former

statements, that casual irregularities are eliminated from the mean of a large number of observations Still I think that a truer result might be obtained by omitting extremely aberrant observations, but it becomes a question to what extent this should be done. I think all may be rejected which exceed four times the largest probable error; in other words, whose probability is less than 0 0228 This is for W—W′ all above 15. The number of these is 58, and the results after their exclusion are given in Table XI.

TABLE XI.

Month	Current	No	P E	Weight	C×W	
January	0 1498	386	±1 592	1·321	0·1979	The mean according to the weights is C″=0·0559.
February	0 2483	366	±1·607	1·317	0·3204	
March	−0·0423	406	±1·726	1·266	−0 0535	
April	−0·0590	381	±1·657	1 289	−0·0760	
May	0·0010	419	±1·262	2 438	0 0024	
June	0 0718	402	±1 204	2 496	0 1753	
	0·0608	2360	±1 443	10·127	0·5665	

The probable errors are less, and the weights greater than in the other case, so that C″ is probably a better value than C′.

It is possible that this mode of proceeding might give the horary changes of the coordinates more correctly than the simple comparison of the numbers in Table III., but the labour of computation would be much greater.

XVI. THE CROONIAN LECTURE.—*Experiments on the Brain of Monkeys* (Second Series).

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Communicated by Dr. SANDERSON, *V.P.R.S.*

Received April 27,—Read May 13, 1875.

IN a former memoir presented to the Royal Society the author described the results of electrical irritation of localized regions of the brain of monkeys. This memoir contains the details of experiments relating chiefly to the ablation or destruction of these localized centres, with the view of determining the significance, as regards motion and sensation, of the phenomena resulting from electrical stimulation, and for the purpose of ascertaining the function of those parts which give no external response to irritation. No originality is claimed either for the idea or method of carrying out these experiments.

The plan chiefly followed in the destruction of localized regions in the hemispheres was the application of the cautery, either in the form of a red-hot iron, or of the galvanic cautery, or of BRUCE's blowpipe cautery, according to special necessities or conditions. The advantage of this method is that destruction of the grey matter can be caused rapidly and effectually, without risk of hæmorrhage or interference with the integrity of surrounding parts. By the same method a part can be severed from the hemispheres without risk of hæmorrhage.

The details of observation are given in full as the best method of indicating the course of events following each operation and the data on which the conclusions are based.

Extirpation of the Frontal Lobes.

It has already been stated that the antero-frontal regions of the hemispheres give no response to electrical stimulation. Only one exception to this statement is to be made (see Exp. I.), viz. that in one case irritation of these regions caused the eyes to be turned to one or other side, according as the electrodes were placed on the opposite hemisphere.

Experiment I.

December 2nd, 1873.—A very lively, active, and intelligent monkey was placed under the influence of chloroform, and the frontal extremities of both hemispheres exposed as far back as the anterior extremity of the supero-frontal sulcus (fig. 1), the infero-frontal regions being exposed to a corresponding extent. On electrical irritation of the upper surface of these regions the eyes were occasionally turned to the opposite side. No results could be observed to follow application of the electrodes to the orbital region.

The exposed portions of the hemispheres were in this instance divided rapidly by

means of a scalpel, and the cut surface touched with perchloride of iron to still the hæmorrhage.

The operation was completed at 4 40 P.M.

When the chloroform stupor had passed off, which occurred in a few minutes, the animal sat up, but nodded off to sleep, opening its eyes faintly when a noise was made.

5 P.M Eagerly drank some sweet tea held to its lips, but immediately went to sleep when it was withdrawn Took a piece of bread and butter held before its face and began to eat, but after a bite or two went to sleep, holding the bread in its hand When it was awakened by cold air blown in its face, at which it expressed annoyance, it woke up and began to eat again greedily

5 15 P.M The scalp was sewn up. The animal retained sensation After the operation the animal took some food and again went to sleep

December 3rd—The animal is alive and well Eats and drinks spontaneously, but frequently subsides into a doze while eating Is constantly tending to sleep when it is not kept awake by external stimuli It pays little or no attention to any thing going on around unless stimuli are applied to it directly Formerly it used to exhibit the utmost curiosity in every thing going on around it. A lighted match held before its face caused it to exhibit some curiosity Touched it several times, each time showing signs of pain and rubbing its fingers vigorously Formerly the sight of fire used to cause it to run away.

December 4th—The animal remains in much the same state, sitting quietly, feeling the wound, which is oozing, and licking its hand Occasionally it runs hither and thither in the cage in an aimless manner Often subsides into a dozing state, but is easily roused by sounds, touch, &c Eats and drinks of its own accord in a mechanical way, frequently going to sleep the while.

Retains all its senses and muscular power

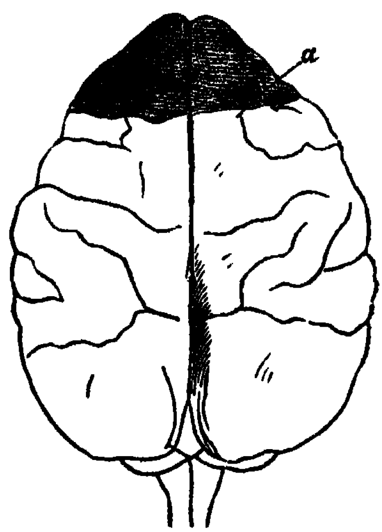
Gives evidence of sight by shrinking and holding its hands to protect its head when threatened with a stick Whatever is placed in its hand is mechanically raised to its mouth.

December 5th—The condition is in all essential respects unaltered Another monkey was placed in the cage beside it Of this it took little or no notice Formerly it took the greatest interest in examining any companion placed beside it

The continual sleepiness continues

The animal died from exhaustion on the 7th without having exhibited any further symptoms

Fig 1



Upper view of the hemispheres of the monkey The shaded part in the frontal lobes indicates the extent to which the brain was destroyed in Experiment I. a=the supero-frontal sulcus.

Post mortem Examination.—The frontal lobes were

found to have been removed by a line corresponding to that described and indicated in fig. 1.

The cut surface had fungated and was protruding through the openings in the skull The rest of the brain had a normal appearance.

Experiment II.

January 13*th*, 1875—A mischievous, good-tempered, and intelligent monkey was placed under the influence of chloroform, and the frontal lobes exposed on both sides By means of the wire cautery the lobes were severed by a transverse line cutting across the anterior extremity of the supero-frontal sulcus The division was carried down to the orbital surface, and the severed portion of brain removed

The operation was finished at 4 P.M.

4 15 P M The animal drank some tea held to its lips, but lay quiet and had not yet attempted to get up

5 P M. Now moves about, which it does rather unsteadily, but evidently sees where it is going, as it avoids obstacles in its path

5.45 P M Sits quietly with its head down when undisturbed, and makes scraping movements with both hands Expresses great annoyance when its face is blown on Tobacco-smoke held to its nostrils caused it to start back and run away

7 P M. Sits with its head down, engaged in picking at imaginary objects in front of it Can find its way in and out of its cage when roused to action Turns its head round and looks when called to, giving full evidence of its sense of hearing

8 10 P M Run out of its cage when the door was opened Runs about and jumps on furniture when roused Otherwise, when left to itself, it sits down and picks at imaginary objects on the floor Took a piece of apple offered to it and ate it.

11 15 P M. Ran about the room when let out of its cage, occasionally stopping to pick up things lying on the floor, and turning round to look when called to. Climbed up a chair and then relapsed into its usual position with its head down, and began to pick away with both hands at nothing

January 14*th.*—10 A.M When taken out of its cage wandered restlessly around the room Took a little food offered to it, and then capsized the dish. When placed in its cage picked up some pieces of bread, and sat and ate them contentedly, then rose and marched round and round After this subsided into a dreamy-like doze, and then after a few minutes began its picking and scraping movements

11 A.M. Is busily engaged picking up pieces of bread lying in its cage, carefully scraping and eating them. Runs about the cage occasionally in a restless manner, and then subsides into its quiet attitude, picking and scraping among the straw &c. in its cage.

5.30 P.M. When let out it ran about the room for some time, jumping on chairs &c, and then after sitting still for a few minutes, picking as usual, started up and ran about again in the same aimless manner.

Took some food offered to it, but after eating a little set to work to scatter it all about.

10.15 P.M. Is found clinging to its cage with hands and feet, apparently asleep, and takes no notice of my approach.

January 15*th*.—10.30 A.M. This morning when taken out seemed unwilling to move about. When roused and pushed, seemed to walk somewhat unsteadily, and as if its limbs were clogged.

Thinking that the motor centres were becoming involved in softening, I chloroformed the animal to death.

Post mortem Examination.—On examination of the brain it was found that the frontal lobes had been cut off on both sides according to the line indicated, viz. in a line passing transversely through the anterior extremities of the supero-frontal sulci. The plane of section sloped somewhat from above downwards, leaving the posterior half of the orbital surface uninjured.

The cut surface was projecting so as to protrude through the openings in the skull and reach the under surface of the scalp.

Some degree of softening had extended from the edges of section on both sides to the proximity of the antero-parietal sulcus, but slightly more on the left than on the right side (see figs. 2 & 3). The rest of the brain was round in appearance. The olfactory bulb and tract on both sides had escaped injury.

Fig. 2

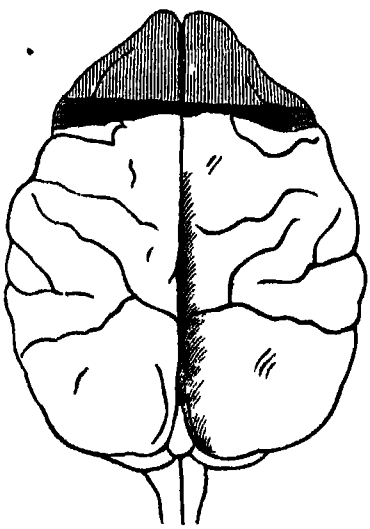
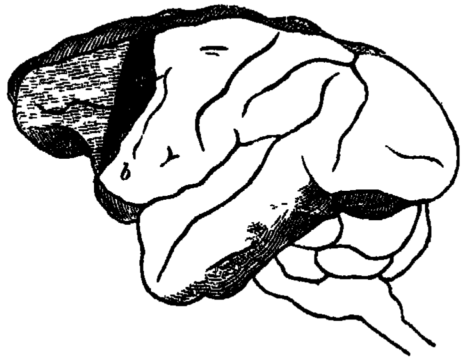


Fig. 3



Figs. 2 & 3 represent the upper surface of both hemispheres and side view of the left hemisphere of the brain of the monkey. The line cutting across the frontal regions at the anterior extremity of the supero-frontal sulci indicates the line of section of the lobes in Experiment II. The parallel lines indicate the extent of brain-substance removed. The shading posterior to the line indicates the extent to which softening had advanced. *b* = the antero-parietal sulcus (HUXLEY).

Experiment III.

March 16*th*, 1875.—The frontal lobes of both hemispheres were exposed in a small active and intelligent monkey, and by means of white-hot wires the frontal lobes were severed from the rest of the hemispheres by a line passing approximately through the anterior extremity of the supero-frontal sulcus on each side. The division, however, was slightly further back on the left than on the right side. The operation was completed at 1 P M

A few minutes after being let loose it sat up, and seeing a piece of cotton-wool lying before it, took it up and began tearing it with its teeth. Offered a piece of apple, it seized it and ate it

1 15 P M. Walks about the room pretty steadily There is no affection of its muscular power nor of sensation, it sees where it goes, turns its head when called to, and smells and eats fruit offered to it

6 P M. Animal is found sitting quietly in its cage It used to be very discontented at being shut up, and kept up a continual whining Offered some fruit, it smelt it and ate it. When let out of the cage it ran about the room, giving full evidence of the retention of all its special senses and powers of motion

7 P M. Sits in the cage diligently occupied in examining and picking its hands and feet and woollen jacket

Has rather a stolid look, and makes no attempt to move away when a hand is put out to lay hold of it

Formerly it was very timid and disliked being touched

8 P.M Eagerly drank some sweet tea

When let out of its cage it walked about a little, and then sat down and went on with its usual employment of examining and picking at its hands, feet, and coat.

9.30 P M Found sitting in same position at the same employment Takes no notice of whether the room is suddenly lightened or darkened, but goes on with its occupation all the same

11.30 P M. Found asleep on its perch On the gas being turned up and the animal awakened, it began to examine its hands &c. as before The cat happening to come into the room caused it to give a shriek and appear terrified.

March 17*th*.—8.30 A M Ate some breakfast, came out of its cage when the door was opened, and marched about the room. An hour or two afterwards another monkey was placed in the cage beside it. Its companion examined it with curiosity, but it sat quietly and made no sign of interest Gradually sidled up to it, however, and sat hugging it, enjoying the warmth of contact

2 P M When let out of its cage it ran about a little followed by its companion. After a few minutes sat down in a corner and began to examine and pick its hands, feet, and tail. Makes no resistance when its companion pulls it about rather roughly and examines its head

5.30 P.M. Condition remains as before. Frequently gets a tug and a bite from its companion, who seems annoyed at its occasional restless movements.

Later in the evening when examined it exhibited no new symptoms.

March 18*th*.—10 30 A M. Remains as before Sensation and voluntary motion are unimpaired Eats and drinks heartily, finding its own food in the cage.

When the door of its cage is opened it comes out and runs about, and then settles down quietly in a corner Allows itself to be touched and taken hold of, which its companion resents very much Has lost its former timidity and shyness Pays no attention to any thing going on around, but sits picking its hands and feet unless directly disturbed, when it gets up and runs about.

8 P M Remains in same condition. Eats and drinks heartily.

March 19*th* —10 A M. Ate breakfast heartily When taken out of its cage it ran about the room in a wild manner, jumping on furniture. Gives a little grunt of recognition when called to by name. After running about it subsided into a dull stupid-like state, scratching its sides occasionally or the edges of the wound, which would seem to itch. The wound looks tolerably dry and healthy

Later it sat down by the fire close to its companion, but occasionally got up and made some restless movements, whereupon it got a tug or bite from its companion, who seemed to lose patience with its waywardness.

5 P M Found in a dozing state, but woke up and drank some tea and ate some bread and butter, after which it again subsided into a dozing condition

March 20*th* —9 A M Ate and drank There is no difference observable as regards sensation or voluntary motion. During the day sat quietly except when roused, when it would get up and run about wildly for a few minutes and then subside into its sleepy condition

5 30 P M. Gave a screech when the cat was brought into the room, but after a short interval walked up to it half in terror and half showing fight.

Continued much the same as before during the rest of the day.

March 21*st*.—11 A.M Ate some breakfast, but appears much less active than before. Inclined to climb about along the inside of its cage

When taken out it ran about a little and then sat down, clinging to some object. Sees and hears as before, and other senses seem unimpaired. A few minutes after it had been let out of its cage it returned, and began climbing restlessly on the sides of its cage, occasionally resting quietly with its eyes closed as in sleep.

1 P M Was found climbing restlessly along the inside of the cage. Pays no attention to its companion, and does not seek to sit beside it as usual Utters a short grunt when called to. When taken out and placed before the fire it sat perfectly still with its head bent On being disturbed by the movements of its companion it would get up and run about a little.

It was observed that its movements were less free than before, and that it walked as if its limbs were clogged.

2 P.M. Was found sprawling against the wall of the room in a corner as if it wished to climb.

When set to move about it picked up things lying on the floor, smelt them, and occasionally put them in its mouth. Eats and drinks as usual.

5 30 P M In attempting to drink some tea, of which it was very fond, its head was observed to shake so that it could scarcely hold its lips to the fluid. When its head was held steady it drank with avidity

Fig 4

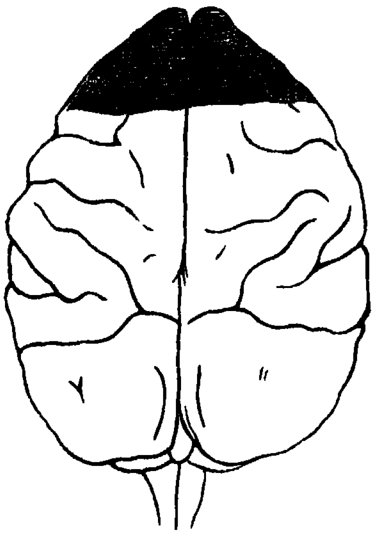


Fig 5

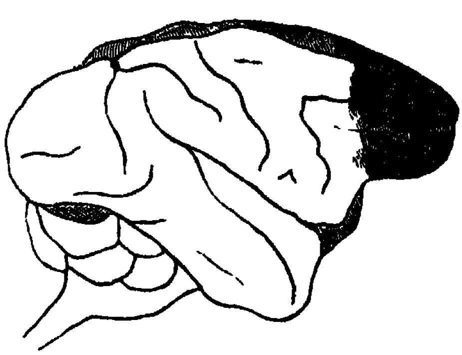


Fig 6

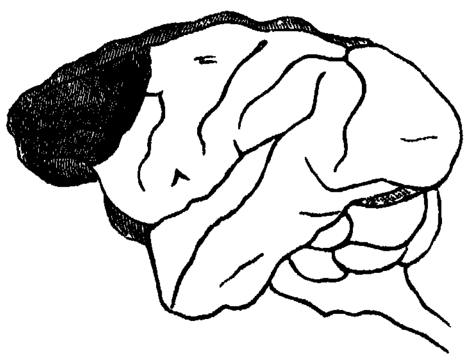


Fig 4 represents the upper surface of the hemispheres, fig 5 the right, and fig 6 the left hemisphere of the brain of the monkey The shaded parts in the frontal lobes in all the figures indicate the extent of destruction of the brain-substance in Experiment III

This paralysis agitans was taken as an implication of motor centres, and therefore the animal was chloroformed to death to prevent complications.

Post mortem Examination —On removal of the scalp the brain was found protruding on each frontal region, the herniæ reaching the under surface of the scalp. The sur-

faces were suppurating slightly. The edges of the bone looked healthy, and there was no œdema of the scalp or surrounding parts

The dura mater was of normal appearance, and stripped readily from the surface of the hemispheres, which looked somewhat "wet" but otherwise normal.

On removal of the brain the base and cranial nerves were all found intact. The olfactory tracts and bulbs had escaped injury, though the bulbs were slightly covered with pus.

On opening the ventricles slight excess of fluid was found in them, but the ganglia were quite normal in appearance The anterior cornua of the ventricles had not been penetrated.

The abnormal appearances were entirely confined to the frontal lobes The hernial prolongations were of the size of the openings in the frontal bone, and were bounded by a sharp line somewhat congested, indicating the line of section of the lobes.

In the right hemisphere the line of section struck the anterior extremity of the supero-frontal sulcus, and sloping somewhat downwards and forwards had struck the orbital surface in a plane anterior to the superior line of section.

In the left hemisphere the line of section was situated slightly posterior to that on the right, cutting across the supero-frontal sulcus, and sloping forwards like the right. The posterior half of the orbital surface was intact on both sides

The softening at the margins of the section did not extend into the antero-parietal sulcus

There was some softening between the lips of the longitudinal fissure at the base, but this did not extend beyond the perpendicular plane of section

The septum lucidum was uninjured

The rest of the brain was intact

An analysis of these three experiments elicits, with individual differences, certain common and fundamental facts. They show conclusively that an animal deprived of its frontal lobes retains all its powers of voluntary motion unimpaired, and that it continues to see, hear, smell, and taste, and to perceive and localize tactile impressions as before. It retains its instincts of self-preservation, retains its appetites, and continues to seek its food. It is also capable of exhibiting various emotions The result, therefore, is almost negative, and the removal of a part of the brain which gives no external response to electric stimulation exercises no striking positive effect, and yet the facts seem to warrant the conclusion that a decided change is produced in the animal's character and disposition For this operation I selected the most active, lively, and intelligent animals which I could obtain. To one seeing the animals after the removal of their frontal lobes little effect might be perceptible, and beyond some dulness and inactivity they might seem fairly up to the average of monkey intelligence. They seemed to me, after having studied their character carefully before and after the operation, to have undergone a great change While conscious of sensory impressions, and retaining voluntary power, they, instead of being actively interested in their surroundings, ceased to exhibit

any interest in aught beyond their own immediate sensations, paid no attention to, or looked vacantly and indifferently at, what formerly would have excited intense curiosity, sat stupidly quiet or went to sleep, varying this with restless and purposeless wanderings to and fro, and generally appeared to have lost the faculty of intelligent and attentive observation.

Perhaps this condition may be attributed to the constitutional disturbance excited by the operative procedure alone, but the effects of this are capable in a great degree of elimination, and in the record of subsequent experiments it will be seen that after operations of equal severity marked differences are observable according to the part of the brain which was destroyed The animals seem to bear the operation with comparatively little constitutional disturbance, and this is testified by the fact that they continue to eat and drink heartily within a few hours, and often less, after a large portion of the brain has been removed

The phenomena occurring towards the latter end of the periods of observation are more to be regarded as signs of constitutional disturbance, and as indications of the advance of inflammatory softening or morbid process into other cerebral regions The spasmodic motor affections, as well as the paretic condition seen in regard to certain movements, are to be explained by the implication of motor centres, the nature and position of which will be illustrated in the next series of experiments.

Destruction of Motor Areas—Regions of the Fissure of Rolando.

In my former Memoir I have related the results of electrical irritation of regions situated in the immediate neighbourhood of the fissure of Rolando, which show that certain definite and purposive movements of the hand, foot, arm, leg, face, and mouth result from the electrical stimulus applied to individual areas capable of more or less exact localization The experiments next to be related have reference to the effect of destruction of these centres, collectively and individually, on the power of voluntary motion.

Experiment IV

June 18*th*, 1873.—The right hemisphere of a monkey had been partially exposed and experimented on for the purpose of localizing the regions of electric stimulation

The part exposed included the ascending parietal and postero-parietal convolutions, the ascending frontal, and the posterior extremities of the three frontal convolutions. After having been under experimentation for eight hours the animal recovered sufficiently to sit up and take food. The wound was sewn up, and the animal placed in its cage.

June 19*th*.—The animal is apparently as well as ever, eating and drinking heartily, and as lively and intelligent as before. No change was perceptible during the whole of this day.

June 20*th*.—The wound was oozing, and the animal was less active, but there was

no diminution of sensation or voluntary motion. It closely watched flies buzzing about, and frequently made attempts to catch them.

Towards the afternoon it began to suffer from choreic spasms of the left angle of the mouth and of the left hand There was no loss of consciousness The animal was apparently annoyed by the spasmodic action of its mouth, and frequently endeavoured to still them by holding its mouth with the other hand

Towards the close of the day the spasms frequently repeated, became more intense, and exhibited an epileptiform nature, the convulsions of the left side of the body becoming general

This state continued till

June 23*rd*—Left hemiplegia had manifested itself The angle of the mouth was drawn to the right, the left cheek-pouch was flaccid and full of food, there was almost total paralysis of the left arm, and partial paralysis of the left leg The pupils were equal, and there was no paralysis of the left eyelids apparently The animal still maintained an intelligent aspect, but seemed disinclined to move on account of the powerlessness of its left side

June 24*th*—Hemiplegia is complete on the left side, hand, foot, and face The animal moved by means of its left limbs, dragging the right after it

The animal died from exhaustion on the 27th

Post mortem Examination—The whole of the exposed part of the brain was in a state of softening and suppuration, projecting through the opening of the skull

The extent is indicated in figure 7

Fig 7

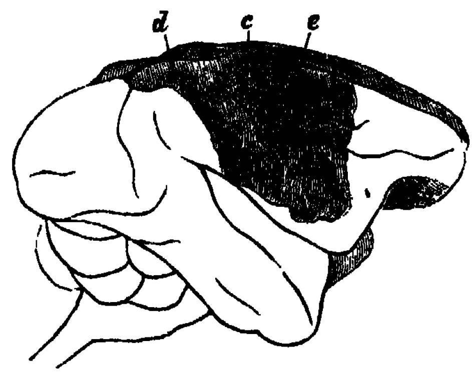


Fig 7 represents the right hemisphere of the brain of the monkey The shaded part indicates the extent of destruction of the grey matter in Experiment IV
c=the fissure of Rolando
d=the postero-parietal lobule or upper end of the ascending parietal convolution
e=the ascending frontal convolution

The brain otherwise was normal The softening was confined to the surface of the hemisphere, and did not extend to the ganglia, which were normal

In this experiment we have a general affection of the whole of the motor region of the right hemisphere, beginning with inflammatory irritation, which showed itself in choreic spasms passing into general epileptiform convulsions, and ending ultimately, as softening advanced, in complete left hemiplegia

This result followed destruction of the cortical motor centres alone

Experiment V

January 5*th*, 1875—A macaque of large size was placed under the influence of chloroform, and the ascending frontal, ascending parietal, and postero-parietal convolutions of the left hemisphere exposed

Electrical irritation was applied, and the movements already related as following

stimulation of these regions produced. The animal was allowed to recover consciousness completely at 5 P.M. It remained for two hours as well as before to all appearance.

At 7 P.M. by means of the blowpipe cautery the surface of the postero-parietal lobule (foot-centre), of the ascending parietal (hand and wrist centre), with a small portion of the upper extremity of the ascending frontal convolution (arm and leg centres) were destroyed

Though the animal was quite conscious it expressed no sign of pain or uneasiness during the process Once during the passage of the cautery along the ascending parietal convolution a partial closure of the fist occurred, seeming as if the heat had caused in some degree the same effect as the electric stimulus

On being set free the animal jumped away, but staggered and fell over on its right side. It was observed that when the animal moved, it did so by the aid of the left arm and left leg, dragging the right leg on the floor. When it rested, the right leg was seen to straddle outwards, as if the power of adduction had been lost There was no muscular resistance to the free movement of the ankle in any direction, but there was resistance to forcible extension of the leg. The right arm was kept flexed at the elbow, but the wrist dropped and the hand hung flaccid. There was no resistance offered to flexion and extension of the wrist, but decided muscular resistance to straightening the arm The animal made no use of its right hand to grasp, or in progression, but it retained the power of flexing the right forearm

The sensibility of the right side was unimpaired, as judged by the expression of pain and annoyance when the limbs were pricked or pinched.

The great difficulty it experienced in walking, or sitting steadily upright, caused the animal to growl in annoyance each time it staggered

Otherwise the animal was well, and ate and drank as before within an hour after the operation.

The animal was then subjected to an experiment for destruction of the angular gyrus (see Exp VIII.), and its further history and the results of the post mortem examination are detailed under Exp. VIII

This experiment demonstrates very conclusively that the destruction of cortical centres, irritation of which by the electric stimulus gave rise to very definite movements of the hand and foot, caused motor paralysis of the same movements and of none other, and, as will be found, the paralysis remained permanent up till the time of death

Experiment VI

February 26*th*, 1875.—A monkey was chloroformed, and the left hemisphere was exposed on the region which former experiments had indicated as the centre for the biceps (*f*, fig. 8). By electrical irritation the region was accurately defined, and the grey matter destroyed by means of the blowpipe cautery The animal was conscious, and lay perfectly quiet during the operation, though unbound When placed on the floor the animal sat very unsteadily; and the cause of this was seen to be that the right

arm hung by the right side in a state of flaccid extension. When urged to move it used the left limbs and the right leg as before, but had lost the power of flexing the right arm. In trying to walk, it frequently fell over on its right side.

An hour after the operation the paralytic condition of the right forearm remained very marked; the loss of voluntary power was confined to the same action as was excited by the electric stimulus.

The animal died from an overdose of chloroform when about to be subjected to a further operation.

Post mortem Examination.—The only lesion in the brain was a cauterized spot of the size of a threepenny bit, corresponding to the bicipital centre in the ascending frontal convolution (see fig. 8).

Fig. 8.

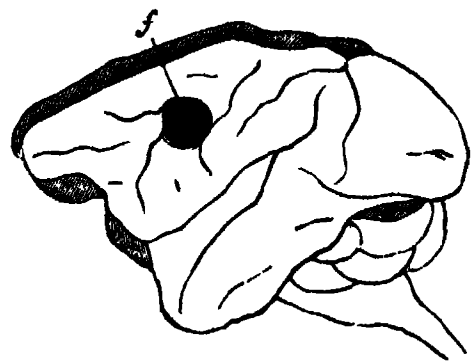


Fig. 8 represents the left hemisphere of the brain of the monkey.

The shaded spot on the ascending frontal convolution marked by the letter *f* indicates the extent to which the grey matter of the surface had been destroyed in Experiment VI.

These three experiments, besides others where the same regions became involved indirectly as the result of other experiments*, afford a simple and conclusive proof that the movements which are excited by the application of the electrodes to the surface of the hemispheres in these regions are due to excitation of the grey matter of the cortex, seeing that destruction of these same areas causes paralysis of the same movements, while sensation remains unaffected†.

In the first experiment the more or less complete destruction of the cortex in the region of the fissure of Rolando caused complete hemiplegia on the opposite side of the body, affecting all the unilateral movements capable of being called into play by the electric irritation. In the next two, only those movements were paralyzed which had their special centres destroyed in the cortex of the opposite hemisphere.

* See Experiments VII. and X.

† I am aware that the conclusion here stated, and which seems to me well established by the above facts, apparently stands in diametric contradiction to the conclusions which HERMANN ('Archiv für Physiologie,' Band x. Hefte 2 & 3, p. 77) has arrived at from a few similar experiments on the motor centres of the brain of dogs. He concludes that because dogs ultimately recover completely from such disturbances of motor functions as are at first caused by the ablation of cortical centres, these centres cannot be motor in the true sense of the term. Experiments on dogs, however, are not strictly comparable with experiments on monkeys, and the relative subordination and association of lower centres in different animals is a fact which ought to be carefully considered. The explanation I have elsewhere given ('West Riding Reports,' vol. iii.) of how associated movements, such as those of the limbs of dogs, can still be carried out through the associated action of lower centres so long as the cortical centres of the other hemisphere are intact, is quite in harmony with the facts HERMANN gives, and is further demonstrated by the complete paralysis of *voluntary* motion which follows the destruction of corresponding regions in both hemispheres in these animals.

*Experiments relating to the Localization of Sensory Perception**.

Certain movements of the eyes, ears, and nostrils, obtained by stimulation of certain convolutions already described, led me to regard them as the external manifestations of sensations thus subjectively aroused, and the following experiments were directed to test the truth of this hypothesis, and to determine to what extent sensory localization in the brain might be possible.

Destruction of the Angular Gyrus.

As already related, electric stimulation of this convolution caused movement of the eyeballs to the opposite side, with a direction upwards or downwards, according as the anterior or posterior division was stimulated, and frequently the pupils contracted and the animal tended to close the eyes

Experiment VII.

November 18*th*, 1873 —The angular gyrus of the left hemisphere of a monkey was exposed, and after electric irritation, causing the movements already described, the whole of this convolution, with the upper part of the superior temporo-sphenoidal convolution situated between the two limbs, was seared and destroyed with the galvanic cautery (see fig 9) The left eye was then securely sealed up with plaster, and the animal left to recover from its chloroform stupor

Fig 9

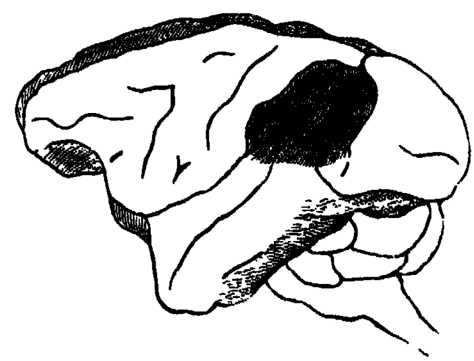


Fig 9 represents the left hemisphere of the brain of the monkey

The shading which occupies the whole of the angular gyrus and the upper angle of the superior temporo-sphenoidal convolution indicates the extent to which the grey matter was cauterized in Experiment VII

A few minutes after it began to struggle a little, as if endeavouring to rise, but was unable to get on its legs Half an hour after it sat up, and began to grope about cautiously, but made no efforts at progression. It made no sign when a light was approximated to its eye It did not flinch when lifted up and its face brought quite up to the light

It had retained its sensation as regards hearing and touch, starting if a noise was made, and expressing annoyance if it was pinched

When placed in its cage beside two other monkeys, it clung to the bars of the cage, and took no notice of its companions It would not stir from the position it assumed A little later sat down in its cage, with its head covered with its hands

An hour having elapsed, it was taken from the cage and the left eye unbandaged.

Immediately on this being done, it looked around, and seeing the door of the cage open, ran nimbly and made its way among its companions.

* By this term, as also by the term "sensation" which I sometimes use, I wish to signify the fact of conscious discrimination of impressions as distinct from the mere sensory impressions themselves

When taken out again, and the door shut, it ran back, looking at its companions, and desirous to gain admittance.

When held up to the light it flinched and averted its head.

The transition after the bandage was removed was of a striking character, and indicated an evident restoration of sight which had been lost.

Next day (Nov. 19) the animal looked perfectly well, running about, eating and drinking as usual

An experiment was then made with the view of ascertaining whether the blindness of the right eye had continued. The left eye was again bandaged up as before, and the animal placed on the floor. It immediately ran up to the cage, and putting its hand through the bars into a dish of water began to lap it.

Sight had therefore returned, notwithstanding the destruction of the angular gyrus on the left side.

The animal died on Nov. 24 from suppuration and necrosis of the skull, having also become paralyzed on the right hand.

Post mortem Examination.—The angular gyrus and the ascending parietal convolutions were softened, and the hemisphere fungating from the orifice in the skull. The abnormal appearances were confined to the surface of the hemisphere. No drawing was made of the exact extent of the softening, but the paralysis of the right hand coincided with the destruction of the ascending parietal convolution. This experiment served to show that destruction of the angular gyrus resulted in blindness of the opposite eye, and that this loss of visual perception was only of temporary duration, compensation having been effected within a period of twenty-four hours.

Experiment VIII

January 5th, 1875.—The subject of this experiment was the same monkey spoken of under the head of Exp. V

Two hours after the destruction of the motor centres alluded to, the animal was again chloroformed, and the angular gyrus clearly exposed, the left eye closed with plaster, and the animal allowed to recover

On returning to consciousness it followed my movements with its right eye, and indicated its sense of hearing by turning its head and looking when called to Took some fruit offered to it in its left hand, and sat contentedly eating it. It seemed disinclined to move on account of the motor paralysis of its right side.

It sat with the right leg doubled up under it, and resting the internal malleolus on the floor. Sometimes it supported the right hand with the left. Expressed annoyance when pinched. The animal having thus recovered from the operation of exposure of the brain, it was taken and the angular gyrus carefully destroyed by means of the cautery, no more than two hours having elapsed since the first operation.

When let loose, it moved about a little when nudged, but would not move of its own accord. When forced to move, it avoided obstacles as if it still saw. On exami-

nation it was found that the bandage had slipped, and that the left eye was partially open. On this defect being remedied, it put up its left hand, and tried to pull the bandage from the eye. On this being prevented, it sat still and would not move When pushed and forced to move on, it ran its head against every thing in its way.

When removed into another room it sat still with its head bent, and would not stir Would not come when called to

When taken back and placed beside its cage it still refused to move, and grunted annoyance if disturbed, or rushed with its head against any thing in its way. After it had remained for an hour in this condition the bandage was removed from its left eye On this being done, it began to look around, and on being called to by name, ran to me and tried to climb on to my knee as it had used to do This it did on three separate occasions.

The difference in its attitude after the bandage was removed was as striking as in Exp VII, and indicated restoration of sight

January 6th.—On account of the paralytic condition of its right side, and the suppuration going on in its wound, it was chloroformed to death

Post mortem Examination—The postero-parietal lobule, ascending parietal, and upper part of the ascending frontal convolutions, with the angular gyrus were softened and disorganized (see fig. 10). The rest of the brain was quite normal in appearance

Fig 10

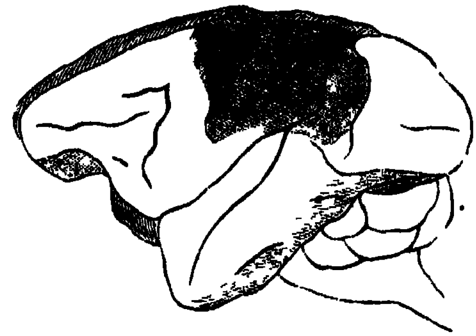


Fig 10 represents the left hemisphere of the brain of the monkey, the shaded part indicating the extent to which the surface was destroyed in Experiment VIII

Experiment IX

April 7th, 1875—This animal was used for an experiment on the superior temporo-sphenoidal convolutions on both sides. These were exposed, but previous to their destruction the angular gyrus was exposed on the left side for the purpose of demonstration of the effects of destruction of this convolution to Dr. BURDON SANDERSON and Dr. LAUDER BRUNTON. At 3 30 P M. the angular gyrus was exposed, and its surface destroyed accurately by means of the blowpipe cautery.

The left eye was securely closed by means of plaster, and the animal placed on the floor

After a few minutes it began to move about, which it did very irregularly, sometimes going backwards, and occasionally turning round and round.

4.20 P.M. The animal is more lively, but sprawls about on the floor, and does not make any regular progression. Drank some tea held to its lips.

4.55 P.M. Answers with a grunt, or makes mouths when called to. Sprawls about on the floor or goes backwards. When placed close to the door of its cage makes no

attempt to enter or seek its companion, who calls for it anxiously. When urged to move, it ran against obstacles held in its path.

It was adjudged to be blind.

5 P.M. The bandage was now removed from the left eye. After a few moments of apparent stupor and unwillingness to move, it ran when touched, avoided obstacles which formerly it had run against, and made its way to its cage and jumped up beside its companion.

The animal had evidently recovered its sight.

On this being established it was again placed under chloroform, and the superior temporo-sphenoidal convolution was destroyed in both hemispheres The results will be recorded subsequently (see Exp. XV p. 461)

Next day (April 8) at 12 noon it was taken out of its cage, and the left eye bandaged up as before, much against the animal's will When let loose it made a spring at me, and then galloped away into the other room and made for its cage Followed its companion out of the cage a short time after, and found its way in again and jumped on the perch Retired from the perch when I approached making mouths.

Vision therefore had returned in the right eye.

The subsequent history and post mortem examination of this animal will be found on p 461 *et seq* under the head of Exp. XV

This experiment completely confirms the former two as to the fact of blindness being caused in one eye on the destruction of the angular gyrus of the opposite hemisphere.

The important fact noted in Exp VII. is also confirmed, viz. that within a very short period visual perception becomes again possible with the same eye, notwithstanding the lesion.

The next experiment relates to the effects of destruction of the angular gyrus on both sides

Experiment X

January 8*th*, 1875—The angular gyrus was exposed accurately and clearly in both hemispheres of a monkey, and the animal allowed to recover from its chloroform-stupor 2.45 P.M.

At 3 P M the animal had almost recovered, but was somewhat unsteady Looks around, and turns its head when called to, and makes mouths as before.

3 30 P M When taken away from the fire before which it had been sitting, it ran back to its position, looking back at me, making grimaces and mouths.

It drank with avidity some sweet tea, of which it was exceedingly fond on all occasions. When the dish was removed to the other side of the room away from the fire, it ran to it and drank it up

When a light was flashed before its eyes, it turned away its head and tried to conceal its face in its hands.

4 P.M. The animal having completely recovered from the operation and being in full

possession of all its powers, it was taken and the angular gyrus destroyed on both sides by means of the cautery.

The operation was finished at 4 35

The animal when placed on the floor uttered a cry and looked about in a scared manner

Pricked up its ears and cried when called to.

Sat up quite steadily, but would not move.

The pupils reacted to light

4 55 P.M. A light flashed before its eyes caused it to wince and erect its head. When placed beside the fire it sat up, enjoying the heat.

When removed from the fire it lay down, and would not move from its position even when nudged

Turned its head sharply when called to by name

When taken hold of clung violently to me, in terror at being placed down again.

When placed beside the fire sat contentedly enjoying the heat Made no sign of perception when the room was suddenly darkened and lightened

5 30 P M. Sits quietly by the fire A piece of apple dropped beside its hand caused it to lay hold of it, and after smelling eat it When taken away from the fire and placed on a chair, lay down and refused to stir.

There is no paralysis of motion or sensation unless of sight; and this is difficult to ascertain beyond all doubt, as no crucial test seems applicable

8 P M. The question of sight was decided in the following manner A dish of sweet tea, of which it was fond, was placed to its lips, whereupon it drank greedily, keeping its mouth in the dish as it was lowered, but on the dish being withdrawn from immediate contact and placed on the floor quite under its nose, the animal was unable to find it, though exhibiting a desire to do so This was repeated several times with the same result. On the dish being raised to its lips it drank eagerly, and followed it with its mouth immersed until every drop was exhausted, the dish being drawn along the floor for some feet.

January 9th.—11 A M The animal is alive and well, and retains its muscular power and senses, except sight It eats and drinks with avidity whatever is brought up to its mouth, but is unable to find its food when it is removed from immediate contact.

Will not move from its place, but remains quite still with its eyes open The pupils are equal and active. An object waved in front of its eyes causes wincing only if closely approximated to the eyes

A threatened blow with a stick causes no reaction, unless when brought almost in contact with its eyes.

The left wrist seemed slightly dropped, and not used like the other. With this exception all the voluntary movements were unimpaired.

To avoid the complication of extension of softening to other regions, the animal was killed with chloroform at 12 noon.

Post mortem Examination.—Slight suppuration existed at the margins of the wound and under the scalp, and there was some œdema of the cellular tissue over the orbits. The skull was deficient in the region of the parietal eminences

The brain-surface corresponding to this opening was slightly elevated above the rest.

The surface of the brain was everywhere normal, except in the region of the angular gyri.

The surface of these convolutions was destroyed on both sides. Slight softening extended about a line into the adjoining margin of the occipital lobe on both sides, slightly more on the left than on the right (see figs. 11, 12).

Fig 11

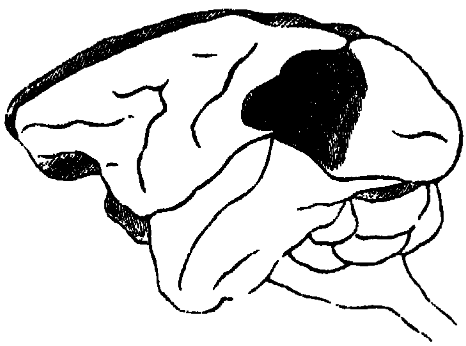
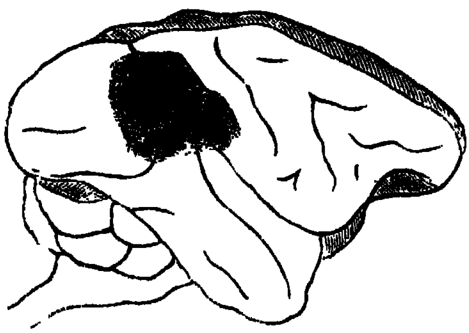


Fig 12.



Figs 11 & 12 represent the left and right hemispheres of the monkey respectively The shaded portions indicate the extent of destruction of the surface of the hemispheres in Experiment X

The lower part of the ascending parietal convolution of the right side was also slightly involved

The base of the brain, the ganglia, and the optic tracts were uninjured

This experiment completely confirms the other three as to the effect of destruction of the angular gyrus or the power of visual perception

The slight affection of the left wrist is explained by slight invasion of the right ascending parietal convolution by the process of softening

These four experiments demonstrate conclusively that unilateral blindness of a complete character results from destruction of the angular gyrus of the opposite hemisphere, and that this unilateral blindness is only of temporary duration, provided the angular gyrus of the other hemisphere remains intact, while permanent blindness results from the destruction of the angular gyrus in both hemispheres Further proof of this will be found in Experiment XXI

The loss of visual perception is the only result of this lesion, the other senses and the powers of voluntary motion being retained so long as the lesion remains confined to the angular gyrus itself.

By the term visual perception I wish to indicate the consciousness of visual impressions, and to distinguish this from mere impressions on the optical apparatus and reactions which are only of a reflex nature, such as the sudden start which an animal really

blind in the sense in which I use the term may make when a light is flashed before its eyes.

Retinal impressions and reflex actions resulting from these are left unaffected by the lesion which abolishes the perception of visual impressions.

Effects of Lesions of the Temporo-sphenoidal Lobe

The experiments recorded under this heading relate to more or less general, as well as limited, lesions of the convolutions of this lobe. As it is difficult to reach and localize lesions in the individual convolutions, the exact effects of the destruction of any one part have to be arrived at in a great measure by a process of exclusion, besides that of direct experiment on each separate region

The effects of electrical stimulation have been already recorded

Irritation of the superior temporo-sphenoidal convolution always gave very definite results, viz. pricking of the opposite ear, opening of the eyes and dilatation of the pupil, with turning of the head and eyes to the opposite side.

That these phenomena were the indications of excitation of subjective auditory sensations seemed probable, both from experiments on monkeys and other animals.

Stimulation of the posterior division of the third external convolution in cats, dogs, and jackals is usually followed by sudden pricking of the opposite ear In rodents a similar effect results from stimulation of an homologous region.

A very marked effect I observed in the case of a wild jackal, on stimulation of the posterior division of the third external convolution The animal suddenly started, pricking up both ears, and would have bounded off the table had it not been securely fixed

The phenomena were just such as would have resulted from a sudden alarm A similar result I observed in a rabbit on which I was experimenting

That the movements resulting from irritation of the superior temporo-sphenoidal convolution in monkeys resemble those caused by a sudden sound is seen by the following experiment.—

A monkey was placed on a table, and a loud whistle made close to its ear. Immediately the ear became pricked up, the animal turned its head to the same side, opening its eyes widely, while the pupils were observed to be dilated. The dilatation of the pupils was not observed in every case when the experiment was repeated, but the other phenomena were the same

The effect of irritation of the lower end of the uncinate convolution (subiculum cornu ammonis), viz. torsion and closure of the nostril of the same side, is evidently to be taken as the indication of excitation of subjective olfactory sensations, and is precisely similar to the effect of irritating the olfactory bulb itself, as I have ascertained by direct experiment.

The following experiments serve to demonstrate the accuracy of the views at which I had arrived.

Experiment XI.

December 10*th*, 1873.—The left hemisphere of a monkey was exposed in the regions of the ascending parietal convolution, the postero-parietal lobule, the angular gyrus, and the upper part of the superior and middle temporo-sphenoidal convolutions

After experimentation by means of electric irritation on these regions, the temporo-sphenoidal lobe was deeply divided with the galvano-cautery in a line nearly coinciding with the direction of the lower temporal fissure (see fig 13), and the substance of the superior temporo-sphenoidal and middle temporo-sphenoidal convolutions destroyed and scooped out throughout their upper two thirds approximately

After the operation the animal retained sight, and apparently heard as before, as judged by its reaction to sounds

The condition as to smell and taste is exceedingly difficult to determine accurately.

As to smell, there is hardly any odour, pure and simple, which will cause distinct manifestation of olfactory sensation in a monkey, and one must study the habits of the animal carefully, or employ some volatile substance which will cause reaction. These, however, such as ammonic and acetic acid, act conjointly on the nerves of common sensation and on the special nerve of smell. I have found, however, by careful experimentation on a patient who had lost both taste and smell as the result of a blow on the head, that ammonic and acetic acid, and particularly the latter, cause much less reaction than they do when both systems of nerves are intact.

Confirmations of this will be found among the experiments narrated.

The reaction to acetic acid, which I frequently used to test the sensibility of the nostrils, is only a comparative test, and reaction caused by it, when applied to the nostril, is not to be regarded as an indication of smell, but the absence of reaction would show that the sensibility of the nostrils had been entirely lost, while a less reaction in one nostril as compared with the other would fairly indicate some abnormal condition of the nostril, the exact cause of which is capable perhaps of explanation by other facts.

In this case the reaction to the vapour of acetic acid was distinctly less in the left nostril than in the right. (The left nostril is, as will be noted, the same side as the lesion in the hemisphere)

As to taste, no exact experiment was made The right side of the tongue was touched with a rod dipped in perchloride of iron; but, owing to the nature of the substance and the diffusion in the mouth, nothing could be ascertained accurately, though I thought that there seemed to be less immediate reaction on the right side than on the left

The animal had not lost its appetite, for it drank milk and ate some food offered to it.

Hearing, as was noted, did not seem affected, as the animal reacted as usual to sounds, turning its head, &c.

As the animal had, however, its left ear and right hemisphere intact, I plugged up the left ear securely by means of cotton-wool, in order to ascertain whether it heard in reality only with the right

On this being done, sounds which formerly caused the animal to prick its ear and look round, failed to cause any reaction or excite its attention.

Sounds made by concussion caused the animal to look round, as well as the making of any sound which likewise attracted its attention by sight.

Whether, therefore, the animal heard or not, it gave no sign of such sensations being aroused.

It was also found that reaction to pricking and pinching was considerably less on the right than on the left side, though not completely abolished

The animal died next day in a comatose condition.

Post mortem Examination.—The injury to the brain involved the convolutions to the extent described.

The division was carried down to the hippocampus, which, however, was not severed, and the lower part of the uncinate convolution and of the temporo-sphenoidal convolutions still remained, though almost severed from the rest of the temporo-sphenoidal lobe

This experiment only gave partial indications of impairment of certain senses, particularly of hearing and smell, and in some degree of tactile sensation, and is chiefly important in relation to the other experiments to be described

Fig 13

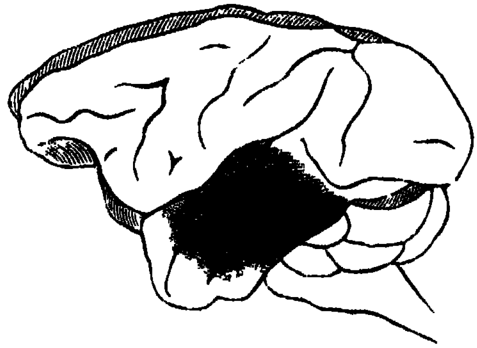


Fig 13 represents the left hemisphere, and the shaded part indicates the seat of lesion in Experiment XI The deep shading in the centre is intended to represent the part at which the temporo-sphenoidal lobe was deeply divided transversely almost as far as the hippocampus major The lighter shading represents the extent to which the surface of the convolutions was destroyed

Experiment XII

January 27*th*, 1875.—The left hemisphere of a lively and intelligent monkey was exposed by a trephine opening in the region of the annectent gyrus connecting the posterior limb of the angular gyrus with the occipital lobe, and the upper part of the superior and middle temporo-sphenoidal convolutions further exposed by the bone-forceps.

With the cautery the convolutions exposed were thoroughly cauterized and the grey matter destroyed scooped out, while the cautery was directed horizontally inwards, so as to divide the lobe transversely as far as possible, taking care to avoid sinking it so deeply as to injure the crus (See fig 14, where the darkest part of the shading indicates the region of the greatest depth)

The operation was completed at 4 P M.

After a few minutes the animal recovered from its stupor, and began to look around. Endeavoured to get up, but staggered towards the right side. Gradually recovered its equilibrium.

On being placed on a chair it gave evident proof of its retention of sight by jumping on to the table, and running to a dish containing milk, and drinking up the contents.

There is distinct reaction on both sides when a hot iron is applied to the skin. The animal starts, and rubs vigorously the part touched.

The extent of its hearing and smell were not ascertained at this time.

January 28th.—10 A.M. The animal is alive and well. Ate its breakfast as usual.

Can walk and jump about, and sees distinctly, as it puts out its hand and lays hold of objects before it.

In order to ascertain its condition as to hearing and smell, the right nostril and the left ear were tightly stopped with cotton-wool.

When offered a piece of apple it hesitated eating it, placing it to its nostrils over and over again, apparently as if it had difficulty in smelling.

Does not pay any attention when a noise is made, such as formerly caused it to respond actively.

Tactile sensation seemed unimpaired on both sides.

5 P.M. The left side of the scalp has become œdematous. The left eye is partially closed by œdema of the eyelids. Eats heartily.

Took a piece of apple offered it in its *left* hand.

On testing the right side by means of the hot iron there was a marked diminution of reaction on the ear, hand, and foot of the right side, as compared with the left.

Sight continues unimpaired. Smell and hearing are considered as impaired, smell on the left, and hearing on the right. It is difficult to ascertain by any crucial test whether they are gone on these sides.

January 29th.—10 A.M. The eyelids are œdematous. Ate some breakfast. When taken out of its cage sat still, unwilling to move. Takes every thing offered to it in its left hand. The animal drinks out of a dish, holding its head sideways, keeping the left side of its lips in contact with the fluid.

On testing with the hot iron there is very marked diminution of reaction over the whole of the right side of the body as compared with the left.

There is no loss of muscular resistance in the limbs of the right side. They do not hang flaccid as in motor paralysis. There is no facial distortion.

The limbs are occasionally moved, but they are not used by the animal in grasping or progression.

The foot and hand are frequently rested on the floor in irregular and what otherwise would be uncomfortable and unnatural positions.

The animal occasionally scratches its left side with its left hand. Occasionally utters a discontented grunt. Retains its intelligent look, and takes notice of what is going on around it.

It was killed with chloroform at 11.15.

Post mortem Examination.—The scalp was œdematous and the wound suppurating. From the opening in the skull a hernia cerebri of the diameter of a walnut was observed.

With the exception of this appearance on the surface, the brain otherwise was perfectly normal in appearance.

The fungus was attached to the superior temporo-sphenoidal convolution along two thirds of its extent

The lower end of this and also of the middle temporo-sphenoidal convolution were not broken down externally, but they were much congested.

The rest of the lobe was completely broken up. The lesion extended inwards, so as to appear on the inner surface of the temporo-sphenoidal lobe, leaving only a continuity of a narrow band between the lower and upper end of the uncinate gyrus (see fig 15)

The hippocampus was much softened.

The occipital lobe was intact, as also the optic thalamus.

The olfactory tract and bulb were uninjured, as also the crura, corpora quadrigemina, and corpus striatum.

This experiment is another link in the chain of evidence pointing to the association of hearing and smell with integrity of the temporo-sphenoidal lobe—hearing on the opposite and smell on the same side. The hypothetical seats of these, the superior temporo-sphenoidal for hearing and the subiculum cornu ammonis for smell, were either disintegrated or cut off by the lesion described. Though the effect is not regarded as conclusive proof of this association, it will be seen to derive importance from conjunction with other experiments to be related At the same time, however, the fact is again noted that tactile sensation was almost completely abolished on the right side This effect was subsequent to the phenomena just observed, and apparently advanced with the process of softening inwards towards the hippocampus and uncinate convolution

Fig 14

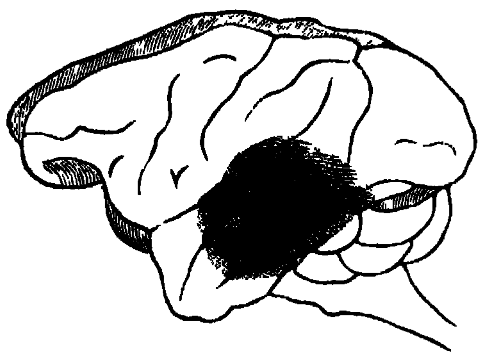


Fig 14 represents by the shaded part the extent of the lesion as seen on the outer aspect of the left hemisphere in Experiment XII The dark shading in the centre indicates the part at which the lobe was deeply injured

Fig 15

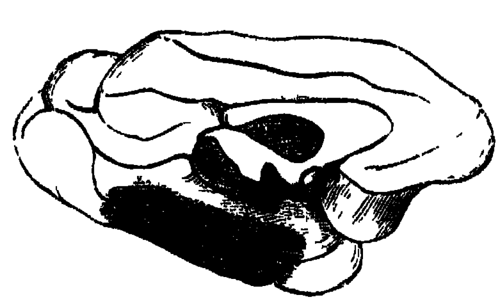


Fig 15 represents the extent of the lesion, as seen on the inner aspect of the temporo-sphenoidal lobe

Experiment XIII

February 2nd, 1875.—The brain of a monkey was exposed by trephining over the

region of the annectent gyrus passing from the posterior limb of the angular gyrus into the occipital lobe on both sides.

By means of hot wires the temporo-sphenoidal lobe was divided transversely in this region, care being taken to avoid crossing the fissure of Sylvius, and also to avoid the crura and optic tracts.

The wires were also directed downwards and forwards, so as to break up the lobe as far as possible in the interior. This was carried out much more completely on the left than on the right side

The operation was completed at 4 P.M.

4.30 P.M. The animal has recovered from its chloroform stupor, and moves about rather unsteadily.

It evidently retained its sight, as it directed its course to the fireplace, where it sat down to warm itself

5 P.M. Drank a dish of tea offered to it It sits still with its head bent on the floor, and seems disinclined to move It has no muscular paralysis, and can hold on by both feet and hands Sits, however, very unsteadily when perched on the back of a chair. Gives no sign of hearing when called to, as it used.

There is distinct reaction to the application of a hot iron to any part of its body, though there seems somewhat less reaction on the right side as compared with the left

11 30 P.M. Is more lively, and looks about intelligently, and seems to walk somewhat more steadily.

February 3rd.—10 30 A.M. The animal was found sitting quietly with its head bent. On being roused and offered some milk, it drank a very little, but kept moving its lips about in the liquid, without continuing to drink.

Made no response when a loud sound was made close to its cage.

When taken out of its cage it moved only when nudged, and then made its way to the fire, where it sat down, holding on to the fender, enjoying the heat.

When tested with the hot iron there was found to be very decided diminution of sensation on the right side, on ears, hands, and feet.

There was no muscular flaccidity of the limbs or distortion of the face A shrill sound made close to its ear caused it to start somewhat.

1 P.M. The animal was fed with milk, as it did not seem inclined to eat of its own accord

Made no sign of reaction when acetic acid was held before its nostrils or placed in its mouth

7 P.M. When acetic acid was placed within its nostrils it appeared to suffer from irritation, and at last a kind of sneeze was effected.

With the left hand it tried to clear away the offending matter from its left nostril, but made only a kind of attempt with the right hand to the right nostril, not succeeding in localizing the seat of irritation. Opened its eyes slightly when loudly called.

It uses its left hand more than the right in laying hold of any thing. Formerly it used the right chiefly.

It is very easily knocked over by a push when it is sitting, often falling quite supine.

It was again fed, as it does not seem able to feed itself.

8 P.M. The eyelids are somewhat œdematous, more so on the right than on the left

The animal sits leaning its weight chiefly on the left arm and leg. When knocked over, which is done by a slight push, it recovers itself chiefly with the left arm and leg. The right leg, when it sits, is sometimes doubled up, and rests on its outer side. It makes no use of its right arm for any voluntary movement. The left arm and leg are moved cautiously Muscular resistance continues in all four limbs There is no facial distortion

On being tested with a red-hot iron there was entire absence of reaction on the right side. The left side seems to react somewhat less than before.

The animal, in struggling when acetic acid was placed in its nostrils, moved all four limbs, but it fell repeatedly while trying to get rid of the irritation

At 9 P M. the animal was killed with chloroform.

Post mortem Examination—The skull was deficient below the parietal eminences, and the brain-substance was protruding slightly from the orifices in the skull. The dura mater stripped readily from the brain, but underneath it there was found a thin layer of extravasation over the region of the right temporo-sphenoidal lobe

In the left hemisphere (see fig 17) there was a surface corresponding to the trephine-

Fig 16

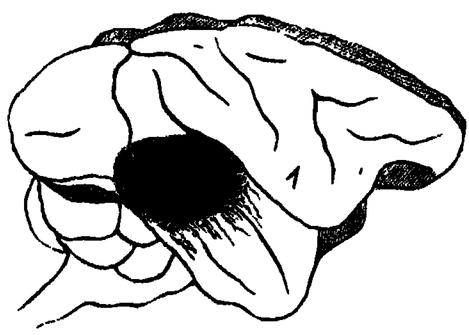


Fig 16 represents the right hemisphere, and the shaded part the extent of superficial injury in Experiment XIII. The dark shading in the centre indicates the point of greatest depth of the lesion The dotted lines indicate the extent of internal softening of the medullary matter

Fig 17

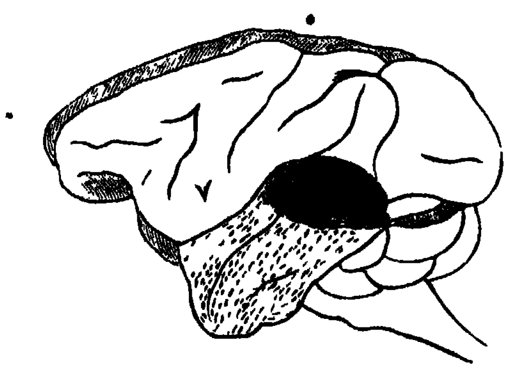


Fig 17 represents the left hemisphere, and the shading the extent of superficial lesion in Experiment XIII The deep shading in the centre indicates the line of deep transverse section, and the dotted lines indicate the extent of internal softening of the interior of the lobe

opening of about the size of a shilling, somewhat elevated above the surrounding surface. The middle of this was deeply excavated, and the division extended from behind the

fissure of Sylvius below the inferior occipital fissure to the edge of the uncinate convolution on the internal aspect.

In the right hemisphere the fungating mass occupied about the same extent as in the left, but extended somewhat further backward into the occipital lobe.

The temporo-sphenoidal lobe was not divided to the same extent transversely; but a deep excavation occupied the centre of the fungating surface, and corresponded to the level of the upper end of the middle temporo-sphenoidal convolution (fig 16) The internal aspect of the temporo-sphenoidal lobe was of normal appearance

The rest of the brain was normal

On examination after the brain had been hardened in spirit for 20 hours, it was found that in the left hemisphere the transverse division extended almost to the hippocampus. The whole of the interior of the lobe below this point was reduced to a pulp, the softening extending to some extent between the lips of the fissure of Sylvius, and affecting the surface of the island of Reil to a slight extent.

The grey matter of the lower half of the temporo-sphenoidal convolutions and of the uncinate gyrus formed a sort of shell, enclosing softened medullary substance. The hippocampus was disorganized as far as the subiculum cornu ammonis. The optic thalamus was not injured

In the right hemisphere the excavation extended to the extraventricular surface of the optic thalamus, but the hippocampus and fornix could still be seen of normal or almost normal appearance The internal or medullary surface of the superior and middle temporo-sphenoidal convolutions was softened to a slight extent below the point of greatest depth of the wound on the hemisphere The subiculum and the lower ends of these convolutions are not injured externally.

In this experiment the results as regards hearing were such as to indicate abolition, or at least considerable impairment, of reaction to stimuli which in the ordinary conditions are responded to actively So far, therefore, the theory that this is dependent on the destruction of the superior temporo-sphenoidal convolution holds good, for this convolution was divided or disintegrated almost completely on both sides.

The reaction to acetic acid in the nostrils is not to be taken as a sign of the retention of true smell, for it in all probability was more due to irritation of nerves of common sensation

The reaction, however, was decidedly diminished, and was not caused when the vapour was held only before the nostrils

The absence of reaction on the tongue points to impairment of the sensation of taste, and perhaps the want of desire to eat may have its explanation in loss of this faculty.

The experiment, however, is not regarded as conclusive, and is to be taken in connexion with other facts. It is brought out more clearly than before that the loss of tactile sensation coincides with lesion of the hippocampus and hippocampal convolution This region was quite destroyed on the left side, and loss of tactile sensation was observed on the opposite side, while on the left side tactile sensation apparently

continued good, the hippocampus and uncinate gyrus remaining intact, or at least not presenting any marked abnormality on the right hemisphere.

Experiment XIV.

March 9th, 1873.—The brain of a monkey was exposed on both sides in the region of the upper part of the superior and middle temporo-sphenoidal convolutions, and red-hot wires were passed from this point downwards and forwards, with the intention of breaking up the grey matter on the outer aspect of the lobes as far as the subiculum cornu ammonis. Owing to hæmorrhage from the left, the destruction was made more deeply than intended into the lobe in attempts to check it The operation was completed at 3 30

4.15 P.M. Is recovering from its stupor, and moves when disturbed.

4.25 P M. Begins to sit up, but seems to have some difficulty in using its right limbs

4.40 P.M Tactile sensation seems gone on the right side There is no reaction to the application of a hot iron to the right hand or foot, but slight on the ear The same heat causes violent reaction on the left side.

The animal has not yet sought to move about

4.50 P M Neither aloes nor citric acid caused any reaction when placed on the tongue Acetic acid caused no reaction when held befor the nostrils.

Tactile sensation, as indicated by reaction, is unimpaired on the left side, but there is no reaction on the right side to hot iron or pinching

Acetic acid caused no reaction when placed on the tongue.

No reaction to the application of a hot iron to the right side of the tongue, and little, if any, on the left.

The animal sits up, supporting itself with its left hand and foot chiefly. Makes no use of its right hand, but clings firmly with its left hand when about to be placed on the floor after being taken up

5.10 P.M. Acetic acid placed within the right nostril caused no reaction and no lacrymation. Placed within the left nostril caused no torsion on turning away the head, but caused a copious flow of tears from the left eye

5 40 P.M Aloes nor acetic acid applied to the tongue caused any reaction.

The animal is perfectly conscious, though it sits still, and is disinclined to move.

It gives no signs of hearing when a noise is made beside its cage

Cutaneous sensibility of the left side remains intact, apparently is quite gone on the right. The animal was placed in its cage, where it lay half asleep, but immediately roused itself when the left hand was touched.

6 P.M. While lying asleep in its cage with the tongue showing between the teeth, acetic acid was applied to the top of the tongue No reaction of any kind ensued. Applied to the left nostril no movement resulted.

A hot wire applied to the tip of the tongue caused no reaction. The same stimulus

applied to the left hand caused a sudden start, opening of the eyes, and withdrawal of the hand.

The left side of the lip and face retained sensibility.

8 P.M. A faint reaction ensued on the application of the hot iron to the right foot.

The same stimulus applied to the hand caused no reaction. The tongue remained absolutely insensible. The left side of the body gives active reaction.

9 P.M I repeated these tests in presence of Dr. LAUDER BRUNTON. The absolute want of reaction on the right side with the exception of slight reaction of the right foot, the retention of sensibility as indicated by reaction on the left side, the absolute insensibility of the tongue to stimuli of any kind, the entire want of reaction to acetic acid placed in the right nostril, and the copious lacrymation of the left eye when it was introduced into the left nostril were confirmed in his presence.

Desirous to avoid further complication after this demonstration, I killed the animal with chloroform

Post mortem Examination.—The brain, except at the points to be described, was everywhere normal. The base of the brain and the cranial nerves were intact. The fifth and the Gasserian ganglion on both sides were specially examined and found intact.

In the left hemisphere there was a wound with blackened edges, of the extent seen in fig. 18, occupying the upper part of the superior and middle temporo-sphenoidal

Fig 18.

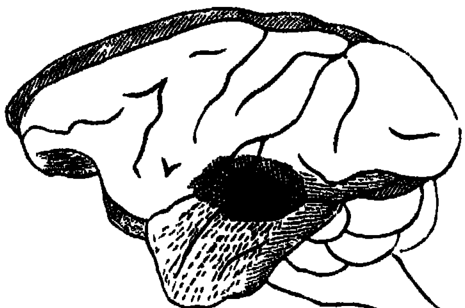


Fig 18 represents the left hemisphere, and the shading the extent of the lesion in Experiment XIV. The deep shading in the centre indicates the point of deepest excavation, and the dotted lines proceeding downwards and forwards are intended to represent the extent of internal disintegration of the lobe

Fig 19

Fig 19 represents the extent of lesion in the right hemisphere in Experiment XIV The shading indicates the extent to which the grey matter was destroyed.

convolutions The middle of this was excavated, and the lesion was continued into the interior of the lobe, the whole of which was converted into a softened mass, enclosed by a shell of grey matter on the outer and inner aspect of the lobe. The hippocampus was injured at a point opposite the wound, and was softened throughout below this. There was some blackened effusion within the lips of the fissure of Sylvius and covering the surface of the island of Reil, which, however, was of normal consistence,

and easily separable from the convolutions overlapping it. The lower part of the ascending frontal convolution was slightly softened and congested.

In the right hemisphere the surface of the lower half of the temporo-sphenoidal convolutions was quite disintegrated and blackened (fig. 19). The subiculum cornu ammonis was broken up The softening did not extend to the fissure of Sylvius. The internal aspect of the lobe, the hippocampus, and uncinate convolution were normal, except at the subiculum, as already described. No other injury existed in any part of the brain.

This experiment was followed by results of a very remarkable character

There was absence of reaction to stimuli of smell, taste, hearing, and of tactile reaction on the right side (almost complete)

As regards the loss of tactile sensibility, we have again the apparent connexion of this with destruction of the hippocampal region.

On both sides the subiculum cornu ammonis was broken down, and on both sides there was absence of any reaction indicating olfactory sensation

A peculiarity, however, existed in the comparative reaction of each nostril to the effect of acetic acid. In the left nostril, *i e* the side on which tactile sensibility remained, acetic acid caused a copious flow of tears from the left eye, while in the right nostril no effect of any kind was produced. This is evidently to be ascribed to the abolition of common sensibility as well as of true smell from the right nostril. The lacrymation was the indication of the reflex excitation of the lacrymal gland through the medium of tactile sensibility, which still continued unimpaired on the left side The absence of motor reaction, however, was an interesting fact, and serves to show how much of the reaction caused by a pungent vapour applied to the nostril is dependent on the integrity of true olfactory sensibility

As regards taste, the results indicated its entire abolition But not only taste, as such, but also the tactile sensibility of the tongue seemed to have been destroyed. This was noted as a remarkable phenomenon, and the tests were frequently repeated in order that no fallacy might be allowed to remain Not only on the right side of the tongue, but also on the left, was this absence of reaction noted The centres for the tactile sensibility of the tongue on the left side seemed to have been destroyed along with those of special sense, a fact apparently indicating their close anatomical relation in the hemisphere. The following experiments serve to narrow the boundaries of the lesions causing these various results as regards hearing and tactile sensation.

Experiment XV

April 7th, 1875 —The subject of this experiment was the same monkey used for Exp. IX.

After the animal had quite recovered from the effect of destruction of the angular gyrus on the left side, it was again chloroformed, and the superior temporo-sphenoidal

convolution was destroyed on both sides throughout the greater part of its extent, by means of the blowpipe cautery passed along the surface.

An hour after the operation (6.30 P.M.) it still staggered while walking, and looked only half awake. Made no sign when a whistle was made close to its ear or when called loudly.

Acute sensibility existed on both sides, as determined by the application of a hot iron. It rubbed vigorously the parts touched

Aloes and citric acid placed on the tongue caused great annoyance and movements of the mouth and tongue to expel the offending substance. The animal also ground its teeth, and then got up and ran about the room, grinding its teeth, and annoyed at the unpleasant sensation in its mouth

Acetic acid held before its nostrils caused it to start and sneeze and rub its nose.

When not disturbed sat quietly with its head down.

8.30 P M Found asleep in its cage. Made no sign of perception till I laid hold of it, when it started with a shriek. Looked up and ran to a dish of water and drank

Again, on trying to rouse its attention, it did not look up when a loud sound was made, though its companion looked terrified

12.30 A M. A loud sound made in the immediate vicinity of its cage caused a slight start.

April 8th.—10 A M. Animal alive and active, and jumped out of its cage when the door was opened Sight was good, and tactile sensation unimpaired Various experiments were made to ascertain the existence or not of hearing; but it was difficult to devise a test, as the animal was continually on the alert; and it was not easy to make a sound without in any way attracting its attention by sight. The following method was tried While the animal was sitting quietly by the fire, I retired to the other room, and while watching through the chink of the half-shut door called loudly, whistled, knocked on the door, tinkled glass, &c , without ever causing it to look round or give any sign of having heard. I then cautiously approached the animal, and not till it saw me did it give any sign of consciousness of my presence.

When the same experiment was repeated, while the monkey and its companion were quietly seated by the fire enjoying the heat, it gave no sign of hearing, while its companion started with alarm, and came with curiosity to ascertain the cause of the sound. At 12 (noon) the test of sight, related under Exp. IX., was made.

8 P.M In presence of Dr BURDON SANDERSON I repeated the various tests with the view of eliciting signs of hearing. To all it remained without response. It seemed unconscious of my presence when speaking close to its ear, and only started when it caught sight of me

April 9th.—The animal was found weak and prostrate, and was killed with chloroform.

Post mortem Examination.—There was a considerable amount of pus underneath the scalp and below the detached surface of the left temporal muscle. Pus was found beneath the dura mater continuous with the collection beneath the muscle. The surface of the brain was otherwise intact, except at the points to be described.

In the left hemisphere the brain-surface corresponding to the opening in the skull to the extent indicated by the dotted line in fig 20 was elevated above the rest and congested The surface of the angular gyrus and of the superior temporo-sphenoidal convolution was disorganized to the extent seen in the figure. The lower part of the shading indicates medullary softening, caused by passing a hot wire into the substance.

In the right hemisphere (fig 21) a similar dotted line indicates the extent of the

Fig 20

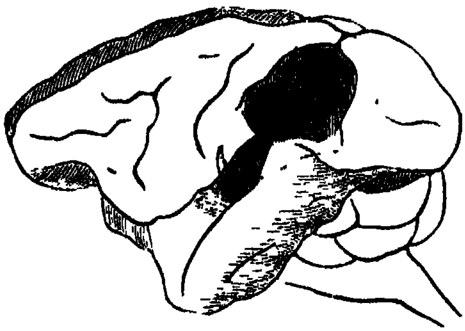


Fig 20 represents by the shading the extent of destruction of the grey matter of the left hemisphere in Exp XV The dotted line indicates the extent of surface exposed by removal of the bone and dura mater

Fig. 21

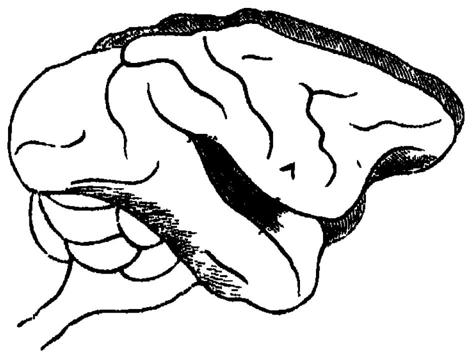


Fig 21 represents the extent of the lesion in the right hemisphere in Exp XV
The dotted line has the same signification as in last figure

opening of the skull, and the extent of congestion and hernia of the surface of the brain. The hernia was only slightly elevated above the rest of the hemisphere The lesion was accurately circumscribed

The grey matter on the surface of the superior temporo-sphenoidal convolution was destroyed throughout the upper two thirds of its extent (*i e.* the extent which reacts to electrical stimulation)

The base of the brain, ganglia, and cranial nerves were intact.

This experiment (besides confirming the fact of loss of sight by destruction of the angular gyrus) serves to localize the effects as to hearing, which were observed to result from extensive lesions of the temporo-sphenoidal lobe. It is obviously more difficult to ascertain the presence or absence of the sense of hearing in the lower animals than in man, on account of the difficulty of distinguishing between reflex action and true sensory perception.

In the above experiments, involving destruction of the superior temporo-sphenoidal convolution, it will be seen that, with the exception of an occasional start to a shrill sound, in general there was an abolition of reaction to sounds which in normal conditions are sufficient to excite active attention, and this while the animals were on the alert and in full possession of their other senses.

If this absence of reaction, except where it might well be the result of reflex action,

following the destruction of this region of the brain, be taken with the phenomena resulting from electrical stimulation of the same part, we have, it appears to me, as satisfactory proof as it is possible to obtain from the lower animals, that the sense of hearing is localized on the superior temporo-sphenoidal convolution.

Having thus eliminated the result of destruction of this convolution from the complex effects caused by more extensive lesions of the temporo-sphenoidal lobe, I proceed to describe experiments tending to fix more definitely the seat of tactile perception.

Several experiments have already been detailed, which rendered it more than probable that the loss of tactile sensation was dependent on lesion of the hippocampus major or uncinate convolution, or both

Experiments were devised for the purpose of destroying this region without injury to the rest of the temporo-sphenoidal lobe To effect this seems almost impossible, considering its deep-seated and concealed position in the internal aspect of the hemisphere.

The method I at last resolved to pursue was to endeavour to reach this from the occipital region by passing heated wires through the posterior aspect of the occipital lobe in the direction of the hippocampus I had first ascertained the negative effects of destruction of the occipital lobe These will be related subsequently.

Having made repeated experiments on the dead brain, so as to acquire knowledge of the direction and extent to which the cautery should be pushed, I proceeded to experiment on the living animal

My first attempts were not quite successful, as will be seen, but ultimately my efforts were rewarded with success.

Experiment XVI.

February 5*th*, 1875 —This, though not successful as regards the object intended, yet presents some interesting phenomena. The left occipital lobe was exposed posteriorly, and penetrated at the posterior extremity of the superior occipital fissure by means of hot wires, which were directed with a view to follow the inner aspect of the temporo-sphenoidal lobe. There was no hæmorrhage from the sinus. During the operation the animal was observed to make sighing respiration. The operation was finished at 4.30 P.M.

The animal lay in a state of stupor for more than an hour, only making slight movements when disturbed, and then with its left limbs.

7 P.M. The animal lies quiet, but indicates consciousness by grunting discontentedly when moved. Struggles with its limbs, chiefly the left, but occasionally with the right.

On testing the cutaneous sensibility with the hot iron, reaction was decisive over the whole of the left side, but quite abolished on the right. The animal occasionally opened its right eye, but the left remained permanently closed. The animal passed into a state of coma, and was found dead at 11.30 P M.

Post mortem Examination (next morning).—It was found that the cautery, as indicated by the blackened sinus, had penetrated the occipital lobe at the point mentioned, where a round hole was situated, and on emerging had ploughed a furrow on the upper

end of the uncinate gyrus, but then leaving the inner aspect of the temporo-sphenoidal lobe, had ploughed off the left tubercles of the corpora quadrigemina, then penetrating the middle of the left optic thalamus had passed inwards and emerged at the longitudinal fissure on its basilar aspect. The corpus striatum was uninjured, as the wire had penetrated to the inside of this ganglion.

There was no effusion into the skull, and, beyond the injury narrated, the rest of the brain had not been injured The optic tract of the left side had of course been destroyed along with the left tubercles of the corpora quadrigemina, and the anterior extremity of the sinus was situated just in front of the optic commissure.

In this the loss of sensation on the opposite side coincided with destruction of the left optic thalamus and the injury to the *tegmentum cruris.*

The ptosis of the left eye indicated the destruction of the nucleus of the third nerve, situated just below the region of the lesion in the corpora quadrigemina. As the optic thalamus was destroyed along with part of the uncinate convolution, this experiment of course does not warrant any conclusion as to the effect of destruction of this convolution itself.

As regards the optic thalamus, and the effect of its destruction, see also Exp XIX

The following experiment is a repetition of the last, and was only partially successful

Experiment XVII

February 9th, 1875 —The left occipital lobe of a monkey was exposed as in last experiment, and hot wires were pushed through the tip of the occipital lobe in a direction downwards and outwards, approximately in the direction of the hippocampus major. There was no hæmorrhage of any extent.

The operation was completed at 3.15 P M The animal was already conscious before the wound was dressed. It was freed and laid before the fire.

3.30 P.M. Lies by the fire breathing quietly. Pupils equal, and both eyes open. Utters a grunt of recognition when called to, and also begins to move its tail and right hand.

Gets up, but sits unsteadily, inclining to fall over on its right side. Reaction to hot iron distinct on both sides of the body.

3.50 P.M. Retains sight unimpaired. Can now sit up more steadily and walk without falling. Took a piece of apple offered to it in its right hand and ate it

5 P M. Took some tea, and ate some fruit. While sitting before the fire accidentally touched the bar of the grate, on which it manifested a lively sense of pain, and rubbed the part. The animal seems to retain all its senses and muscular power unimpaired.

9 P.M. Continues as before. Clings with right as well as left hand to its cage when laid hold of.

When offered any thing to eat, it now uses its *left* hand, whereas formerly it almost invariably employed the right. There is a distinct reaction to heat on the right side.

February 10th.—10 A.M. Remains as before. Eats and drinks heartily. Sees and hears perfectly. Reaction to hot iron still continues on both sides.

No difference observed in the animal when again tested at 7 P.M.

February 11*th*.—10 A.M. The animal looks much as before. The wound is suppurating freely. Can see and hear, and move about Takes every thing offered to it in its left hand. Reaction to hot iron still continues on both sides. A sore on its right foot seems to cause it great trouble, as it is continually biting and scratching it.

February 12*th*.—10 A.M. The animal ate and drank as before. There appeared to be slight twitching of the right side of the body. Reaction to heat still observed on both sides

10 45 A.M. The animal had again a return of the twitching of its right side. The animal was quite conscious, and did not fall After a few minutes the animal walked back to the fire, whence it had been removed for observation It was now seen to drag its right limbs somewhat.

11.40 A.M. In climbing in its cage seems to have great hesitation in using the right hand. When taken out had a slight return of the twitching When it had ceased some food was placed in its right hand. Failed to grasp it, but took it with its left hand, raised it to its mouth and ate

4 P.M Still continues to drag its right limbs in walking, and cannot grasp with the right hand There is marked diminution of reaction on the right side, as compared with the left, when a hot iron was applied

After this there was a return of the spasmodic twitching of the right side

In the interval of the fits the right leg was again tested with the hot iron, and reaction seemed to have entirely disappeared, while reaction was active when the stimulus was applied to the left

Towards evening the animal began to exhibit symptoms of basilar meningitis, suffering from frequent convulsive seizures It became comatose, and died in convulsions on February 13.

Post mortem Examination (February 13th, 10.30 A.M).—The exposed posterior extremity of the left occipital lobe was fungating The dura mater stripped easily from the surface of the brain; but the vessels of the pia mater were injected on the left hemisphere, particularly on the postero-parietal region.

The course of the wire was easily traced by the sinus it had caused, and by a line joining the points of entrance and exit After penetrating the occipital lobe it had ploughed a furrow on the upper extremity of the uncinate gyrus (see fig. 22), and then, instead of following the inner aspect of the temporo-sphenoidal lobe, had made its way horizontally outwards through the lobe, and emerged on the outer aspect at the extremity of the superior temporo-sphenoidal fissure (see fig. 23). On examination of the brain after hardening in spirit, it was found that softening had extended from the track of the wire, and that the hippocampus was in great measure softened down and disorganized.

In addition to these appearances there were signs of inflammation of the membranes at the base of the brain, on the pons. and anterior surface of the medulla. The left

optic tract was adherent to the hippocampal convolution The dura mater in the left sphenoidal fossa, and on the left petrous bone, and on the basilar process had a yellowish

Fig. 22.

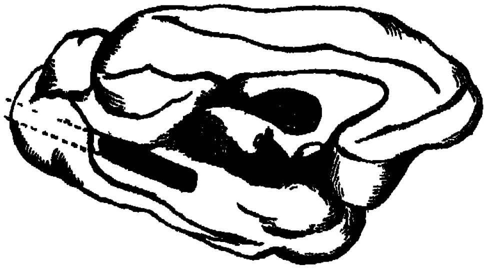


Fig. 22 represents the lesion of the uncinate gyrus and the direction of the sinus caused by the cautery in Exp XVII

Fig. 23.

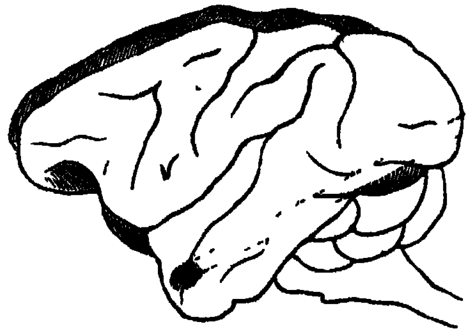


Fig 23 represents the outer surface of the same hemisphere, and the dotted lines indicate the track of the cautery in Exp. XVII. The black dot at the extremity of the upper temp sph fissure indicates the point at which the track terminated externally

aspect The inflammation appeared to have spread from the point in the left sphenoidal fossa where the cautery had emerged from the brain The optic thalamus and other ganglia were normal, except perhaps slight extension of the inflammation up the left optic tract to the left nates and testes

This experiment became complicated by the results of basilar inflammation, but it is possible to trace the course of the phenomena

The cautery, as determined by the points of entrance and exit, seems, after ploughing along the upper end of the uncinate gyrus, just to have missed the hippocampus and descending pillar of the fornix At first the effects were negative or nearly so, but gradually the animal began to exhibit failing sensation, as indicated by the diminution of reaction to tactile impressions and inability to use the right hand, until ultimately sensation became to all appearance abolished, or nearly so, on the right side This would coincide with the advance of softening into the hippocampus, as was found to be the case after death.

Whether the spasmodic affection of the right side is to be attributed to sensory irritation excited by the progress of inflammatory softening is a question, but it had also a possible origin in the basilar inflammation which, extending from the left sphenoidal fossa, naturally would affect the left half of the pons and medulla first, and show its effect by convulsive action of the opposite side of the body in the first instance.

This complication renders it difficult to estimate the exact effect of the lesion in the temporo-sphenoidal lobe, but the difference observed in the reaction of the two sides to the hot iron strongly confirms the view that this was dependent on the lesion of the hippocampus in the left hemisphere.

The following experiment serves to confirm this view.

Experiment XVIII

March 2nd, 1875 —A large monkey of the baboon type was chosen for this experiment. As it seemed to be usually left-handed, the right hemisphere was operated on.

The right occipital lobe having been exposed, hot irons were passed through the posterior aspect of the lobe in the direction of the hippocampal gyrus.

There was no hæmorrhage of any moment.

The operation was completed at 3.15 P.M

3.20 P M. The animal lies by the fire, having recovered consciousness, and moves its limbs, but has not yet attempted to get up.

3 25 P.M. When moved it opened its eyes, whence it was concluded that the crus cerebri had not been injured.

While lying by the fire scratched its right leg with its right hand Does not move its left arm or left leg, whether laid on its right or left side.

3 35 P M. There is no reaction to a hot iron on the left side. The same stimulus causes active manifestation of pain and rubbing when applied to the right side.

3.45 P M Begins to sit up and look about.

Moves only the limbs of the right side, and in sitting up occasionally falls over on its back

The test of the hot iron was again applied.

On the right side the slightest touch caused active reaction, and caused the animal to rub the part touched Applied to the left foot, the iron was kept in contact several seconds without causing the slightest reaction, but when kept up longer a slight retraction was caused. The same result was obtained on the left hand and the right ear

Pinching of the right hand and right foot caused violent reaction and slight cry. No effect followed the strongest pinching of the left hand and foot The left ear gave slight reaction to pinching.

4 P.M The animal is sitting up and looking about Grunts when called to. Occasionally falls over. Recovers itself by the aid of its right limbs. It can draw the legs together, but the left foot is generally allowed to straddle outwards and rest on the internal malleolus. The left arm is kept motionless in a semiflexed condition. Muscular resistance continues

4.15 P M. When offered food it took it with the right hand, and raised it to its mouth.

Occasionally moves its left arm and leg while sitting still, but does not use them to grasp or in progression

The reaction to the hot iron is still persistent on the right, but gone on the left side.

The animal was occasionally seen to give a jerk of its head and grind its teeth, which I attributed to some irritation of the fifth nerves, probably caused by inflammation of the dura mater in the neighbourhood of the Gasserian ganglion, set up by contact of the cautery.

4.30 P.M. The reaction to heat was again tried. A hot iron is allowed to remain in contact with the right side without causing any reaction, except when kept so long as to burn the part, while the slightest contact with the right side causes violent reaction and active rubbing of the part This was observed on the ear, hand, and foot.

There is no facial distortion.

5.45 P M. On being placed in its cage the animal mounted its perch with difficulty, and sat unsteadily with its head down. On turning its body a little the left leg slipped off the perch. The animal, in recovering itself, clutched hold of the bars of the cage with its right hand, and though the left was placed on the bars no grasp was made with it Aided by its teeth and the right hand, it ultimately regained its equilibrium, and dragged up its left leg, after having fairly got hold of the perch with its right.

Sits now holding on firmly to the perch with the right foot.

After this, on the animal shutting its eyes and going to sleep, the left foot frequently slipped off, causing sudden grasping with the right hand on the cage until it recovered its equilibrium.

8 P M. The anæsthesia of the left side being again firmly established, and the animal being otherwise well and apparently in possession of all its other senses, the animal was killed with chloroform, in order to avoid complication by the extension of the lesion

Post mortem Examination —The exposed surface of the right occipital lobe was slightly congested The surface of the brain, except at the point of entry of the cautery, was everywhere else normal There was no effusion within the skull. There was injection of the vessels of the dura mater in the right sphenoidal fossa and over the region of the Gasserian ganglion, extending from an inflamed spot with which the point of the wire had come in contact

The base of the brain and cranial nerves were normal in appearance. The crura, the corpora quadrigemina, the optic thalami, corpora striata, pons, and medulla were uninjured and normal in appearance.

The cerebellum was just grazed on its right upper lobe, where the cautery had come in contact with the tentorium in its course

The track of the cautery was clearly traceable. It had penetrated the right occipital lobe just at the posterior extremity of the superior occipital sulcus Here there was a round hole with blackened edges, about a quarter of an inch in diameter

Emerging on the under surface of the lobe, the track appeared as a deep furrow, commencing at the posterior termination of the calcarine fissure, and running along the uncinate gyrus for about an inch Thence following a concealed course below the surface of the uncinate convolution, which yielded to pressure, it emerged at the tip of the temporo-sphenoidal lobe on the orbital aspect of the lower end of the superior temporo-sphenoidal convolution, two lines external to the subiculum cornu ammonis. On cross section of the lobe it was found that the cautery had ploughed along the hippocampus major.

The track of the cautery was followed with precision by the discoloration caused.

This experiment indicates with precision the region in the temporo-sphenoidal whose destruction is followed by impairment or total abolition of tactile sensation.

In the various cases in which this result followed extensive lesion of the temporo-sphenoidal lobe, it was found that the hippocampus major and the hippocampal convolution were more or less extensively involved. The destruction of these convolutions alone, as shown by this experiment, abolishes tactile sensation alone.

Fig. 24.

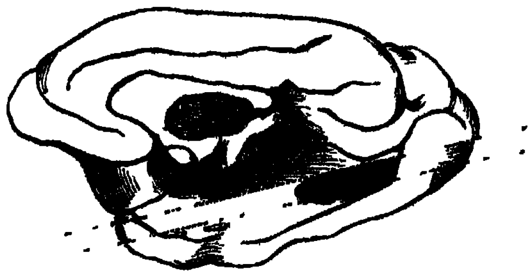


Fig 24 represents by the shading the external extent of the lesion in the uncinate convolution, and the dotted lines the track of the sinus caused by the cautery in Experiment XVIII.

To ascertain the existence or absence of this sense is surrounded with some difficulty, owing to the fact that reflex reaction may simulate the appearance of tactile sensation, properly so called. The mere fact of reaction to a stimulus is no proof of the existence of sensation

The entire absence of reaction, however, observed in some of the preceding experiments, where the hippocampal region was completely destroyed, is a strong proof of the abolition of sensation, when it is considered that reaction was lively and marked on the opposite side of the body at the same time.

But the loss of tactile sensation is even more conclusively indicated by the fact that monkeys in whom the hippocampal region was destroyed ceased to use the opposite limbs for the purposes of prehension or the exercise of the faculty of touch.

To react to tactile stimuli may signify reflex action or tactile sensation, or both, *to touch* necessarily implies the possession of the power of tactile perception.

The condition of the limbs in these cases was such as to simulate motor paralysis, and it is well known that Sir Charles Bell mistook the immobility of the side of the face resulting from anæsthesia caused by division of sensory branches of the fifth for real motor paralysis It was pointed out by Mayo that, owing to the loss of tactile sensation, an animal has no indication for the regulation and adaptation of its muscular movements, and hence ceases to make them. That anæsthesia, and not motor paralysis, existed on the side opposite the destruction of the hippocampus, is shown by the fact that a certain degree of voluntary motion was retained. The animal (Exp. XVIII) whose leg was anæsthetic could replace it on the perch, though it continually tended to slip off when the animal withdrew its attention from it.

There was no muscular flaccidity as in true motor paralysis, nor was there any appearance of facial distortion, such as would have been produced by motor paralysis of one side.

It is impossible to differentiate between lesion of the hippocampus itself and of the hippocampal or uncinate convolution A lesion involving the hippocampus necessarily involves the medullary aspect of the uncinate convolution, and it is impossible to destroy the uncinate convolution without injuring the hippocampus.

In the above-mentioned experiments both these convolutions were more or less conjointly involved.

Owing to this difficulty, I shall speak of the two together as the "hippocampal fold," and regard this as the seat of tactile perception.

We are now in a position to differentiate the various effects on sensation caused by general destruction of the temporo-sphenoidal lobe.

As regards hearing, separate evidence is given in Experiment XV. of the localization of this faculty in the superior temporo-sphenoidal convolution.

The absence of reaction to the usual auditory stimuli, combined with the effects of stimulation of this convolution, afford evidence of the strongest possible character of the localization of this sense.

The localization of smell is no less clearly indicated Anatomically, the connexion between the olfactory tract and the subiculum cornu ammonis, though less evident in man, is clear in the monkey, and very apparent in the lower animals

The effects of irritation of this region are very constant and characteristic, and are of the same nature as direct irritation of the nostril or of the olfactory bulb itself

Destruction of this region causes abolition or diminution of reaction to stimuli on the same side as the lesion.

Taken together, these facts establish the localization of the sense of smell in the subiculum, or tip of the temporo-sphenoidal lobe.

As to the sense of taste, the positive indications are less distinct than those of smell or hearing

Yet the phenomena occasionally observed on stimulation of the lower part of the middle temporo-sphenoidal convolution, viz movements of the lips and cheek-pouches, may be taken in connexion with lesions affecting this region, and accompanied by loss of reaction to stimuli of taste, to afford evidence of no weak character for the localization of taste in or near this region

That the centres of gastatory and olfactory perception are closely related to each other anatomically is rendered probable by the fact, often observed, of loss of taste and smell following severe blows on the head, and particularly of the vertex. It is not at all likely that one and the same cause should simultaneously directly affect all the nerves which are involved in the sensations of smell and taste, but it is easy to understand that a *contre-coup* might readily affect the integrity and functional activity of the lower end of the temporo-sphenoidal lobe, in which the above experiments serve to localize the central seats of these faculties.

We have thus accounted for the senses of sight, hearing, taste, smell, and touch, and given evidence for the localization of each and all of these in the central convolutions

Whether they are all integrated in the optic thalamus is a subject on which the experiments I have yet made do not furnish sufficient evidence; but the following experiment serves to prove that, in regard to tactile sensation, this is the case.

Destruction of the Optic Thalamus.

Experiment XIX.

February 12*th*, 1875.—The left hemisphere was exposed by a trephine-opening in the region of the pli de passage from the posterior division of the angular gyrus to the occipital lobe.

With a small trocar and cannula (after the method adopted by NOTHNAGEL in his experiments on rabbits) the anterior extremity of the annectent gyrus was perforated horizontally in the direction which experiments on the dead brain had taught me to reach and destroy the optic thalamus.

After withdrawal of the trocar, a stilette with expanding wings was passed through the cannula, and rotated so as to break up the parts with which it should come in contact.

There was some hæmorrhage from the cannula.

The operation was completed at 5 30 P M.

5 50 P.M The animal now is sitting up, leaning towards the right side. Makes some movements with its left limbs

7 P M. The animal looks quite active and intelligent. Can move about pretty freely, but seems weak on the right side. Does not use the right hand in taking hold of any thing presented to it. A hot iron applied to the right hand caused the animal to wince and rub the part touched.

8 P.M. Animal can move about. Took a piece of apple offered to it in its left hand, and held it to its mouth with both hands. Sight and other senses do not seem affected.

8.45 P.M. A bandage was placed on the left eye in order to ascertain the condition as to vision on the right The bandage could not be maintained, as the animal bounced about, knocking its head against furniture, and tearing at the bandage till it got it off. Owing to this the condition as to sight could not be definitely tested, though the running against obstacles seemed to indicate affection of sight in the right eye.

February 13*th*.—11 A.M. The animal is much in the same condition as yesterday. Uses all four limbs in walking, but the movements of the right are made with caution and hesitation, nor does it use the right hand in grasping, taking every thing offered to it with the left.

3 P M. Thinking that the optic thalamus had been only partially destroyed, I passed a hot wire in the track of the cannula, so as to completely traverse the optic thalamus, the distance &c. being carefully calculated from the result of experiments on the dead brain.

Before the animal recovered from chloroform the left eye was bandaged, and the animal laid before the fire.

3 10 P.M. The animal, while lying before the fire, begins to make some movements with its left limbs. The right remain motionless. The right eye was open, and the pupil dilated.

Active reaction followed the application of a hot wire to the left side, hand, foot, and ear.

No reaction followed application of the iron to the same points on the right side

3.24 P M. Begins to move about, turning towards the right side. When placed on the back of a chair the animal clung tenaciously with the left hand and foot, but did not grasp with the right.

The right side is completely anæsthetic. The animal, though keeping its right eye open, apparently does not see, as it runs its head against obstacles in its way. When placed on a chair it tumbled off, with its eye open. Muscular resistance is considerable in the limbs of the right side There is no trace of facial distortion.

3.40 P.M Can flex and extend the right leg. Does so when lying down and in trying to get up. Does not move the right limbs in walking, but drags them after the left Turns about aimlessly, and knocks its head against furniture &c. Sometimes goes backwards. There is no reaction on the right side, but active on the left to hot iron

3 55 P.M The animal was placed on the floor, and surrounded by a circle of battery-jars It turned round and round, knocking its head against them, and apparently unable to find its way out between them.

The bandage was then removed from the left eye. The animal still remained quiet for a few minutes When placed on the back of a chair, it quickly found its way down. When placed beside its cage it looked about and then went in. Sight was therefore improved or restored since the removal of the bandage.

5 P.M The animal was observed to flex the right arm and partially close the fist while it was sitting still. Entire abolition of reaction still continues in right After some minutes the animal seemed to be animated by all its former vivacity. Ate and drank heartily. Makes active movements, turning round and round frequently to the left, using its left limbs only.

At 5 30 P.M. the animal was chloroformed to death, so that the exact seat of the lesion might be ascertained.

Post mortem Examination.—From the opening in the skull below the parietal eminence there was a hernia cerebri involving the upper part of the middle temporo-sphenoidal, annectent gyrus, and lower part of the angular, and upper end of the superior temporo-sphenoidal convolution (see fig. 25).

In the centre of this was an opening, almost circular, with softened edges, indicating the point of entrance of the cannula. The surface and base of the brain were everywhere else normal. The cranial nerves were intact.

Fig. 25

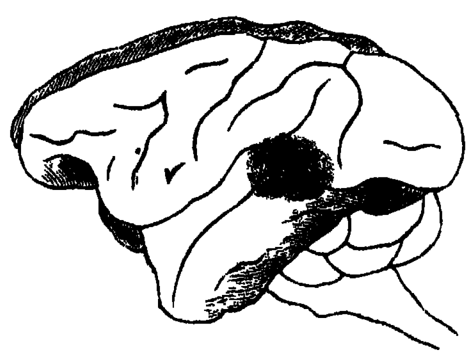


Fig 25 represents by the shading the area of superficial injury of the left hemisphere of the monkey in Experiment XIX The dark central shading indicates the orifice of the wound leading into the optic thalamus.

On opening the ventricles they were found free from effusion. The left optic thalamus was disorganized.

The track of the hot iron was easily traced by its blackened appearance. It had passed horizontally almost in the centre of the ventricular aspect of the ganglion, a line or so beneath the surface, and completely traversing the left thalamus, had just crossed the third ventricle and made a slight indentation on the opposite right thalamus.

Besides this wound there was another lacerated surface situated more towards the extraventricular aspect of the thalamus. This had been caused by the spring stilette, which, as it had been conjectured, had not penetrated the body of the ganglion Round this discoloured laceration softening had extended somewhat, but had not quite invaded the body of the thalamus The anterior and posterior extremities of the thalamus were almost of normal appearance. The intervening portion was quite broken up.

The corpora striata and corpora quadrigemina were uninjured The crura cerebri were intact

In this experiment the lesion was confined to the optic thalamus, or as far as can be effected by such a method of experimentation.

This result, and the result of Experiment XVI., show that complete disorganization of the optic thalamus in monkeys abolishes cutaneous sensation on the opposite side (As I am restricting my conclusions to monkeys, I do not here stay to discuss in detail the results of NOTHNAGEL'S experiments on rabbits (VIRCHOW'S Archiv, 1874, p 201), which lead him to apparently contradictory conclusions. I will merely remark, on the ground of experiment, that NOTHNAGEL, in my opinion, is not warranted in asserting that true sensation continues in rabbits after total destruction of the optic thalami Reaction to tactile stimuli, in all respects resembling sensory, such as springing forward when the tail is pinched, or uttering screams, still continues to be manifested by these animals after complete removal of the hemispheres)

The retention of reaction to stimulation in the first instance in this experiment may have been due partly to reflex action, partly to the retention of sensation, but that sensation was impaired was evidenced by the fact that the animal ceased to use its right limbs as before for the purposes of prehension and touch Here also, as in destruction of the hippocampal fold, there was apparent muscular paralysis—but not so in reality, as the animal could still move the limbs in some degree, and the muscles retained their tonicity and resistance

The interference with vision may have been due to the proximity of the lesion to the angular gyrus and its medullary connexions, as much as to the lesion of the optic thalamus, and therefore no definite conclusion is built on this fact. With regard to the circular movements of the animal which were occasionally made, the body seemed to go to the right or left according as the left arm was adducted or abducted

The next experiments relate to the effects resulting from destruction or complete removal of the occipital lobes.

It has before been stated that the occipital lobes do not give any external response to the electric stimulus.

Destruction of the Occipital Lobes

Experiment XX.

November 21st, 1873.—The occipital lobes were exposed on both sides in an active and intelligent monkey.

By means of the galvano-cautery the upper surface of the exposed lobes was disorganized as far back as their posterior extremity, while the left was further almost severed from the rest of the brain by carrying the cautery perpendicularly downwards towards the tentorium. It was not removed, however.

The operation was completed at 6 P.M.

6.15 P.M. The animal sat up spontaneously, which it did in a very unsteady manner, and kept its head bent on the chest. Some milk was poured down its throat. Gave evidence of retention of sight.

6.25 P.M. Moves about a little, looking about. Shows signs of pain and annoyance when its tail is pinched. Grunts discontentedly when nudged and made to move.

8 P.M. Made to swallow some more food. When placed in the cage beside the other monkeys it sat with its head bent, grunting when disturbed by them, and screaming when they began roughly to examine its head.

Being obliged to be absent from London for a few days, I found on my return that the animal had survived till the 25th. During the whole period it had maintained its dejected and melancholy attitude, paying no attention to its surroundings, and had shown no desire to eat or drink.

After death the occipital lobes were found disorganized, while the rest of the brain was uninjured. The stomach and intestines were completely empty. The other viscera were normal. No drawing was made of the brain.

In this case it might be supposed that the effects were merely due to the severity of the operation, but a review of the foregoing experiments will serve to indicate that experiments involving quite as serious surgical operations were not followed by the same depression, the animals still retaining their appetite, and eating and drinking as before.

The results as regards motion and sensation were negative, and the only effect which could be noted was the general depression, and the abolition of the animal's appetite.

Experiment XXI.

January 16th, 1875.—The occipital lobes of a monkey were exposed on both sides, and the dura mater removed from both. Owing to the rupture of a venous sinus on the right side, cotton-wool, soaked in perchloride of iron, had to be used to stanch the hæmorrhage, and there was reason to fear that it had in some degree injured the brain.

At 4 P.M. the left occipital lobe was separated from the hemisphere by means of

white-hot wires passed perpendicularly downwards close to the sulcus, separating this lobe from the angular gyrus.

4.35 P.M The animal was let loose and laid down. After a few minutes it attempted to sit up, and uttered a croaking sort of sound.

5 P.M. Moves about the room rather unsteadily, occasionally uttering a short cry. Turns its head when called.

7 P.M. The animal appears to be blind When placed on the back of a chair it would not move, though the chair was shaken, and the animal evidently felt uncomfortable. A piece of apple was held before it. It smelt it, and wished to lay hold of it, but made futile grasps after it. It could not find the way into the cage when placed close to the door.

8 P.M. It had been intended also to remove the right occipital lobe, but owing to the uncertainty as to the cause of the blindness, it was thought advisable to leave the right side undisturbed, so that if the blindness were due to affection of the left angular gyrus during the process of removing the left occipital lobe, time should be allowed for compensation. The wound was therefore sewed up and dressed.

The animal, when placed on the floor, wished to return to me, but could not find its way.

January 17*th*, 10 A.M —The animal refuses to eat. Drank some water in which its mouth was forcibly immersed. When taken out of its cage it is seen to retain its muscular power, but gropes about on the floor. The pupils are equal and of medium size, and react sluggishly to light.

1 P.M Tries to climb up whatever it comes in contact with. Likes to be taken up and caressed, but cannot find its way Still continues blind.

An ophthalmoscopic examination was attempted, but could not be carried out, on account of the animal's restlessness.

January 18*th* —10 A M. The animal looks somewhat more lively today. Ate a fig and drank some water, but refused other food. Still continues blind, and moves about in a groping manner.

At 10.30 A.M. the animal was killed with chloroform, in order to ascertain the cause of the blindness.

Post mortem Examination.—The wound was suppurating freely.

The cut surface of the left occipital lobe was found projecting almost to the orifice in the skull.

The exposed surface of the right occipital lobe was soft and pulpy. There was slight extravasation on the surface of the dura mater on the right parietal region, caused by the rupture of the sinus above alluded to.

The left occipital lobe had been cut off by a line passing perpendicularly through its junction with the left angular gyrus (see fig. 26).

The angular gyrus was softened all along its posterior division, and just beyond the curve (see fig. 28).

The right occipital lobe, besides being softened on its upper aspect, was discoloured and covered by a layer of extravasation, which likewise covered the right nates. The nates themselves were, however, uninjured, and of normal consistence.

Fig 26.

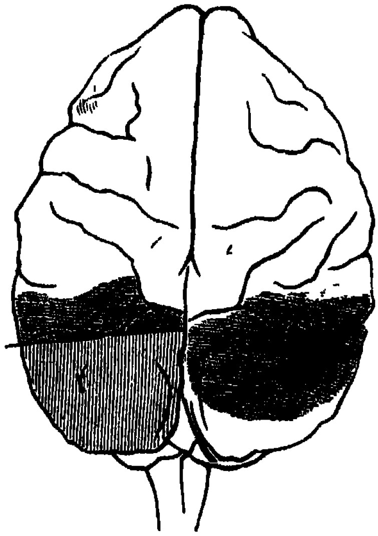


Fig. 26 represents by shading the extent of softening on both hemispheres of the monkey in Experiment XXI The transverse line on the left occipital lobe is the line of section, and the part marked by paralle lines is the part entirely removed

Fig 27

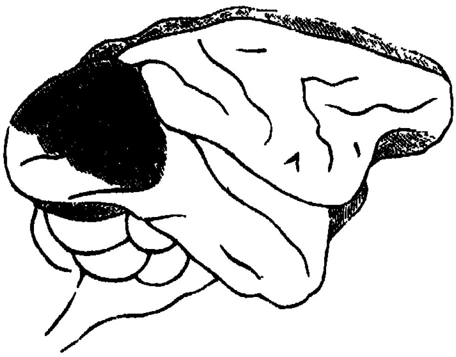


Fig. 27 represents the extent of softening in the right hemisphere of the monkey in Experiment XXI.

Fig 28.

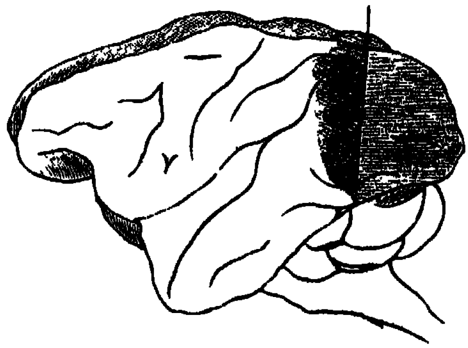


Fig 28 represents by the perpendicular line the line of section of the occipital lobe The part marked by parallel lines was cut off The shading indicates the extent of lesion of the surface.

The posterior limb of the right angular gyrus was softened and discoloured yellow, owing to the contact with the cotton-wool soaked with perchloride of iron which had pressed on this part of the right hemisphere The posterior extremity of the right occipital lobe was normal in appearance, both on its upper and under surface (see fig. 27)

The optic tracts and cranial nerves were intact.

The upper part of the tentorium cerebelli on the left side was covered with pus, and

the cerebellar surface beneath was yellowish in colour, but not softened. No other lesion existed in the brain.

This experiment was unsuccessful in so far as the object of localizing the lesion in the occipital lobes was concerned, but is a valuable confirmation of the results obtained by former experiments on the angular gyri In this experiment, besides the complete removal of the left occipital lobe and extensive injury to the right, the angular gyrus was deeply involved on both sides, not throughout, however. The lesion was, however, extensive enough to produce total blindness; and it further illustrates the fact that when the angular gyrus is destroyed on both sides no compensation of visual perception occurs.

Beyond the fact of loss of sight, which is to be attributed to the lesion of the angular gyri, the lesions of the occipital lobes were in a great measure negative, the animal retaining its muscular powers, and apparently other senses, and still exhibiting, though to a less extent than before, its desire for food.

Experiment XXII.

January 22*nd*, 1875.—The occipital lobes were exposed on both sides in a monkey, and the surface exposed destroyed by the cautery, which was also passed deeply into the interior of the lobes, in order to cause as much disorganization as possible. Care was taken not to injure the angular gyri

The operation was completed at 3 30 P M

4.10 P M. The animal after lying in a state of stupor till now begins to move, but staggers a good deal. The eyes are open and the pupils dilated.

It indicates consciousness by turning its head when called to

4.45 P.M. Sits quietly with its head down on its chest. It drank a little tea in which its mouth was kept immersed Turned fiercely round on its tail being pinched.

5.45 P M. Gives emphatic evidence of sight. Ran away when I approached it, carefully avoiding obstacles. Seeing its cage door open, it entered and mounted on its perch, carefully avoiding the cat which had taken up its quarters there.

Tried to escape my hand when I offered to lay hold of it, but picked up a raisin which I had left on the perch

8 P.M. When not disturbed sits quietly with its head bent on its chest. Easily roused. Does not take any food or drink offered to it.

12 midnight. Is sound asleep on its perch. Has not eaten any of the food left in the cage.

January 23*rd*.—10 A.M. Animal found sitting in the cage with the head bent as before. Drank a little milk held up to its lips. When removed from the cage walked about somewhat unsteadily, and then sat down as before. The eyes are partially closed from œdema of the eyelids. Sight continues. Made for a warm corner by the fire. Wakes up and grunts when called to. There is no loss of motion or sensation as far as can be seen.

8 P.M. Still continues sitting as before. When disturbed moves very unwillingly and apparently with great caution, as if its sight were impaired, occasionally knocking its head against obstacles. Drank some water, but would not eat.

9 P M. The animal remains as when last seen Has taken no food.

January 24th —11 A.M Found lying prostrate in the cage. Killed with chloroform

Post mortem Examination.—The exposed surface of the occipital lobes on their superior and lateral aspect was soft and pulpy and suppurating. The extent is marked by the shading in figures 29, 30, 31 The softening extended deeply into the interior, but did not affect the under or inner aspect of the lobes

The angular gyrus on both sides was of normal consistence, but the grey matter had a yellowish tint in the posterior half

There was no effusion into the lateral ventricles.

The rest of the brain was quite normal

Fig 29

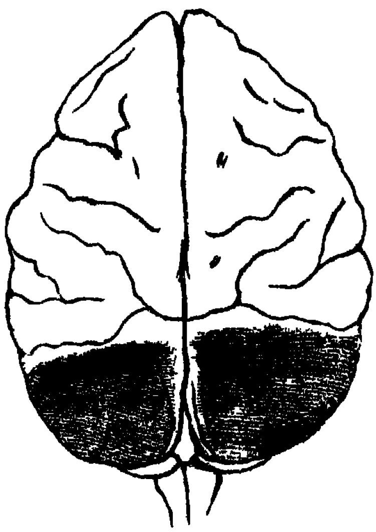


Fig 29 represents by the shading the extent of destruction of the grey matter of the occipital lobes in Exp XXII

Fig 30

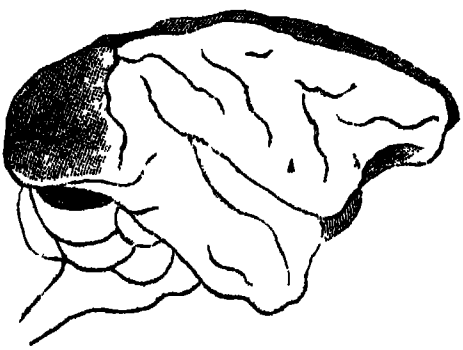


Fig 30 indicates the extent of softening in the right occipital lobe in Exp. XXII

Fig 31

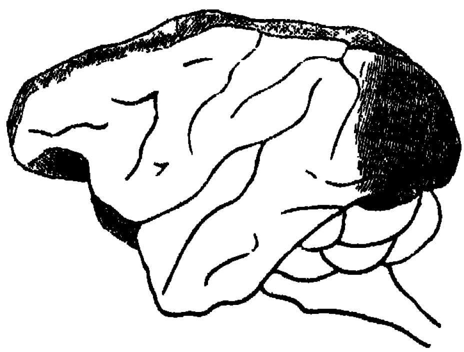


Fig 31 indicates the extent of softening in the left occipital lobe in Exp XXII.

In this experiment the results as regards sensation and voluntary motion were entirely negative. Sight became affected later; and this can be accounted for by the proximity of the angular gyrus to the lesion, and the commencement of pathological change in its substance. Nothing further was to be observed, except the dull dejection and melancholy attitude of the animal and its persistent refusal of food.

Experiment XXIII.

March 10*th*, 1875.—The occipital lobes were exposed in a small and rather weakly monkey, and the lobes severed by perpendicular section with hot wires about a quarter of an inch posterior to the parieto-occipital fissure, so as to avoid all interference with the angular gyrus. The operation was completed at 4 30 P.M., the animal having by the time the wound was dressed almost completely regained consciousness

4 45 P.M. Begins to move about in rather a staggering manner, but exhibiting no muscular paralysis.

4.55 P.M. Can see quite well, as it avoids obstacles, and when removed regains its place by the fire. Twitches its ear and turns its head when called to, or a noise made. Can sit quite steadily.

7 P.M. Sits still looking about vacantly. Will only move when nudged. Tactile sensation is unimpaired. Sight and hearing continue. Withdrew its head sharply when acetic acid was held before its nose. Made movements of tongue and mouth, as if to expel it when colocynth was placed in its mouth. Circulation and respiration regular and normal.

The animal has refused food and drink

7.45 P.M. Drank a few teaspoonfuls of tea held up to its lips, and accidentally placing its hand in the dish stooped and drank up the contents.

When left to itself, takes up a position with its head bent on its chest and covered with its hands.

8.45 P.M. Remains as before. Refuses to eat or drink. When a dish of milk was held before it in such a manner that it could not hold its head down without immersing its mouth in the liquid, it sipped a little but wished to avert its head.

9.40 P.M. Reaction to taste again tried with aloes, and again discomfort manifested. Turned away its head when assafœtida was held before its nostrils. Active reaction to acetic acid. Smelt at its hand on which some assafœtida had been spilt.

12 midnight. Lies asleep in cage breathing quietly. Easily roused by a touch on its hand, which caused it to open its eyes. Animal weak.

March 11*th*.—9.30 A.M. Found dead in its cage and rigid, death having occurred in the night.

Post mortem Examination.—The brain was everywhere normal except in the region of the occipital lobes. The occipital lobes had been completely divided and removed on both sides, but more on the right than on the left. The parts removed are indicated in figures 32, 33, 34 by the shading.

The lungs were normal, of pinkish colour. The heart was dilated, and its cavities full. The stomach contained a few coagula of milk which it had swallowed The other viscera presented no abnormal appearance.

There was therefore nothing to account for death in the animal except the prostration consequent on the operation in an animal of weakly constitution.

The only facts, therefore, which can be relied on as proved by this experiment are the negative results as regards the individual senses and the powers of motion. The abolition of appetite was not absolute, but nearly so. The occipital lobes were not entirely removed, as will be seen by the figures.

Fig 32.

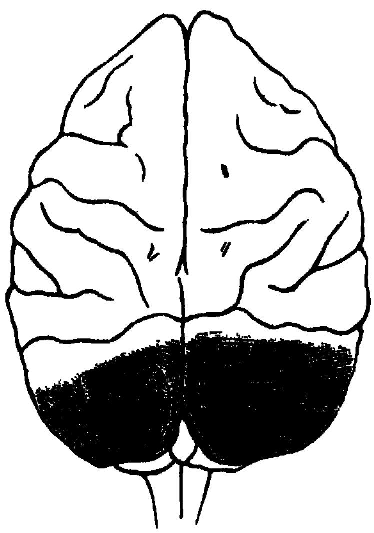


Fig 32 indicates by the shading the extent of removal of the occipital lobes in Exp XXIII

Fig. 33

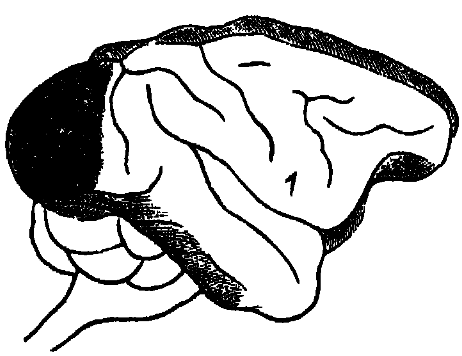


Fig. 33 indicates the extent of removal of the right occipital lobe in Exp. XXIII.

Fig 34

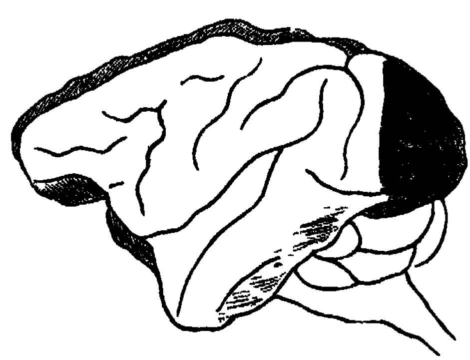


Fig 34 indicates the extent of removal of the left occipital lobe in Exp. XXIII.

Experiment XXIV.

March 18*th*, 1875.—The occipital lobes were exposed in a large and strong macaque, lively and active, but of rather a timid disposition and unwilling to be handled. With

hot wires the lobes were divided and removed by a line somewhat in advance of the anterior extremity of the superior occipital sulcus, but the exact line was doubtful. The left section sloped posteriorly, the right was almost perpendicular to the tentorium.

There was very little hæmorrhage, and the operation was rapidly completed at 11 30 A.M., the animal being almost conscious during the dressing of its wound.

11.45 A.M. The animal has been lying quietly looking about, but has not moved. While this note was being written the animal sat up spontaneously, but feeling weak and unsteady lay down again. Turned its head and looked when called by name. Got up and tried to walk, but staggered and fell.

12 10 P.M Oscillates while sitting up and totters when it tries to walk.

Sits near the fire, rubbing its nose and ears when they become too hot. Followed its companion with its eyes, but cannot succeed in walking steadily to join it.

12.20 P.M On my approaching it and making a threatening grimace at it, it turned away making mouths at me as usual A few minutes after ran away when I approached it, moving now almost quite steadily.

7 P M. Can move about freely, but there seems to be some confusion or defect of vision, as the animal puts out its hand to reach things without appreciating distance. Can see its way, however, tolerably well Smells at various kinds of food offered to it, but refuses to eat. Refused tea, of which it formerly was very fond. Objects to being disturbed, and sits hugging its companion, which it occasionally salutes with a tug or a bite when it does not sit quiet.

March 19th.—10 A.M. Refused all food. Looks rather dejected, but otherwise is well, retaining its muscular powers and sensation unimpaired, with the exception of slight defect in vision, as above noticed

The wound looks healthy, and the animal vigorous.

11.15 A.M. Licked at a piece of orange offered it, but will not eat any thing else. Frequently treats its companion to a rough shake or bite.

5 P M. Still refuses to eat or drink. Has taken nothing since the operation but the piece of orange.

March 20th.—9 A.M. Still refuses food or drink. Sits quietly and takes little or no interest in its companion, which runs about

Otherwise there is no change in the symptoms, as to motion or sensation.

7 P M Drank eagerly a large quantity of water. Refused all kind of food.

March 21st.—11 A M. The animal is well and in seemingly good health. The wound is oozing only slightly at one part, the greater part having healed up.

Came out of the cage when the door was opened and walked to the fire, before which it sat down with a contented grunt. Still refuses to eat.

1 P.M Greedily accepted and ate a piece of orange, which is the only thing it seems to have any desire for. Incidentally it was observed to seize hold of its companion (a male) and make the movements of coitus. This occurred twice. (The testicles existed, but the penis had been amputated.)

2 P M. Drank water, but refused food

7 P.M. Again eagerly drank cold water Does not exhibit any desire to eat the food, of which there is a plentiful supply in the cage. Goes occasionally and takes a draught of water. It was once at this time observed to nibble a crust of bread, but further did not manifest any sign of hunger.

March 22nd —10 A M. Looks very dejected, sitting quietly in a corner of the cage Took a little water held up to its mouth, but would not eat.

7 P.M. For the first time since the operation has exhibited a distinct desire to eat by accepting and eating a piece of bread and then drinking largely of water This was at the end of the fifth day. Otherwise the animal is as before

8 P M. Refused its former beverage tea, of which it used to be fond Sits dejectedly in a corner of the cage, feeling its head and licking its hand occasionally The wound looks well, only oozing slightly

11 P.M Again offered food, but refused all the food the other monkeys seemed to enjoy. At last, on being offered a cold potato, it took it in its hands, smelt it carefully, and then, as if suddenly struck by a new idea, began to eat with great gusto.

March 23rd.—The animal looks well and less dejected than before Walked out of the cage when the door was opened Retains its muscular power and senses as before Ate and drank several times during the day Seems to have recovered its appetite for its former food

March 24th —The animal continues well and took its breakfast as usual

Today it was placed in a hamper and taken to the country, to be under my observation during a short absence from London

April 10th —Since last observation the animal has continued well The wound gradually healed up completely The animal retained its appetite, eating and drinking heartily With the exception of the defect of vision, seen particularly in the want of appreciation of distance, the animal had recovered perfectly to all appearance It would be difficult to say what alteration in its disposition had occurred, yet it looked duller and less active than it used to be.

It had, however, entirely recovered from the effect of the operation, and was used for another experiment to be recorded next (see Exp XXV).

This experiment is remarkable as being the only successful case I have observed of recovery taking place after removal of a large portion of the skull and a considerable quantity of the brain-substance

The history of the animal offers some interesting features, and is a further illustration of the entirely negative effect as regards motion and sensation of destruction of the occipital lobes. The only exception was with reference to vision, which continued impaired throughout In the other cases when vision was lost or impaired, it was found on post mortem examination that the angular gyrus was more or less affected In this case also, as will be seen from the post mortem examination (p. 486), the angular gyrus was again the seat of lesion.

This animal exhibited less of that dejection and depression which characterized the other animals similarly operated on.

It is difficult to single out any one positive result of the destruction of this part of the brain, except the remarkable aversion to food which was observed almost invariably. This may be regarded as due to the constitutional disturbance consequent on such severe mutilation, but if so, it will be difficult to account for the fact that equally severe mutilation of the frontal lobes and other parts of the brain caused little or no impairment of the appetite for food.

I am disposed to think, therefore, that the aversion to food stands in causal relation to the destruction of the occipital lobes as such, and that these lobes are somehow related to the systemic sensations. The other animals did not live long enough to decide as to whether this condition should remain permanent, but in experiment XXIV. the animal, otherwise exceptional, after remaining without food for a period of five days, again recovered its appetite and continued to eat as before.

Thirst did not seem to have been affected to the same extent as the appetite for food.

If the systemic sensation of hunger has its seat in the occipital lobes, it is difficult to account for the restoration of this appetite after these lobes have been removed. Yet it is possible that compensation may have occurred by association with other senses, such as of taste and smell This is offered as a possible explanation, but it must be admitted that neither the electrical irritation of the occipital lobes nor their destruction suffice to indicate clearly the functions which these lobes perform.

It would appear from experiment XXIV. that their destruction does not abolish the sexual appetite The exhibition of this appetite may perhaps have been due to irritation of some centre in proximity to the seat of lesion. Some interesting speculations might be made with reference to these results, but as my object in this paper has been to restrict myself to conclusions directly deducible from my experiments, to enter on such would be foreign to the subject.

The following experiment is interesting, and one perhaps not often capable of repetition.

Conjoint removal of Frontal and Occipital Lobes.

Experiment XXV.

April 10*th*, 1875.—The monkey which had had its occipital lobes removed on March 18th (exp. XXIV), *i e.* twenty-three days previously, and which had apparently quite recovered, was placed under the influence of chloroform, and the frontal lobes removed on both sides by a line approximately traversing the anterior extremity of the supero-frontal sulcus.

The operation was completed at 12 noon.

The animal had regained consciousness before the wound had been quite dressed.

12 10 P.M. On being let loose and placed on the floor, it sat up and began to move about in a tottering manner. When it shook itself it fell over on its side.

12.20 P.M. Is sitting up somewhat unsteadily and gnawing at whatever comes within

its reach. Occasionally suddenly puts out its hand, and frequently rubs its nostrils as if there were some source of irritation in them.

It gives complete evidence of retaining hearing.

2 P.M. Runs away when I approach. Is not quite steady in its movements. Can find its way into its cage as before when taken out. Sight continues as formerly.

2 30 P.M. Drank some water and ate some fruit. Sits on its perch occupied in feeling its head and licking its hand. Seems much less timid than before; does not seek to move off its perch when about to be laid hold of, but resists and offers to bite. When not disturbed sits in a dreamy sort of state, taking no notice of any thing.

4 P.M. Found sitting in the same position as when last seen, with its head bent Looks vacantly, and does not seem to mind an attempt to lay hold of it. When seized it resisted, and attempted to bite, exhibiting great anger.

Ate some food offered to it When removed from the cage it walked about restlessly and without seeming to have any purpose. Ran away on being approached, but did not as usual make for its cage.

12 midnight. The animal sits still and is evidently feverish, the head being swollen and hands and feet hot.

April 11th —11 A.M. Found asleep in a corner of the cage. When removed it subsides into a deep sleep and nearly falls over, but recovers itself suddenly. There is no motor paralysis, and sensation is unaffected.

This state continued during the day, and towards night the animal fell into a state of semistupor, and did not seem able to support itself on its legs, sprawling about occasionally when disturbed. No convulsions were observed.

April 12th.—The animal was found dead in its cage at 10 A M partially rigid, so that death must have occurred some hours before.

Post mortem Examination.—The scalp was œdematous, and there was a considerable amount of pus oozing from the wound. The skull was deficient over the region of the frontal and occipital lobes The brain-substance at the occipital openings was adherent by adventitious membranes to the under surface of the scalp The left looked of normal colour and not congested The right was congested, and appeared as if it had received a contusion from a fall

From the frontal openings there protruded two livid herniæ cerebri On removal of the dura mater, a layer of pus was found coating its under surface. This was not adherent to the brain-substance, from which it stripped entirely.

The brain-substance had normal colour and consistence.

The roof of the orbit was also covered with pus, which extended as a thick layer into the sphenoidal fossæ, but was easily detachable and of recent formation.

The base of the brain and cranial nerves were free from signs of inflammation. There were traces of inflammation and some degree of suppuration between the longitudinal fissure at the occipital region and over the tentorium cerebelli. These were to all appearance of older formation than those in the anterior part of the skull. The cerebellum had a normal appearance.

After the brain had been hardened in spirit, it was found that the frontal lobes had been removed by a line crossing the anterior extremity of the supero-frontal sulcus on both sides. The plane of section sloped somewhat forwards, and the under surface of the orbital region remained where it conceals the olfactory tracts and bulbs The cut surface bulged considerably and the edges of the section were softened nearly as far back as the antero-parietal sulcus on both sides (see figures 35, 36, 37). The edges were raised, and the vessels were injected for some distance posterior to the cut surface.

Fig 35

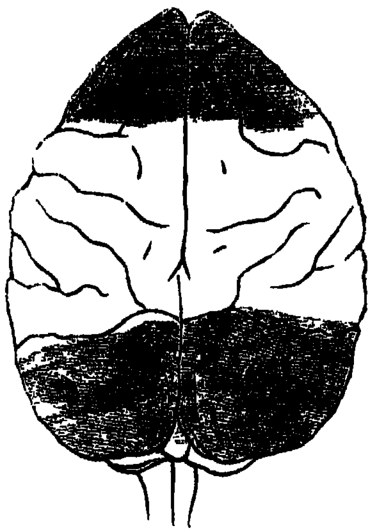


Fig 35 represents by shading the extent of destruction of the frontal and occipital regions in Exps XXIV & XXV

Fig 36

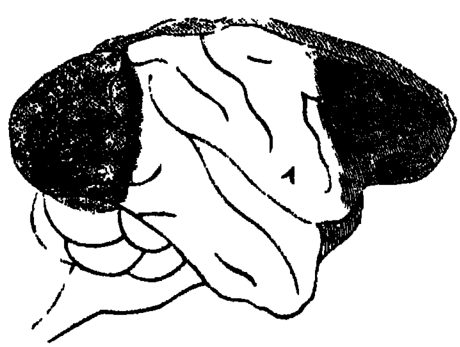


Fig 36 represents by shading the extent of destruction of the right hemisphere in Exps XXIV & XXV

Fig 37.

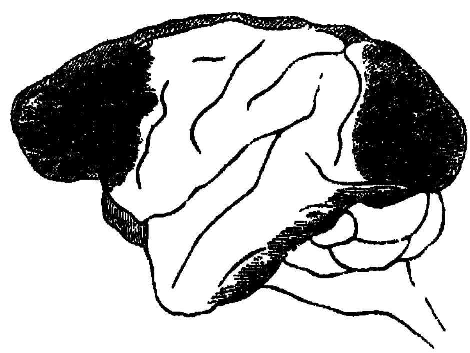


Fig 37 represents by shading the extent of destruction of the left hemisphere in Exps XXIV & XXV.

The occipital lobes had been removed almost completely. On the left side the hemisphere became rounded off just behind the posterior limb of the angular gyrus, which was intact. On the right side the posterior limb of the angular gyrus was ragged and torn, and formed part of the edge of the plane of section (see fig. 36).

With the exception of the injection of the vessels of the pia mater behind the frontal section, the rest of the brain had a normal aspect.

The most important fact demonstrated by this experiment is that the conjoint removal of the frontal and occipital lobes exercises no effect on the powers of voluntary motion or of sensory perception.

The results of the post mortem examination indicate that the phenomena of the second day are not to be regarded as the effect of the removal of the frontal lobes as such, but as due to the inflammatory complications which resulted in death.

But the fact that for many hours after the operation the animal continued to retain its powers of sensation and of volition, proves that these faculties are independent of the frontal and occipital lobes, and that they are associated with those parts of the brain which, by other experiments, I have shown to be specially related to sensation and motion.

What the positive effects were, as distinguished from the merely negative, it would be difficult to state in exact terms They are quite in accordance with experiments already related as to the effect of destruction of the frontal lobes

Without entering further into the psychological aspects of these results, I would sum up the conclusions which seem to me to be legitimately deducible from them as follows —

(1) Ablation of the frontal regions of the brain which give no reaction to electrical irritation is without effect on the powers of sensation or voluntary motion, but causes marked impairment of intelligence and of the faculty of attentive observation

(2) Destruction of the grey matter of the convolutions bounding the fissure of Rolando causes paralysis of voluntary motion on the opposite side of the body, sensation remaining unaffected, while lesions circumscribed to special areas in these convolutions, previously localized by the author, cause paralysis of voluntary motion limited to the muscular actions excited by electrical stimulation of the same parts

(3) Destruction of the angular gyrus (pli courbe) causes blindness of the opposite eye, the other senses and voluntary motion remaining unaffected This blindness is only of temporary duration, provided the angular gyrus of the other hemisphere remains intact When both are destroyed the loss of visual perception is total and permanent

(4) The effects of electrical stimulation and the results of destruction of the superior temporo-sphenoidal convolution indicate that this region is the centre of auditory perception

(5) Destruction of the hippocampus major and hippocampal convolution abolishes the sense of touch on the opposite side of the body.

(6) The sense of smell has its centre in the subiculum cornu ammonis or tip of the uncinate convolution on the same side.

(7) The sense of taste is localized in a region in close anatomical relation to the centre of smell, and is abolished by lesion of the lower part of the temporo-sphenoidal lobe.

(8) Destruction of the optic thalamus causes complete anæsthesia of the opposite side of the body.

(9.) Destruction of the occipital lobes produces no effect on the special senses, nor on the powers of voluntary motion, but is followed by a state of depression and refusal of food not to be accounted for by mere constitutional disturbance consequent on the operation. The function of these lobes is regarded as obscure, but considered as being in some way related to the systemic sensations. Their destruction does not abolish the sexual appetite

(10) After removal both of the frontal and occipital lobes an animal still retains its faculties of special sense and the powers of voluntary motion.

XVII. *On a Class of Identical Relations in the Theory of Elliptic Functions.*

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Communicated by JAMES GLAISHER, *F.R.S.*

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§ 1. THE object of the present paper is to notice certain forms into which the series for the primary elliptic functions admit of being thrown, and to discuss the identical relations to which they give rise. These latter, it will be shown, may be obtained directly by the aid of FOURIER'S theorem, or in a less straightforward manner by ordinary algebra.

§ 2 Whenever we have a periodic function of x, say ψx, such that $\psi x=\psi(x+\mu)$, it is well known that we may assume, for all values of x,

$$\psi x=A_0+A_1\cos\frac{2\pi x}{\mu}+A_2\cos\frac{4\pi x}{\mu}+\text{\&c.}$$
$$+B_1\sin\frac{2\pi x}{\mu}+B_2\sin\frac{4\pi x}{\mu}+\text{\&c.};$$

and if ψx be even, so that $\psi x=\psi(-x)$, then B_1, B_2, &c. all vanish, while if ψx is uneven, so that $\psi x=-\psi(-x)$, A_0, A_1, &c. vanish. If ψx is such that $\psi x=-\psi(x+\mu)$, then we have

$$\psi x=A_1\cos\frac{\pi x}{\mu}+A_3\cos\frac{3\pi x}{\mu}+\text{\&c.}$$

or

$$=B_1\sin\frac{\pi x}{\mu}+B_3\sin\frac{3\pi x}{\mu}+\text{\&c},$$

according as ψx is even or uneven

But there is another totally different form in which ψx may generally be exhibited, viz.

$$\psi x=\phi x+\phi(x-\mu)+\phi(x+\mu)+\phi(x-2\mu)+\phi(x+2\mu)+\text{\&c}$$

or

$$=\phi x-\phi(x-\mu)-\phi(x+\mu)+\phi(x-2\mu)+\phi(x+2\mu)-\text{\&c},$$

according as $\psi(x+\mu)=\psi x$ or $=-\psi x$.

The sine and cosine cannot be so expressed, but the other primary circular functions do admit of this form, as, *ex. gr.*, in the formulæ

$$\cot x=\frac{1}{x}+\frac{1}{x-\pi}+\frac{1}{x+\pi}+\frac{1}{x-2\pi}+\frac{1}{x+2\pi}+\text{\&c.},$$

$$\operatorname{cosec} x=\frac{1}{x}-\frac{1}{x-\pi}-\frac{1}{x+\pi}+\frac{1}{x-2\pi}+\frac{1}{x+2\pi}-\text{\&c}$$

(in which, after the first term, the series proceed by pairs of terms, so that for every term $\frac{1}{x-n\pi}$ there is a term $\frac{1}{x+n\pi}$)

Thus in general (although the sine and cosine are, as just mentioned, exceptions) we shall have, by equating the different forms of ψx, identities such as *ex. gr* (if ψ is even)

$$\phi x+\phi(x-\mu)+\phi(x+\mu)+\ \&c=A_0+A_1\cos\frac{2\pi x}{\mu}+A_2\cos\frac{4\pi x}{\mu}+\ \&c.$$

Also, it will be seen in § 10 that in certain cases even when ψx is not periodic it may be exhibited in the form $\phi x+\phi(x-\mu)+\phi(x+\mu)+$ &c., and we shall obtain identities in which the two sides of the equation are non-periodic.

§ 3 Before applying these principles to the elliptic functions, it is convenient to write down at once the following eight formulæ, which are to be found in the 'Fundamenta Nova' (pp 101, 102, &c), and which are all placed together in DURÈGE's 'Theorie der elliptischen Functionen' (Leipzig, 1861), pp. 226, 227.—

$$\sin\operatorname{am}u=\frac{2\pi}{k\mathrm{K}}\left\{\frac{q^{\frac{1}{2}}}{1-q}\sin\frac{\pi u}{2\mathrm{K}}+\frac{q^{\frac{3}{2}}}{1-q^3}\sin\frac{3\pi u}{2\mathrm{K}}+\ \&c.\right\}, \qquad (1)$$

$$\cos\operatorname{am}u=\frac{2\pi}{k\mathrm{K}}\left\{\frac{q}{1+q}\cos\frac{\pi u}{2\mathrm{K}}+\frac{q^{\frac{3}{2}}}{1+q^3}\cos\frac{3\pi u}{2\mathrm{K}}+\ \&c.\right\}, \qquad (2)$$

$$\Delta\operatorname{am}u=\frac{\pi}{2\mathrm{K}}\left\{1+\frac{4q}{1+q^2}\cos\frac{\pi u}{\mathrm{K}}+\frac{4q^2}{1+q^4}\cos\frac{2\pi u}{\mathrm{K}}+\ \&c\right\}, \qquad (3)$$

$$\tan\operatorname{am}u=\frac{\pi}{2k'\mathrm{K}}\left\{\tan\frac{\pi u}{2\mathrm{K}}-\frac{4q^2}{1+q^2}\sin\frac{\pi u}{\mathrm{K}}+\frac{4q^4}{1+q^4}\sin\frac{2\pi u}{\mathrm{K}}-\ \&c\right\}, \qquad (4)$$

$$\operatorname{cosec}\operatorname{am}u=\frac{\pi}{2\mathrm{K}}\left\{\operatorname{cosec}\frac{\pi u}{2\mathrm{K}}+\frac{4q}{1-q}\sin\frac{\pi u}{2\mathrm{K}}+\frac{4q^3}{1-q^3}\sin\frac{3\pi u}{2\mathrm{K}}+\ \&c.\right\}, \qquad (5)$$

$$\sec\operatorname{am}u=\frac{\pi}{2k'\mathrm{K}}\left\{\sec\frac{\pi u}{2\mathrm{K}}-\frac{4q}{1+q}\cos\frac{\pi u}{2\mathrm{K}}+\frac{4q^3}{1+q^3}\cos\frac{3\pi u}{2\mathrm{K}}-\ \&c\right\}, \qquad (6)$$

$$\frac{1}{\Delta\operatorname{am}u}=\frac{\pi}{2k'\mathrm{K}}\left\{1-\frac{4q}{1+q^2}\cos\frac{\pi u}{\mathrm{K}}+\frac{4q^2}{1+q^4}\cos\frac{2\pi u}{\mathrm{K}}-\ \&c.\right\}, \qquad (7)$$

$$\cot\operatorname{am}u=\frac{\pi}{2\mathrm{K}}\left\{\cot\frac{\pi u}{2\mathrm{K}}-\frac{4q^2}{1+q^2}\sin\frac{\pi u}{\mathrm{K}}-\frac{4q^4}{1+q^4}\sin\frac{2\pi u}{\mathrm{K}}-\ \&c.\right\}, \qquad (8)$$

wherein, of course, $q=e^{-\frac{\pi\mathrm{K}'}{\mathrm{K}}}$

In what follows, let $r=e^{-\frac{\pi\mathrm{K}}{\mathrm{K}'}}$, and take

$$\mu=\frac{\pi\mathrm{K}'}{\mathrm{K}},\quad \nu=\frac{\pi\mathrm{K}}{\mathrm{K}'},$$

so that

$$q=e^{-\mu},\quad r=e^{-\nu},\quad \text{and } \mu\nu=\pi^2.$$

Also let $x=\frac{\pi u}{2K}$ and $z=\frac{\pi u}{2K'}$, so that $z=\frac{\pi x}{\mu}=\frac{\nu x}{\pi}$.

§ 4. The process of transformation into the form

$$\phi x\pm\phi(x-\mu)\pm\phi(x+\mu)+\text{\&c.}$$

may be conveniently exhibited on (2), we have

$\cos\operatorname{am}\frac{2Kx}{\pi}=\cos\operatorname{am}u=\sec\operatorname{am}(ui,k')$, which, from (6),

$$=\frac{\pi}{2kK'}\left\{\frac{2}{e^z+e^{-z}}-\frac{4r}{1+r}\,\frac{e^z+e^{-z}}{2}+\frac{4r^3}{1+r^3}\,\frac{e^{3z}+e^{-3z}}{2}-\text{\&c.}\right\}$$

$$=\frac{\pi}{kK'}\left\{\frac{1}{e^z+e^{-z}}-(e^z+e^{-z})(r-r^3+r^5-\text{\&c.})+(e^{3z}+e^{-3z})(r^3-r^6+r^9-\text{\&c.})-\text{\&c}\right\}$$

$$=\frac{\pi}{kK'}\left\{\frac{1}{e^z+e^{-z}}-\frac{re^z}{1+r^2e^{2z}}-\frac{re^{-z}}{1+r^2e^{-2z}}+\frac{r^2e^z}{1+r^4e^{2z}}+\frac{r^2e^{-z}}{1+r^4e^{-2z}}-\text{\&c.}\right\}$$

$$=\frac{\pi}{kK'}\left\{\frac{1}{e^z+e^{-z}}-\frac{1}{re^z+r^{-1}e^{-z}}-\frac{1}{r^{-1}e^z+re^{-z}}+\frac{1}{r^2e^z+r^{-2}e^{-z}}+\frac{1}{r^{-2}e^{2z}+r^2e^{-2z}}-\text{\&c}\right\}$$

$$=\frac{\pi}{kK'}\left\{\frac{1}{r^{\frac{x}{\pi}}+r^{-\frac{x}{\pi}}}-\frac{1}{r^{\frac{x}{\pi}-1}+r^{-\left(\frac{x}{\pi}-1\right)}}-\frac{1}{r^{\frac{x}{\pi}+1}+r^{-\left(\frac{x}{\pi}+1\right)}}+\frac{1}{r^{\frac{x}{\pi}-2}+r^{-\left(\frac{x}{\pi}-2\right)}}+\frac{1}{r^{\frac{x}{\pi}+2}+r^{-\left(\frac{x}{\pi}+2\right)}}-\text{\&c.}\right\}. \quad (9)$$

The process requires that re^z should be <1, that is, that u should be $<2K$, but as both sides of the equation are such that they change sign without being altered in value when $u+2K$ is written for u, we see that the result obtained is true for all values of u. Thus we have

$$\cos\operatorname{am}2Kx=\frac{\pi}{kK'}\left\{\frac{1}{r^x+r^{-x}}-\frac{1}{r^{x-1}+r^{-(x-1)}}-\frac{1}{r^{x+1}+r^{-(x+1)}}+\frac{1}{r^{x-2}+r^{-(x-2)}}+\text{\&c}\right\}. \qquad (10)$$

for all values of x.

If in (10) we take $x=0$, we have

$$\frac{kK'}{\pi}=\tfrac{1}{2}-\frac{2}{r^{-1}+r}+\frac{2}{r^{-2}+r^2}-\text{\&c.},$$

or, writing K and k' for K' and k, and therefore q for r,

$$\frac{2k'K}{\pi}=1-\frac{4q}{1+q^2}+\frac{4q^2}{1+q^4}-\text{\&c},$$

which is at once seen to follow from (7), and is given by JACOBI, 'Fundamenta Nova,' p. 103.

It is, of course, easy to deduce (9) directly from the infinite product

$$\sqrt{\left(\frac{1-\cos\operatorname{am}u}{1+\cos\operatorname{am}u}\right)}=\tan\frac{x}{2}\prod_1^\infty\frac{(1-2q^{2n}\cos x+q^{4n})(1+2q^{2n-1}\cos x+q^{4n-2})}{(1+2q^{2n}\cos x+q^{4n})(1-2q^{2n-1}\cos x+q^{4n-2})},$$

for consider

$$\frac{1-2q^{2n}\cos x+q^{4n}}{1+2q^{2n}\cos x+q^{4n}},\text{ which }=\frac{(1-q^{2n}e^{ix})(1-q^{2n}e^{-ix})}{(1+q^{2n}e^{ix})(1+q^{2n}e^{-ix})}$$

Taking the logarithm and differentiating, we obtain, after a little reduction,

$$\frac{\pi i}{\mathrm{K}}\left\{\frac{1}{q^{2n}e^{ix}-q^{-2n}e^{-ix}}+\frac{1}{q^{-2n}e^{ix}-q^{2n}e^{-ix}}\right\}.$$

Similarly, from the uneven factor we get

$$-\frac{\pi i}{\mathrm{K}}\left\{\frac{1}{q^{2n-1}e^{ix}-q^{-(2n-1)}e^{-ix}}+\frac{1}{q^{-(2n-1)}e^{ix}-q^{2n-1}e^{-ix}}\right\};$$

thus

$$\frac{k'}{\cos\operatorname{am}(\mathrm{K}-u)}=\frac{\pi}{2\mathrm{K}}\operatorname{cosec}x+\frac{\pi i}{\mathrm{K}}\Sigma\left\{\frac{1}{q^{2n}e^{ix}-q^{-2n}e^{-ix}}+\frac{1}{q^{-2n}e^{ix}-q^{2n}e^{-ix}}-\frac{1}{q^{2n-1}e^{ix}-q^{-(2n-1)}e^{-ix}}\right.$$
$$\left.-\frac{1}{q^{-(2n-1)}e^{ix}-q^{2n-1}e^{-ix}}\right\}.$$

Replace u by $\mathrm{K}-u$, that is to say x by $\frac{1}{2}\pi-x$, and remembering that $e^{\frac{1}{2}i\pi}=i$, $e^{-\frac{1}{2}i\pi}=-i$, we find

$$\sec\operatorname{am}u=\frac{\pi}{2k'\mathrm{K}}\left\{\sec x+2\Sigma\left(\frac{1}{q^{2n}e^{ix}+q^{-2n}e^{-ix}}+\cdot\quad-\frac{1}{q^{2n-1}e^{ix}+q^{-(2n-1)}e^{-ix}}-\ldots\right)\right\}$$

Herein write ui for u and k' for k, and we obtain the value on the right-hand side of (9) for $\sec\operatorname{am}(ui, k')$, that is, for $\cos\operatorname{am}u$.

If the other formulæ in the group (1) to (8) be transformed in the same way, viz. by use of the identical equations

$$\sin\operatorname{am}u=-i\tan(ui, k'),$$
$$\Delta\operatorname{am}u=\operatorname{cosec}\operatorname{am}(ui+\mathrm{K}', k'),$$

we obtain the following seven formulæ:—

$$\sin\operatorname{am}2\mathrm{K}x=-\frac{\pi}{2k\mathrm{K}'}\left\{\frac{r^x-r^{-x}}{r^x+r^{-x}}-\frac{r^{x-1}-r^{-(x-1)}}{r^{x-1}+r^{-(x-1)}}-\frac{r^{x+1}-r^{-(x+1)}}{r^{x+1}+r^{-(x+1)}}\right.$$
$$\left.+\frac{r^{x-2}-r^{-(x-2)}}{r^{x-2}+r^{-(x-2)}}+\frac{r^{x+2}-r^{-(x+2)}}{r^{x+2}+r^{-(x+2)}}-\&c\right\}, \quad . \quad . \quad . \quad . \qquad (11)$$

$$\Delta\operatorname{am}2\mathrm{K}x=\frac{\pi}{\mathrm{K}'}\left\{\frac{1}{r^x+r^{-x}}+\frac{1}{r^{x-1}+r^{-(x-1)}}+\frac{1}{r^{x+1}+r^{-(x+1)}}+\&c.\right\}, \quad . \quad . \quad . \qquad (12)$$

$$\tan\operatorname{am}2\mathrm{K}x=\frac{\pi}{k'\mathrm{K}'}\left\{\frac{1}{r^{x-\frac{1}{2}}-r^{-(x-\frac{1}{2})}}+\frac{1}{r^{x+\frac{1}{2}}-r^{-(x+\frac{1}{2})}}+\frac{1}{r^{x-\frac{3}{2}}-r^{-(x-\frac{3}{2})}}+\&c.\right\}, \quad . \qquad (13)$$

$$\frac{1}{\sin\operatorname{am}2\mathrm{K}x}=-\frac{\pi}{2\mathrm{K}'}\left\{\frac{r^x+r^{-x}}{r^x-r^{-x}}-\frac{r^{x-1}+r^{-(x-1)}}{r^{x-1}-r^{-(x-1)}}-\frac{r^{x+1}+r^{-(x+1)}}{r^{x+1}-r^{-(x+1)}}+\&c.\right\}, \quad . \qquad (14)$$

$$\frac{1}{\cos\operatorname{am}2\mathrm{K}x}=\frac{\pi}{k'\mathrm{K}'}\left\{\frac{1}{r^{x-\frac{1}{2}}-r^{-(x-\frac{1}{2})}}-\frac{1}{r^{x+\frac{1}{2}}-r^{-(x+\frac{1}{2})}}-\frac{1}{r^{x-\frac{3}{2}}-r^{-(x-\frac{3}{2})}}+\&c.\right\}, \quad . \qquad (15)$$

$$\frac{1}{\Delta\operatorname{am}2\mathrm{K}x}=\frac{\pi}{k'\mathrm{K}'}\left\{\frac{1}{r^{x-\frac{1}{2}}+r^{-(x-\frac{1}{2})}}+\frac{1}{r^{x+\frac{1}{2}}+r^{-(x+\frac{1}{2})}}+\frac{1}{r^{x-\frac{3}{2}}+r^{-(x-\frac{3}{2})}}+\&c.\right\}, \quad . \quad . \qquad (16)$$

$$\cot\operatorname{am}2\mathrm{K}x=-\frac{\pi}{\mathrm{K}'}\left\{\frac{1}{r^x-r^{-x}}+\frac{1}{r^{x-1}-r^{-(x-1)}}+\frac{1}{r^{x+1}-r^{-(x+1)}}+\&c.\right\}. \quad . \quad . \quad . \quad . \qquad (17)$$

It must be remarked that in (11) and (14) the number of terms must always be uneven; this point will be noticed at greater length further on (§ 10).

§ 5. Writing the hyperbolic sine, cosine, &c. as sinh, cosh, &c., these formulæ may also be written in a somewhat different form: thus

$$\cos \operatorname{am} u=\frac{\pi}{2kK'}\left\{\operatorname{sech}\frac{\pi u}{2K'}-\operatorname{sech}\frac{\pi}{2K'}(u-2K)-\operatorname{sech}\frac{\pi}{2K'}(u+2K)+\&c.\right\},$$

$$\sin \operatorname{am} u=\frac{\pi}{2kK'}\left\{\tanh\frac{\pi u}{2K'}-\tanh\frac{\pi}{2K'}(u-2K)-\tanh\frac{\pi}{2K'}(u+2K)+\&c.\right\},$$

and similarly for the others

I do not think it likely that the formulæ (10) to (17) are new, but I have not succeeded in finding them anywhere. SCHELLBACH ('Die Lehre von den elliptischen Integralen . . ' Berlin, 1864, p 33) gives the corresponding forms for θu, $\theta_1 u$, &c, but he does not allude to the similar expressions for the elliptic functions. It would, however, in any case have been necessary for the explanation of the rest of this paper to have written down the latter and demonstrated one of them.

§ 6 By equating the values of sin am u, cos am u, &c, as given by (1) to (8) and by (10) to (17), we obtain a series of identities of an algebraical character (*i e.* which are independent of the notation of elliptic functions) Thus from (2) and (10) we have (remembering the definitions of μ, ν, &c at the end of § 3)

$$\frac{\pi}{kK}\left\{\frac{\cos x}{\cosh\frac{\mu}{2}}+\frac{\cos 3x}{\cosh\frac{3\mu}{2}}+\frac{\cos 5x}{\cosh\frac{5\mu}{2}}+\&c.\right\}=\frac{\pi}{2kK'}\{\operatorname{sech} z-\operatorname{sech}(z-\nu)-\operatorname{sech}(z+\nu)+\&c\},$$

viz

$$\frac{\cos x}{\cosh\frac{\mu}{2}}+\frac{\cos 3x}{\cosh\frac{3\mu}{2}}+\frac{\cos 5x}{\cosh\frac{5\mu}{2}}+\&c.=\frac{\pi}{2\mu}\left\{\operatorname{sech}\frac{\pi x}{\mu}-\operatorname{sech}\frac{\pi}{\mu}(x-\pi)-\operatorname{sech}\frac{\pi}{\mu}(x+\pi)+\&c.\right\}$$

This may be written (by interchanging x and z, μ and ν) in the rather more convenient form

$$\operatorname{sech} x-\operatorname{sech}(x-\mu)-\operatorname{sech}(x+\mu)+\operatorname{sech}(x-2\mu)+\operatorname{sech}(x+2\mu)-\&c.$$

$$=\frac{2\pi}{\mu}\left\{\frac{\cos\frac{\pi x}{\mu}}{\cosh\frac{\pi^2}{2\mu}}+\frac{\cos\frac{3\pi x}{\mu}}{\cosh\frac{3\pi^2}{2\mu}}+\frac{\cos\frac{5\pi x}{\mu}}{\cosh\frac{5\pi^2}{\mu}}+\&c\right\}. \quad . \quad . \quad . \quad . \quad . \quad (18)$$

In the same way, by comparing (1) and (11), we find

$$\tanh x-\tanh(x-\mu)-\tanh(x+\mu)+\&c.=\frac{2\pi}{\mu}\left\{\frac{\sin\frac{\pi x}{\mu}}{\sinh\frac{\pi^2}{2\mu}}+\frac{\sin\frac{3\pi x}{\mu}}{\sinh\frac{3\pi^2}{2\mu}}+\&c\right\}, \quad . \quad . \quad (19)$$

and by comparing (3) and (12),

$$\operatorname{sech} x+\operatorname{sech}(x-\mu)+\operatorname{sech}(x+\mu)+\&c.=\frac{\pi}{\mu}\left\{1+\frac{2\cos\frac{2\pi x}{\mu}}{\cosh\frac{\pi^2}{\mu}}+\frac{2\cos\frac{4\pi x}{\mu}}{\cosh\frac{2\pi^2}{\mu}}+\&c.\right\}. \quad . \quad (20)$$

The comparison of (4) and (13) gives

$$\operatorname{cosech}\left(x-\frac{\mu}{2}\right)+\operatorname{cosech}\left(x+\frac{\mu}{2}\right)+\operatorname{cosech}\left(x-\frac{3\mu}{2}\right)+\operatorname{cosech}\left(x+\frac{3\mu}{2}\right)+\&c.$$

$$=\frac{\pi}{\mu}\left\{-\tan\frac{\pi x}{\mu}+\frac{4\sin\frac{2\pi x}{\mu}}{e^{\frac{2\pi^2}{\mu}}+1}-\frac{4\sin\frac{4\pi x}{\mu}}{e^{\frac{4\pi^2}{\mu}}+1}+\&c.\right\},$$

which, on replacing x by $x+\frac{1}{2}\mu$, becomes

$$\operatorname{cosech} x+\operatorname{cosech}(x-\mu)+\operatorname{cosech}(x+\mu)+\&c.=\frac{\pi}{\mu}\left\{\cot\frac{\pi x}{\mu}-\frac{4\sin\frac{2\pi x}{\mu}}{e^{\frac{2\pi^2}{\mu}}+1}-\frac{4\sin\frac{4\pi x}{\mu}}{e^{\frac{4\pi^2}{\mu}}+1}-\&c.\right\}. \qquad (21)$$

From (5) and (14) we deduce

$$\coth x-\coth(x-\mu)-\coth(x+\mu)+\&c.=\frac{\pi}{\mu}\left\{\operatorname{cosec}\frac{\pi x}{\mu}+\frac{4\sin\frac{\pi x}{\mu}}{e^{\frac{\pi^2}{\mu}}-1}+\frac{4\sin\frac{3\pi x}{\mu}}{e^{\frac{3\pi^2}{\mu}}-1}+\&c.\right\}. \qquad (22)$$

The comparison of (6) and (15) gives

$$-\operatorname{cosech}\left(x-\frac{\mu}{2}\right)+\operatorname{cosech}\left(x+\frac{\mu}{2}\right)+\operatorname{cosech}\left(x-\frac{3\mu}{2}\right)-\&c$$

$$=\frac{\pi}{\mu}\left\{\sec\frac{\pi x}{\mu}-\frac{4\cos\frac{\pi x}{\mu}}{e^{\frac{\pi^2}{\mu}}+1}+\frac{4\cos\frac{3\pi x}{\mu}}{e^{\frac{3\pi^2}{\mu}}+1}-\&c.\right\},$$

which, on replacing x by $x+\frac{1}{2}\mu$, becomes

$$\operatorname{cosech} x-\operatorname{cosech}(x-\mu)-\operatorname{cosech}(x+\mu)+\&c.=\frac{\pi}{\mu}\left\{\operatorname{cosec}\frac{\pi x}{\mu}-\frac{4\sin\frac{\pi x}{\mu}}{e^{\frac{\pi^2}{\mu}}+1}-\frac{4\sin\frac{3\pi x}{\mu}}{e^{\frac{3\pi^2}{\mu}}+1}-\&c.\right\}. \qquad (23)$$

The comparison of the forms for $\frac{1}{\Delta \operatorname{am} u}$, (7) and (16), merely gives an equation which, on replacement of x by $x+\frac{1}{2}\mu$, is identical with that resulting from $\Delta \operatorname{am} u$, viz. (20), while the forms of $\cot \operatorname{am} u$, (8) and (17), lead at once to (21).

In the expressions on the left-hand side of (19) and (22) the number of terms included must be uneven.

It is proper to remark that the formulæ for $\phi x-\phi(x-\mu)-\phi(x+\mu)+\&c.$ can be readily deduced from those for $\phi x+\phi(x-\mu)+\phi(x+\mu)+\&c.$; thus (18) is a consequence of (20) and (23) of (21). For *ex. gr.* in (20) write 2μ for μ, and we have

$$\operatorname{sech} x+\operatorname{sech}(x-2\mu)+\operatorname{sech}(x+2\mu)+\&c.=\frac{\pi}{2\mu}\left\{1+\frac{2\cos\frac{\pi x}{\mu}}{\cosh\frac{\pi^2}{2\mu}}+\frac{2\cos\frac{2\pi x}{\mu}}{\cosh\frac{\pi^2}{\mu}}+\&c.\right\}.$$

Double this result and subtract (20) from it, and we have (18). In a similar way (23) follows from (21).

The converse proposition is not true, viz. given the value of $\phi x-\phi(x-\mu)-\phi(x+\mu)+\&c.$, we cannot deduce the value of $\phi x+\phi(x-\mu)+\phi(x+\mu)+\&c$

§ 7. The results admit of being connected directly with FOURIER's theorem in the following manner: it is of course well known that every integral of the form

$$\int_0^\pi \phi(x)\cos nx\,dx=A'_n,$$

or, let us write,

$$\int_0^\mu \phi(x)\cos\frac{n\pi x}{\mu}\,dx=A_n,$$

gives rise to a series

$$\phi x=\frac{1}{\mu}\left\{A_0+2A_1\cos\frac{\pi x}{\mu}+2A_2\cos\frac{2\pi x}{\mu}+\&c.\right\},$$

and that similarly from

$$\int_0^\mu \phi(x)\sin\frac{n\pi x}{\mu}\,dx=B_n$$

there follows

$$\phi x=\frac{2}{\mu}\left\{B_1\sin\frac{\pi x}{\mu}+B_2\sin\frac{2\pi x}{\mu}+\&c\right\},$$

and it will now be shown that if ϕx is an even function of x, and if

$$\int_0^\infty \phi(x)\cos\frac{n\pi x}{\mu}\,dx=A_n,$$

then

$$\phi x+\phi(x-\mu)+\phi(x+\mu)+\phi(x-2\mu)+\phi(x+2\mu)+\&c.=\frac{2}{\mu}\left\{A_0+2A_2\cos\frac{2\pi x}{\mu}+2A_4\cos\frac{4\pi x}{\mu}+\&c\right\}, \quad (24)$$

and

$$\phi x-\phi(x-\mu)-\phi(x+\mu)+\phi(x-2\mu)+\phi(x+2\mu)-\&c=\frac{4}{\mu}\left\{A_1\cos\frac{\pi x}{\mu}+A_3\cos\frac{3\pi x}{\mu}+\&c.\right\}, \quad . \quad (25)$$

also, that if ϕx is an uneven function of x, and if

$$\int_0^\infty \phi(x)\sin\frac{n\pi x}{\mu}\,dx=B_n,$$

then

$$\phi x+\phi(x-\mu)+\phi(x+\mu)+\&c.=\frac{4}{\mu}\left\{B_2\sin\frac{2\pi x}{\mu}+B_4\sin\frac{4\pi x}{\mu}+\&c.\right\}, \quad . \quad (26)$$

and

$$\phi x-\phi(x-\mu)-\phi(x+\mu)+\&c.=\frac{4}{\mu}\left\{B_1\sin\frac{\pi x}{\mu}+B_3\sin\frac{3\pi x}{\mu}+\&c\right\} \quad . \quad . \quad (27)$$

It is sufficient to prove one of these formulæ; take (24). Since ϕx is an even function, $\phi x+\phi(x-\mu)+\phi(x+\mu)+\&c.$ (which call ψx) is a periodic function with period μ, and

the right-hand side of (24) must be of the form

$$A'_0+A'_2\cos\frac{2\pi x}{\mu}+A'_4\cos\frac{4\pi x}{\mu}+\&c.$$

Now, ϕ being even,

$$2A_{2m}=\int_{-\infty}^{\infty}\phi(x)\cos\frac{2m\pi x}{\mu}\,dx$$

$$=\left\{\ldots+\int_{-\mu}^{0}+\int_{0}^{\mu}+\int_{\mu}^{2\mu}+\ldots\right\}\phi(x)\cos\frac{2m\pi x}{\mu}\,dx.$$

But

$$\int_{-\mu}^{0}\phi(x)\cos\frac{2m\pi x}{\mu}\,dx=\int_{0}^{\mu}\phi(\xi-\mu)\cos\frac{2m\pi\xi}{\mu}\,d\xi,\text{ on taking }x=\xi-\mu,$$

and

$$\int_{\mu}^{2\mu}\phi(x)\cos\frac{2m\pi x}{\mu}\,dx=\int_{0}^{\mu}\phi(\xi+\mu)\cos\frac{2m\pi\xi}{\mu}\,d\xi,\text{ on taking }x=\xi+\mu,$$

thus

$$2A_{2m}=\int_{0}^{\mu}\{\phi\xi+\phi(\xi-\mu)+\phi(\xi+\mu)+\ldots\}\cos\frac{2m\pi\xi}{\mu}\,d\xi$$

$$=\int_{0}^{\mu}\psi(x)\cos\frac{2m\pi x}{\mu}\,dx=A'_{2m}\ \frac{\mu}{2},$$

unless $m=0$, in which case

$$2A_0=A'_0\,.\,\mu,$$

so that (24) is proved Formula (25) may be either obtained independently by a similar treatment of the integral

$$2A_{2m+1}=\int_{-\infty}^{\infty}\phi(x)\cos\frac{(2m+1)\pi x}{\mu}\,dx,$$

or it may be deduced from (24) by writing therein 2μ for μ (remarking that by this substitution A_{2m} becomes A_m) and subtracting (24) from the double of the equation so formed. Similar processes apply to (26) and (27).

The method by which the formulæ (24) to (27) have been just obtained is the same as that by which Sir W THOMSON (Quarterly Journal of Mathematics, t i. p. 316) deduced the theorem

$$e^{-x^2}-e^{-(x-\mu)^2}-e^{-(x+\mu)^2}+\&c.=\frac{2\sqrt{\pi}}{\mu}\left\{e^{-\frac{\pi^2}{4\mu^2}}\cos\frac{\pi x}{\mu}+e^{-\frac{9\pi^2}{4\mu^2}}\cos\frac{3\pi x}{\mu}+\&c\right\}\quad.\quad(28)$$

from the integral

$$\int_0^{\infty}e^{-x^2}\cos nx\,dx=\frac{\sqrt{\pi}}{2}e^{-\frac{n^2}{4}}.$$

It was after reading Sir W. THOMSON'S paper three or four years ago, that I made a list of all the suitable integrals of the form

$$\int_0^{\infty}\phi(x)\cos nx\,dx$$

that were given in Professor DE HAAN'S 'Nouvelles Tables d'Intégrales définies'

(Leyden, 1867), and deduced therefrom the resulting identities. The only formulæ so obtained which appeared of interest were, in fact, those which are given in the present paper, viz. (18) to (23), but at the time I was not aware of their connexion with the theory of Elliptic Functions. It was only recently, after obtaining the values of sin am x &c. in (10) to (17), that I remarked that the resulting identities were the same as those which I had previously deduced by the aid of Sir W. THOMSON's principle.

It was shown by CAYLEY at the end of Sir W. THOMSON's paper that the identity (28) corresponds to

$$\Theta(ui, k)=\sqrt{\left(\frac{\mathrm{K}}{\mathrm{K}'}\right)}e^{\frac{\pi u^2}{4\mathrm{KK}'}}\,\mathrm{H}(u+\mathrm{K}', k'); \qquad . \quad . \qquad (29)$$

and it is singular that all the identities that follow from the method of this section thus appear to correspond either to elliptic or theta-function transformations Speaking generally, the only evaluable integrals of the requisite form are derived from

$$\int_0^\infty e^{-a^2x^2}\cos 2bx\,dx=\frac{\sqrt{\pi}}{2a}e^{-\frac{b^2}{a^2}} \text{ and } \int_0^\infty e^{-ax}\cos bx\,dx=\frac{a}{a^2+b^2}$$

(including as derivations the corresponding sine formulæ), of which the former give rise to theta-function relations, and the latter to elliptic-function relations.

§ 8 The integrals that produce the formulæ (18) to (23), and the manner in which the latter are obtained from them, deserve some attention. Thus

$$\begin{aligned}\int_0^\infty \frac{\cos nx}{e^x+e^{-x}}\,dx&=\int_0^\infty \cos nx\,(e^{-x}-e^{-3x}+e^{-5x}-\&\text{c})dx\\&=\frac{1}{n^2+1^2}-\frac{3}{n^2+3^2}+\frac{5}{n^2+5^2}-\&\text{c}\\&=\frac{\pi}{4}\operatorname{sech}\frac{n\pi}{2},\end{aligned}$$

whereby (18) and (20) follow at once from (25) and (24).

In a similar way we can show that

$$\int_0^\infty \frac{\sin nx}{e^x-e^{-x}}\,dx=\frac{\pi}{4}\tanh\frac{n\pi}{2}=\frac{\pi}{4}\,\frac{e^{n\pi}-1}{e^{n\pi}+1},$$

but the series obtained from the direct application of this integral would not converge, and in order to deduce (21) and (23) from (26) and (27), it is necessary to express the integral in the form

$$\frac{\pi}{4}\left\{1-\frac{2}{e^{n\pi}+1}\right\},$$

and to make use of the formulæ

$$\begin{aligned}\tfrac{1}{2}\cot\tfrac{1}{2}\theta\ &=\sin\theta+\sin 2\theta+\sin 3\theta+\&\text{c},\\ \tfrac{1}{2}\operatorname{cosec}\theta&=\sin\theta+\sin 3\theta+\sin 5\theta+\&\text{c}.\end{aligned}$$

This renders the process not so satisfactory from a logical point of view; but practi-

cally our knowledge that $\sin\theta+\sin 2\theta+\&c.$ and $\sin\theta+\sin 3\theta+\&c.$ are the FOURIER'S-theorem equivalents of $\frac{1}{2}\cot\frac{1}{2}\theta$ and $\frac{1}{2}\operatorname{cosec}\theta$ would be sufficient to leave no doubt of the accuracy of the formulæ so obtained.

In regard to the other two integrals required for (19) and (22), viz.

$$\int_0^\infty \tanh x \sin nx\, dx \quad \text{and} \quad \int_0^\infty \coth x \sin nx\, dx,$$

it is to be observed that, stated in this form, their values are indeterminate; for the former

$$=\int_0^\infty \left(1-\frac{2}{e^{2x}+1}\right)\sin nx\, dx,$$

and the latter

$$=\int_0^\infty \left(1+\frac{2}{e^{2x}-1}\right)\sin nx\, dx,$$

both of which involve $\cos\infty$. But in point of fact for our purpose the ∞ of the limit of the integral is not arbitrary, but is to be of the form $(m+1)\pi$, the lower limit being $-m\pi$ (or if we replace $\sin nx$ by $\sin\frac{n\pi x}{\mu}$, the limits are $(m+1)\mu$ and $-m\mu$). Taking then m infinite,

$$\int_0^{(m+1)\pi}\tanh x\sin nx\,dx=\int_0^{(m+1)\pi}\sin nx\,dx-2\int_0^\infty\frac{\sin nx}{e^{2x}+1}\,dx$$

$$=\left[-\frac{\cos nx}{n}\right]_0^{(m+1)\pi}-2\left\{\frac{n}{n^2+2^2}-\frac{n}{n^2+4^2}+\frac{n}{n^2+6^2}-\&c.\right\}$$

$$=\left[-\frac{\cos nx}{n}\right]_0^{(m+1)\pi}-\frac{n}{2}\left\{\frac{1}{(\frac{1}{2}n)^2+1^2}-\frac{1}{(\frac{1}{2}n)^2+2^2}+\frac{1}{(\frac{1}{2}n)^2+3^2}-\&c.\right\}$$

$$=\left[-\frac{\cos nx}{n}\right]_0^{(m+1)\pi}-\frac{1}{n}+\frac{\pi}{2}\operatorname{cosech}\frac{n\pi}{2}$$

Similarly

$$\int_{-m\pi}^0\tanh x\sin nx\,dx=\int_0^{m\pi}\tanh x\sin nx\,dx$$

$$=\left[-\frac{\cos nx}{n}\right]_0^{m\pi}-\frac{1}{n}+\frac{\pi}{2}\operatorname{cosech}\frac{n\pi}{2},$$

and therefore

$$\int_{-m\pi}^{(m+1)\pi}\tanh x\sin nx\,dx=\left[-\frac{\cos nx}{n}\right]_0^{(m+1)\pi}+\left[-\frac{\cos nx}{n}\right]_0^{m\pi}-\frac{2}{n}+\pi\operatorname{cosech}\frac{n\pi}{2}$$

$$=\pi\operatorname{cosech}\frac{n\pi}{2}, \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

whether m be even or uneven, if n is uneven, whence the result in (19) follows directly.

A similar course of procedure shows that

$$\int_{-m\pi}^{(m+1)\pi}\coth x\sin nx\,dx=\pi\coth\frac{n\pi}{2}=\pi\left\{1+\frac{2}{e^{n\pi}-1}\right\},$$

from which (22) may be derived.

In his 'Nouvelles Tables,' T. 265, Prof. De Haan assigns definite values to the indeterminate integrals

$$\int_0^\infty\tanh x\sin nx\,dx\ \text{and}\ \int_0^\infty\coth x\sin nx\,dx\,;$$

and it is noticeable that, if these values be used, they lead to the same results as those just investigated. The reason is that the integrals in De Haan are in effect evaluated on the assumption that $\cos\infty=0$, and if in (30) we had, in place of the first two terms, viz.

$$\pm\frac{1}{n}+\frac{1}{n}\mp\frac{1}{n}+\frac{1}{n},$$

written

$$0+\frac{1}{n}+0+\frac{1}{n},$$

it is clear that the final result would have been the same

It may be remarked that the identities (19) and (22) may be somewhat generalized by means of the integrals

$$\int_0^\infty\frac{\sinh ax}{\cosh bx}\sin nx\,dx=\frac{\pi}{b}\,\frac{\sinh\frac{n\pi}{2b}\sin\frac{a\pi}{2b}}{\cosh\frac{n\pi}{b}+\cos\frac{a\pi}{b}},$$

$$\int_0^\infty\frac{\cosh ax}{\sinh bx}\sin nx\,dx=\frac{\pi}{2b}\,\frac{\sinh\frac{n\pi}{b}}{\cosh\frac{n\pi}{b}+\cos\frac{a\pi}{b}}\,;$$

while other identities may be derived from

$$\int_0^\infty\frac{\cosh ax}{\cosh bx}\cos nx\,dx=\frac{\pi}{b}\,\frac{\cosh\frac{n\pi}{2b}\cos\frac{a\pi}{2b}}{\cosh\frac{n\pi}{b}+\cos\frac{a\pi}{b}},$$

$$\int_0^\infty\frac{\sinh ax}{\sinh bx}\cos nx\,dx=\frac{\pi}{2b}\,\frac{\sin\frac{a\pi}{b}}{\cosh\frac{n\pi}{b}+\cos\frac{a\pi}{a}},$$

in which, of course, a is to be supposed less than b.

§ 9. The well-known reciprocity of f and ϕ in the formulæ

$$f(n)=\sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty\phi(x)\cos nx\,dx,\quad f(n)=\sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty\phi(x)\sin nx\,dx$$

leads to a corresponding reciprocity in the formulæ (24) to (27). Thus from the first of the integrals we deduce that, ϕ and f being both even functions, if

$$\phi x+\phi(x-\mu)+\phi(x+\mu)+\&c=\frac{\sqrt{(2\pi)}}{\mu}\left\{f(0)+2f\left(\frac{2\pi}{\mu}\right)\cos\frac{2\pi x}{\mu}+2f\left(\frac{4\pi}{\mu}\right)\cos\frac{4\pi x}{\mu}+\&c.\right\},$$

then

$$fx+f(x-\mu)+f(x+\mu)+\&c=\frac{\sqrt{(2\pi)}}{\mu}\left\{\phi(0)+2\phi\left(\frac{2\pi}{\mu}\right)\cos\frac{2\pi x}{\mu}+2\phi\left(\frac{4\pi}{\mu}\right)\cos\frac{4\pi x}{\mu}+\&c.\right\};$$

and if

$$\phi x-\phi(x-\mu)-\phi(x+\mu)+\&c=\frac{2\sqrt{(2\pi)}}{\mu}\left\{f\left(\frac{\pi}{\mu}\right)\cos\frac{\pi x}{\mu}+f\left(\frac{3\pi}{\mu}\right)\cos\frac{3\pi x}{\mu}+\&c.\right\},$$

then

$$fx-f(x-\mu)-f(x+\mu)+\&c=\frac{2\sqrt{(2\pi)}}{\mu}\left\{\phi\left(\frac{\pi}{\mu}\right)\cos\frac{\pi x}{\mu}+\phi\left(\frac{3\pi}{\mu}\right)\cos\frac{3\pi x}{\mu}+\&c\right\}.$$

Also, from the second integral, ϕ and f being uneven, if

$$\phi x+\phi(x-\mu)+\phi(x+\mu)+\&c=\frac{2\sqrt{(2\pi)}}{\mu}\left\{f\left(\frac{2\pi}{\mu}\right)\sin\frac{2\pi x}{\mu}+f\left(\frac{4\pi}{\mu}\right)\sin\frac{4\pi x}{\mu}+\&c\right\},$$

then

$$fx+f(x-\mu)+f(x+\mu)+\&c.=\frac{2\sqrt{(2\pi)}}{\mu}\left\{\phi\left(\frac{2\pi}{\mu}\right)\sin\frac{2\pi x}{\mu}+\phi\left(\frac{4\pi}{\mu}\right)\sin\frac{4\pi x}{\mu}+\&c.\right\},$$

and if

$$\phi x-\phi(x-\mu)-\phi(x+\mu)+\&c=\frac{2\sqrt{(2\pi)}}{\mu}\left\{f\left(\frac{\pi}{\mu}\right)\sin\frac{\pi x}{\mu}+f\left(\frac{3\pi}{\mu}\right)\sin\frac{3\pi x}{\mu}+\&c\right\},$$

then

$$fx-f(x-\mu)-f(x+\mu)+\&c.=\frac{2\sqrt{(2\pi)}}{\mu}\left\{\phi\left(\frac{\pi}{\mu}\right)\sin\frac{\pi x}{\mu}+\phi\left(\frac{3\pi}{\mu}\right)\sin\frac{3\pi x}{\mu}+\&c.\right\}.$$

Applying these formulæ to the identities (18) to (23), we see that (20) is its own reciprocal, as also is the case with (18), (22), and (23), while (19) and (23) are reciprocal to one another Although CAUCHY, in his memoir "Sur les Fonctions réciproques" (Exercices de Mathématiques, seconde année, 1827), has deduced, by means of his calculus of residues, a theorem which is in fact (24), he does not appear to have specially remarked the reciprocal character of the equations.

The application of the formulæ presents no difficulty For example, comparing (18) with the first of the second pair, we have

$$\phi x=\text{sech }x, \qquad fx=\sqrt{\left(\frac{\pi}{2}\right)}\ \text{sech}\frac{\pi x}{2},$$

whence the reciprocal formula is

$$\sqrt{\left(\frac{\pi}{2}\right)}\left\{\text{sech}\frac{\pi x}{2}-\text{sech}\frac{\pi(x-\mu)}{2}-\text{sech}\frac{\pi(x+\mu)}{2}+\&c.\right\}$$
$$=\frac{2\sqrt{(2\pi)}}{\mu}\left\{\text{sech}\frac{\pi}{\mu}\cos\frac{\pi x}{\mu}+\text{sech}\frac{3\pi}{\mu}\cos\frac{3\pi x}{\mu}+\&c.\right\},$$

which, on replacing $\frac{1}{2}\pi x$ and $\frac{1}{2}\pi\mu$ by x and μ respectively, coincides with the original formula (18)

§ 10. On looking at the formulæ (18) to (23) it appears that although we have transformations for sech $x\pm$ sech $(x-\mu)\pm$ sech $(x+\mu)+$&c., cosech $x\pm$ cosech $(x-\mu)$

$\pm$cosech$(x+\mu)+$&c., tanh $x-$tanh $(x-\mu)-$tanh$(x+\mu)+$&c , and coth $x-$coth $(x-\mu)$ $-$coth $(x+\mu)+$&c., there is none for either

$$\tanh x+\tanh (x-\mu)+\tanh (x+\mu)+\&c.$$

or

$$\coth x+\coth (x-\mu)+\coth (x+\mu)+\&c.,$$

it is therefore interesting to inquire what are the corresponding formulæ in these cases If we write (21) in the form

$$\operatorname{cosech} x+\operatorname{cosech}(x-\mu)+\operatorname{cosech}(x+\mu)+\&c=\frac{2\pi}{\mu}\left\{\tanh\frac{\pi^2}{\mu}\sin\frac{2\pi x}{\mu}+\tanh\frac{2^2}{\mu}\sin\frac{4\pi x}{\mu}+\&c\right\},$$

and reciprocate it by the third pair of formulæ of § 9, we obtain the following result,

$$\tanh x+\tanh (x-\mu)+\tanh (x+\mu)+\&c$$
$$=\frac{2\pi}{\mu}\left\{\operatorname{cosech}\frac{\pi^2}{\mu}\sin\frac{2\pi x}{\mu}+\operatorname{cosech}\frac{2\pi^2}{\mu}\sin\frac{4\pi x}{\mu}+\&c\right\}, \quad . \quad . \qquad (31)$$

which apparently ought to be the first of the two formulæ sought, but in point of fact this equation (as can be shown by actual calculation, see § 16) is not true

It seems natural to recur to the integral (30), viz

$$\int_{-m\pi}^{(m+1)\pi}\tanh x\sin nx\,dx=\left[-\frac{\cos nx}{n}\right]_0^{(m+1)\pi}+\left[-\frac{\cos nx}{n}\right]_0^{m\pi}-\frac{2}{n}+\pi\operatorname{cosech}\frac{n\pi}{2},$$

from which, since the first two terms of the right-hand member vanish when n is even, we have

$$\int_{-m\mu}^{(m+1)\mu}\tanh x\sin\frac{2n\pi x}{\mu}=-\frac{\mu}{n\pi}+\pi\operatorname{cosech}\frac{n\pi^2}{\mu},$$

whence ultimately, since $\frac{1}{2}\pi-\frac{1}{2}\theta=\sin\theta+\frac{1}{2}\sin 2\theta+\frac{1}{3}\sin 3\theta+\&c$,

$$\tanh x+\tanh (x-\mu)+\tanh (x+\mu)+\&c$$
$$=\frac{2x}{\mu}-1+\frac{2\pi}{\mu}\left\{\operatorname{cosech}\frac{\pi^2}{\mu}\sin\frac{2\pi x}{\mu}+\operatorname{cosech}\frac{2\pi^2}{\mu}\sin\frac{4\pi x}{\mu}+\&c.\,.\,.\right\}, \qquad (32)$$

but this result is not true either, and for the following reason —Let

$$\psi x=\phi x-\phi (x-\mu)-\phi (x+\mu)\ldots\pm\phi (x-n\mu)\pm\phi (x+n\mu),$$

and

$$\chi x=\phi x+\phi (x-\mu)+\phi (x+\mu)\ldots+\phi (x-n\mu)+\phi (x+n\mu)$$

(n infinite), and suppose ϕx is an uneven function of x which $=1$, when $x=\infty$. Then

$$\psi (x+\mu)=-\psi x\pm\phi (x-n\mu)\pm\phi (x+(n+1)\mu)$$
$$=-\psi x,$$

so that ψx is periodic; but

$$\chi(x+\mu)=\chi x-\phi(x-n\mu)+\phi(x+(n+1)\mu)$$
$$=\chi x+2,$$

so that χx is not periodic. Therefore we have no right to assume that between the limits 0 and μ of x

$$\tanh x+\tanh(x-\mu)+\tanh(x+\mu)+\&c.$$

can be expressed in the form

$$A_1\sin\frac{2\pi x}{\mu}+A_2\sin\frac{4\pi x}{\mu}+A_3\sin\frac{6\pi x}{\mu}+\&c,$$

the true form being

$$B_1\sin\frac{\pi x}{\mu}+B_2\sin\frac{2\pi x}{\mu}+B_3\sin\frac{3\pi x}{\mu}+\&c$$

We may, however, assume that between the limits 0 and $\frac{1}{2}\mu$ of x

$$\tanh x+\tanh(x-\mu)+\tanh(x+\mu)+\&c=A_1\sin\frac{2\pi x}{\mu}+A_2\sin\frac{4\pi x}{\mu}+\&c.;$$

and then

$$\frac{\mu}{4}A_n=\int_0^{\frac{1}{2}\mu}\{\tanh x+\tanh(x-\mu)+\tanh(x+\mu)+\&c\}\sin\frac{2n\pi x}{\mu}\,dx$$

$$=\left\{\int_0^{\frac{1}{2}\mu}+\int_{-\mu}^{-\frac{1}{2}\mu}+\int_\mu^{\frac{3}{2}\mu}+\&c.\right\}\tanh x\sin\frac{2n\pi x}{\mu}\,dx$$

$$=\int_0^{(2m+1)\frac{\mu}{2}}\tanh x\sin\frac{2n\pi x}{\mu}\,dx$$

$$=\left[-\frac{\mu}{2n\pi}\cos\frac{2n\pi x}{\mu}\right]_0^{(2m+1)\frac{\mu}{2}}-\int_0^\infty\frac{2}{e^{2x}+1}\sin\frac{2n\pi x}{\mu}\,dx$$

$$=(-)^{n+1}\frac{\mu}{2n\pi}+\frac{\pi}{2}\operatorname{cosech}\frac{n\pi^2}{\mu}.$$

We thus find that between the limits 0 and $\frac{1}{2}\mu$ of x (and therefore also between the limits $-\frac{1}{2}\mu$ and $\frac{1}{2}\mu$ of x)

$$\tanh x+\tanh(x-\mu)+\tanh(x+\mu)+\&c=\frac{4}{\mu}\cdot\frac{\mu}{2\pi}\left\{\sin\frac{2\pi x}{\mu}-\tfrac{1}{2}\sin\frac{4\pi x}{\mu}+\&c\right\}$$
$$+\frac{4}{\mu}\cdot\frac{\pi}{2}\left\{\operatorname{cosech}\frac{\pi^2}{\mu}\sin\frac{2\pi x}{\mu}+\operatorname{cosech}\frac{2\pi^2}{\mu}\sin\frac{4\pi x}{\mu}+\&c.\right\}$$
$$=\frac{2x}{\mu}+\frac{2\pi}{\mu}\left\{\operatorname{cosech}\frac{\pi^2}{\mu}\sin\frac{2\pi x}{\mu}+\operatorname{cosech}\frac{2\pi^2}{\mu}\sin\frac{4\pi x}{\mu}+\&c.\right\},\quad . \quad (33)$$

the terms on the left-hand side being uneven in number, and such that for every term $\tanh(x-n\mu)$ there is also a term $\tanh(x+n\mu)$.

If we write $x+\mu$ for x in this formula (33) we increase the left-hand side by $\tanh\infty+\tanh\infty$, that is by 2, while the right-hand side is increased by $\frac{2}{\mu}\cdot\mu$, that is

by 2 also; while if we replace x by $x-\mu$ both sides are diminished by 2, so that (33) is true universally for all values of x, on the understanding that the left-hand side is

$$\tanh x+\{\tanh(x-\mu)+\tanh(x+\mu)\}+\{\tanh(x-2\mu)+\tanh(x+2\mu)\}+\&c,$$

viz. that after the first term the series is to proceed by pairs of terms; so that for every term $\tanh(x\pm n\mu)$ there is also a term $\tanh(x\mp n\mu)$, and the whole number of terms included is uneven. Thus for $x=\frac{1}{2}\mu$ the series is

$$\tanh\tfrac{1}{2}\mu+\{-\tanh\tfrac{1}{2}\mu+\tanh\tfrac{3}{2}\mu\}+\{-\tanh\tfrac{3}{2}\mu+\tanh\tfrac{5}{2}\mu\}+\&c,$$

the value of which is unity; and not

$$\{\tanh\tfrac{1}{2}\mu-\tanh\tfrac{1}{2}\mu\}+\{\tanh\tfrac{3}{2}\mu-\tanh\tfrac{3}{2}\mu\}+\&c,$$

which is equal to zero

If we write $x+\frac{1}{2}\mu$ for x, and suppose the terms arranged in pairs from the beginning, we find

$$\{\tanh(x+\tfrac{1}{2}\mu)+\tanh(x-\tfrac{1}{2}\mu)\}+\{\tanh(x+\tfrac{3}{2}\mu)+\tanh(x-\tfrac{3}{2}\mu)\}+\&c$$

$$=\frac{2x}{\mu}-\frac{2\pi}{\mu}\left\{\operatorname{cosech}\frac{\pi^2}{\mu}\sin\frac{2\pi x}{\mu}-\operatorname{cosech}\frac{2\pi^2}{\mu}\sin\frac{4\pi x}{\mu}+\&c\right\} \qquad (34)$$

as the unity which is introduced on the right-hand side by the change is cancelled by the unity on the left-hand side, which results from the supposition that the number of terms is even.

The last equation is, in fact, the relation

$$i\mathrm{Z}(ui+\mathrm{K})=\frac{\pi u}{2\mathrm{KK}'}+\mathrm{Z}(u+\mathrm{K}',k') \quad . \quad . \qquad (35)$$

(Fundamenta Nova, p 165, and DURÈGE, § 69), for

$$\mathrm{Z}(u)=\frac{2\pi}{\mathrm{K}}\left\{\frac{q}{1-q^2}\sin 2x+\frac{q^2}{1-q^4}\sin 4x+\frac{q^3}{1-q^6}\sin 6x+\&c.\right\};$$

so that (35) becomes

$$\frac{2\pi i}{\mathrm{K}}\left\{-\frac{q}{1-q^2}\sin 2xi+\frac{q^2}{1-q^4}\sin 4xi-\frac{q^3}{1-q^6}\sin 6xi+\&c.\right\}$$

$$=\frac{\pi u}{2\mathrm{KK}'}+\frac{2\pi}{\mathrm{K}'}\left\{-\frac{r}{1-r^2}\sin 2z+\frac{r^2}{1-r^4}\sin 4z-\&c.\right\},$$

of which the left-hand side

$$=\frac{\pi}{\mathrm{K}}\{(e^{2x}-e^{-2x})(q+q^3+q^5+\&c.)-(e^{4x}-e^{-4x})(q^2+q^6+q^{10}+\&c.)+\&c\}$$

$$=\frac{\pi}{\mathrm{K}}\left\{\frac{qe^{2x}}{1+qe^{2x}}-\frac{qe^{-2x}}{1+qe^{-2x}}+\frac{q^3e^{2x}}{1+q^3e^{2x}}-\frac{q^3e^{-2x}}{1+q^3e^{-2x}}+\&c.\right\}$$

$$=-\frac{\pi}{2\mathrm{K}}\left\{\frac{1-qe^{2x}}{1+qe^{2x}}-\frac{1-qe^{-2x}}{1+qe^{-2x}}+\frac{1-q^3e^{2x}}{1+q^3e^{2x}}-\frac{1-q^3e^{-2x}}{1+q^3e^{-2x}}+\&c.\right\};$$

and the identity becomes

$$\tanh(x-\tfrac{1}{2}\mu)+\tanh(x+\tfrac{1}{2}\mu)+\tanh(x-\tfrac{3}{2}\mu)+\tanh(x+\tfrac{3}{2}\mu)+\&c.$$
$$=\frac{2x}{\mu}-\frac{2\pi}{\mu}\left\{\frac{\sin 2z}{\sinh\nu}-\frac{\sin 4z}{\sinh 2\nu}+\&c.\right\},$$

z being $\frac{\pi x}{\mu}$ and ν being $\frac{\pi^2}{\mu}$ We see from this investigation also that the left-hand side must consist of an even number of pairs of terms.

As (35) is obtained by differentiating logarithmically the formula

$$\Theta(ui+\mathrm{K})=\sqrt{\left(\frac{\mathrm{K}}{\mathrm{K}'}\right)}e^{\frac{\pi u^2}{4\mathrm{KK}'}}\Theta(u+\mathrm{K}',k'),$$

it follows that (34) is a form of the identity that results from differentiating logarithmically

$$e^{-x^2}+e^{-(x-\mu)^2}+e^{-(x+\mu)^2}+\&c.=\frac{\sqrt{\pi}}{\mu}\left\{1+2e^{-\frac{\pi^2}{\mu^2}}\cos\frac{2\pi x}{\mu}+2e^{-\frac{4\pi^2}{\mu^2}}\cos\frac{4\pi x}{\mu}+\&c.\right\}.$$

The formula corresponding to (33) for the hyperbolic cotangent can be shown, by a process similar to that by which (33) was itself established, to be

$$\coth x+\coth(x-\mu)+\coth(x+\mu)+\&c=\frac{2x}{\mu}+\frac{\pi}{\mu}\left\{\cot\frac{\pi x}{\mu}+\frac{4\sin\frac{2\pi x}{\mu}}{e^{\frac{2\pi^2}{\mu}}-1}+\frac{4\sin\frac{4\pi x}{\mu}}{e^{\frac{4\pi^2}{\mu}}-1}+\&c\right\},\quad(36)$$

which holds good universally, on the same understanding, with regard to the number and order of the terms, as that which was found requisite for the truth of (33)

§ 11. I now proceed to show how the identities which have been obtained in the preceding sections by elliptic functions, or by FOURIER's theorem, can be deduced from the ordinary formulæ for the cotangent and cosecant, viz.

$$\cot x=\frac{1}{x}+\frac{1}{x-\pi}+\frac{1}{x+\pi}+\frac{1}{x-2\pi}+\frac{1}{x+2\pi}+\&c.,\quad . \quad . \quad . \quad (37)$$

$$\operatorname{cosec} x=\frac{1}{x}-\frac{1}{x-\pi}-\frac{1}{x+\pi}+\frac{1}{x-2\pi}+\frac{1}{x+2\pi}-\&c.,\quad . \quad . \quad (38)$$

by elementary algebra and trigonometry.

Thus to prove (18) we have

$$\operatorname{cosec}(x+ai)=\frac{1}{x+ai}-\frac{1}{x+ai-\pi}-\frac{1}{x+ai+\pi}+\&c.,$$

$$\operatorname{cosec}(x-ai)=\frac{1}{x+ai}-\frac{1}{x-ai-\pi}-\frac{1}{x-ai+\pi}+\&c.;$$

whence, by subtraction,

$$\frac{a}{x^2+a^2}-\frac{a}{(x-\pi)^2+a^2}-\frac{a}{(x+\pi)^2+a^2}+\&c.=-\frac{1}{2i}\{\operatorname{cosec}(x+ai)-\operatorname{cosec}(x-ai)\}.$$

Now

$$\operatorname{cosec} u=\frac{2i}{e^{ui}-e^{-ui}}=\frac{2i\,e^{ui}}{e^{2ui}-1}$$

$$=-2ie^{ui}(1+e^{2ui}+e^{4ui}+\&c.);$$

whence

$$-\frac{1}{2i}\{\operatorname{cosec}(x+ai)-\operatorname{cosec}(x-ai)\}=e^{xi-a}+e^{3xi-3a}+e^{5xi-5a}+\&c.$$

$$+e^{-xi-a}+e^{-3xi-3a}+e^{-5xi-5a}+\&c$$

$$=2\{e^{-a}\cos x+e^{-3a}\cos 3x+\&c.\},$$

and, on replacing x and a by $\frac{\pi x}{\mu}$ and $\frac{\pi a}{\mu}$, we obtain the formula

$$\frac{a}{x^2+a^2}-\frac{a}{(x-\mu)^2+a^2}-\frac{a}{(x+\mu)^2+a^2}+\frac{a}{(x-2\mu)^2+a^2}+\frac{a}{(x+2\mu)^2+a^2}-\&c.$$

$$=\frac{2\pi}{\mu}\left(e^{-\frac{\pi a}{\mu}}\cos\frac{\pi x}{\mu}+e^{-\frac{3\pi a}{\mu}}\cos\frac{3\pi x}{\mu}+\&c.\right). \qquad (39)$$

Now from (38)

$$-\sec x=\frac{1}{x-\frac{1}{2}\pi}-\frac{1}{x-\frac{3}{2}\pi}-\frac{1}{x+\frac{1}{2}\pi}+\frac{1}{x-\frac{5}{2}\pi}+\&c$$

$$=\frac{\pi}{x^2-(\frac{1}{2}\pi)^2}-\frac{3\pi}{x^2-(\frac{3}{2}\pi)^2}+\frac{5\pi}{x^2-(\frac{5}{2}\pi)^2}-\&c;$$

whence, writing xi for x,

$$\operatorname{sech} x=\frac{\pi}{x^2+(\frac{1}{2}\pi)^2}-\frac{3\pi}{x^2+(\frac{3}{2}\pi)^2}+\frac{5\pi}{x^2+(\frac{5}{2}\pi)^2}-\&c.,$$

and

$$-\operatorname{sech}(x-\mu)=-\frac{\pi}{(x-\mu)^2+(\frac{1}{2}\pi)^2}+\frac{3\pi}{(x-\mu)^2+(\frac{3}{2}\pi)^2}-\frac{5\pi}{(x-\mu)^2+(\frac{5}{2}\pi)^2}+\&c$$

$$-\operatorname{sech}(x+\mu)=-\frac{\pi}{(x+\mu)^2+(\frac{1}{2}\pi)^2}+\frac{3\pi}{(x+\mu)^2+(\frac{3}{2}\pi)^2}-\frac{5\pi}{(x+\mu)^2+(\frac{5}{2}\pi)^2}+\&c.$$

$$+\operatorname{sech}(x-2\mu)=\frac{\pi}{(x-2\mu)^2+(\frac{1}{2}\pi)^2}-\frac{3\pi}{(x-2\mu)^2+(\frac{3}{2}\pi)^2}+\frac{5\pi}{(x-2\mu)^2+(\frac{5}{2}\pi)^2}-\&c$$

. . . .

Adding these expressions together in columns, and transforming each column by (39), we find

$$\operatorname{sech} x-\operatorname{sech}(x-\mu)-\operatorname{sech}(x+\mu)+\operatorname{sech}(x-2\mu)+\operatorname{sech}(x+2\mu)-\&c$$

$$=\frac{4\pi}{\mu}\left(e^{-\frac{\pi^2}{2\mu}}\cos\frac{\pi x}{\mu}+e^{-\frac{3\pi^2}{2\mu}}\cos\frac{3\pi x}{\mu}+e^{-\frac{5\pi^2}{2\mu}}\cos\frac{5\pi x}{\mu}+\&c.\right)$$

$$-\frac{4\pi}{\mu}\left(e^{-\frac{3\pi^2}{2\mu}}\cos\frac{\pi x}{\mu}+e^{-\frac{9\pi^2}{2\mu}}\cos\frac{3\pi x}{\mu}+e^{-\frac{15\pi^2}{2\mu}}\cos\frac{5\pi x}{\mu}+\&c.\right)$$

$$+\frac{4\pi}{\mu}\left(e^{-\frac{5\pi^2}{2\mu}}\cos\frac{\pi x}{\mu}+e^{-\frac{15\pi^2}{2\mu}}\cos\frac{3\pi x}{\mu}+e^{-\frac{25\pi^2}{2\mu}}\cos\frac{5\pi x}{\mu}+\&c.\right)$$

. . . . ,

which, after summation of the columns,

$$=\frac{4\pi}{\mu}\left(\frac{e^{-\frac{\pi^2}{2\mu}}}{1+e^{-\frac{\pi^2}{\mu}}}\cos\frac{\pi x}{\mu}+\frac{e^{-\frac{3\pi^2}{2\mu}}}{1+e^{-\frac{3\pi^2}{\mu}}}\cos\frac{3\pi x}{\mu}+\frac{e^{-\frac{5\pi^2}{2\mu}}}{1+e^{-\frac{5\pi^2}{\mu}}}\cos\frac{5\pi x}{\mu}+\&c\right)$$

$$=\frac{2\pi}{\mu}\left(\operatorname{sech}\frac{\pi^2}{2\mu}\cos\frac{\pi x}{\mu}+\operatorname{sech}\frac{3\pi^2}{2\mu}\cos\frac{3\pi x}{\mu}+\operatorname{sech}\frac{5\pi^2}{2\mu}\cos\frac{5\pi x}{\mu}+\&c.\right),$$

which is the identity (18), that was in § 6 deduced from the formula

$$\cos \operatorname{am} u=\sec \operatorname{am}(ui, k'),$$

and in § 8 from the integral

$$\int_0^\infty \operatorname{sech} x \cos nx\, dx=\frac{\pi}{2}\operatorname{sech}\frac{n\pi}{2}$$

§ 12 The other identities, (19) to (23), admit of being demonstrated in exactly the same way The formulæ of transformation, similar to (39), that are required are

$$\frac{x}{x^2+a^2}-\frac{x-\mu}{(x-\mu)^2+a^2}-\frac{x+\mu}{(x+\mu)^2+a^2}+\&c=\frac{2\pi}{\mu}\left(e^{-\frac{\pi a}{\mu}}\sin\frac{\pi x}{\mu}+e^{-\frac{3\pi a}{\mu}}\sin\frac{3\pi x}{\mu}+\&c\right),$$

$$\frac{a}{x^2+a^2}+\frac{a}{(x-\mu)^2+a^2}+\frac{a}{(x+\mu)^2+a^2}+\&c=\frac{\pi}{\mu}\left(1+2e^{-\frac{2\pi a}{\mu}}\cos\frac{2\pi x}{\mu}+2e^{-\frac{4\pi a}{\mu}}\cos\frac{4\pi x}{\mu}+\&c\right),$$

$$\frac{x}{x^2+a^2}+\frac{x-\mu}{(x-\mu)^2+a^2}+\frac{x+\mu}{(x+\mu)^2+a^2}+\&c.=\frac{2\pi}{\mu}\left(e^{-\frac{2\pi a}{\mu}}\sin\frac{2\pi x}{\mu}+e^{-\frac{4\pi a}{\mu}}\sin\frac{4\pi x}{\mu}+\&c.\right),$$

the first resulting from $\operatorname{cosec}(x+ai)+\operatorname{cosec}(x-ai)$, and the other two from $\cot(x+ai)\mp\cot(x-ai)$ The following expressions, which are analogous to that used for sech x in the last section, are also needed —

$$\tanh x=\frac{2x}{x^2+(\frac{1}{2}\pi)^2}+\frac{2x}{x^2+(\frac{3}{2}\pi)^2}+\frac{2x}{x^2+(\frac{5}{2}\pi)^2}+\&c,$$

$$\coth x=\frac{1}{x}+\frac{2x}{x^2+\pi^2}+\frac{2x}{x^2+(2\pi)^2}+\frac{2x}{x^2+(3\pi)^2}+\&c,$$

$$\operatorname{cosech} x=\frac{1}{x}-\frac{2x}{x^2+\pi^2}+\frac{2x}{x^2+(2\pi)^2}-\frac{2x}{x^2+(3\pi)^2}+\&c,$$

all of which follow from (37) and (38) at once in the same way as that by which the formula for sech x was obtained.

Only one point calls for notice in these demonstrations, viz. in the proof of (20) we find

$$\begin{aligned}&\operatorname{sech} x+\operatorname{sech}(x-\mu)+\operatorname{sech}(x+\mu)+\&c.\\&=\frac{2\pi}{\mu}\left(1+2e^{-\frac{\pi^2}{\mu}}\cos\frac{2\pi x}{\mu}+2e^{-\frac{4\pi^2}{\mu}}\cos\frac{4\pi x}{\mu}+\&c.\right)\\&-\frac{2\pi}{\mu}\left(1+2e^{-\frac{3\pi^2}{\mu}}\cos\frac{2\pi x}{\mu}+2e^{-\frac{6\pi^2}{\mu}}\cos\frac{4\pi x}{\mu}+\&c.\right)\\&+\frac{2\pi}{\mu}\left(1+2e^{-\frac{5\pi^2}{\mu}}\cos\frac{2\pi x}{\mu}+2e^{-\frac{10\pi^2}{\mu}}\cos\frac{4\pi x}{\mu}+\&c.\right)\\&-\ \cdots\cdots\cdots,\end{aligned}$$

and in order to obtain the correct result we must replace the indeterminate series, $1-1+1-1+1-$&c., by $\frac{1}{2}$. Cases in which the method gives results absolutely erroneous will be noticed in § 16

It will have been seen that the process of § 11 consists in replacing each term of the original series by n terms (n infinite), and therefore the original expression itself by n^2 terms Each series of n terms formed by adding the vertical columns is transformed into another series of n terms, so that we thus replace the first scheme of n^2 terms by a second scheme of n^2 terms, which latter system, being such that the columns admit of being summed as ordinary geometrical progressions, gives the second side of the identity to be proved.

§ 13. A question that naturally arises is to inquire what are the results which we should obtain if, instead of using (39) and the similar formulæ for the conversion of one series into another, we were to replace at once these series by their finite summations, *i. e* instead of (39) to take

$$\frac{a}{x^2+a^2}-\frac{a}{(x-\mu)^2+a^2}-\frac{a}{(x+\mu)^2+a^2}+\&c=\frac{\pi}{2\mu i}\left\{\operatorname{cosec}\frac{\pi}{\mu}(x-ai)-\operatorname{cosec}\frac{\pi}{\mu}(x+ai)\right\}$$

$$=\frac{\pi}{2\mu i}\cdot\frac{2\cos\dfrac{\pi x}{\mu}\sin\dfrac{\pi ai}{\mu}}{\sin^2\dfrac{\pi x}{\mu}+\sin^2\dfrac{\pi ai}{\mu}}$$

$$=\frac{\pi}{\mu}\cdot\frac{\cos\dfrac{\pi x}{\mu}\sinh\dfrac{\pi a}{\mu}}{\sin^2\dfrac{\pi x}{\mu}+\sinh^2\dfrac{\pi a}{\mu}}$$

We thus find

$$\operatorname{sech} x-\operatorname{sech}(x-\mu)-\operatorname{sech}(x+\mu)+\&c$$

$$=\frac{2\pi}{\mu}\cos\frac{\pi x}{\mu}\left\{\frac{\sinh\dfrac{\pi^2}{2\mu}}{\sin^2\dfrac{\pi x}{\mu}+\sinh^2\dfrac{\pi^2}{2\mu}}-\frac{\sinh\dfrac{3\pi^2}{2\mu}}{\sin^2\dfrac{\pi x}{\mu}+\sinh^2\dfrac{3\pi^2}{2\mu}}+\&c\right\}, \tag{40}$$

while the left-hand side also

$$=\frac{2\pi}{\mu}\left(\operatorname{sech}\frac{\pi^2}{2\mu}\cos\frac{\pi x}{\mu}+\operatorname{sech}\frac{3\pi^2}{2\mu}\cos\frac{3\pi x}{\mu}+\&c.\right) \quad . \quad . \tag{41}$$

from (18). Although (40) is the identity which we have absolutely proved, we may regard the fresh identity as being that which follows from (40) and (41), viz (writing for the moment x in place of $\frac{\pi x}{\mu}$, and μ in place of $\frac{\pi^2}{\mu}$)

$$\frac{\cos x}{\cosh\frac{1}{2}\mu}+\frac{\cos 3x}{\cosh\frac{3}{2}\mu}+\&c=\cos x\left\{\frac{\sinh\frac{1}{2}\mu}{\sin^2 x+\sinh^2\frac{1}{2}\mu}-\frac{\sinh\frac{3}{2}\mu}{\sin^2 x+\sinh^2\frac{3}{2}\mu}+\&c\right\} \quad . \quad . \quad . \tag{42}$$

This result follows immediately from another form of the series for the cosine

amplitude; for on p. 113 of his 'Lehre von den elliptischen Integralen und den Theta-Functionen' (Berlin, 1864), SCHELLBACH finds

$$\theta o\, \theta_2 o\, gx = 4 \cos x\, \Sigma_0^\infty \frac{(-)^s q^{s+\frac{1}{2}}(1-q^{2s+1})}{1-2q^{2s+1}\cos 2x+q^{4s+2}}. \qquad (43)$$

We easily see that

$$\theta o\, \theta_2 o\, gx = \frac{2k\mathrm{K}}{\pi}\cos \mathrm{am}\, \frac{2\mathrm{K}x}{\pi} = 4\left\{\frac{q^{\frac{1}{2}}}{1+q}\cos x + \frac{q^{\frac{3}{2}}}{1+q^3}\cos 3x + \&\mathrm{c.}\right\}, \qquad (44)$$

and the comparison of (43) and (44) at once gives (42), since $\sin^2 x + \sinh^2 a = \frac{1}{2}(\cosh 2a - \cos 2x)$ The result (43) is also given in the 'Fundamenta Nova,' p. 102.

It thus appears that by absolutely summing, instead of transforming, in the process of § 11 we obtain the series of formulæ which SCHELLBACH has given on pp. 113, 114 of his treatise, so that all the formulæ and identities which arise from the transformations of the elliptic functions are algebraically exhibited by the method of § 11. It is unnecessary to write down the series of identities analogous to (42) for the other functions, as they can be easily derived as above from the values in SCHELLBACH. It may be remarked that (40) is a transformation of sec am (u, k') = cos am u, but (42) is merely a transformation of cos am u = cos am u If, therefore, we perform the process of § 11 in reverse order (*i. e.* starting with the trigonometrical side of the identity to be proved, sum the rows instead of transforming them) we obtain (42) at once

It appears at first sight as if SCHELLBACH's formula

$$\frac{2k'\mathrm{K}}{\pi}\sec \mathrm{am}\, \frac{2\mathrm{K}x}{\pi} = \sec x + 4\cos x\, \Sigma_1^\infty \frac{(-)^s q^s (1+q^{2s})}{1+2q^{2s}\cos 2x + q^{4s}} \qquad (45)$$

gave rise to another formula for the cosine amplitude, by writing xi for x and changing the modulus from k to k', but this, in fact, merely gives an expression already obtained; for the right-hand side of (45), on writing xi for x and $e^{-\mu}$ for q, becomes

$$\operatorname{sech} x + 4\cosh x\, \Sigma_1^\infty \frac{(-)^s \cosh s\mu}{\cosh 2x + \cosh 2s\mu},$$

which

$$= \operatorname{sech} x + \Sigma_1^\infty (-)^s \frac{\cosh(x-s\mu)+\cosh(x+s\mu)}{\cosh(x-s\mu)\cosh(x+s\mu)}$$

$$= \operatorname{sech} x + \Sigma_1^\infty (-)^s\{\operatorname{sech}(x-s\mu)+\operatorname{sech}(x+s\mu)\}.$$

Formulæ such as (45) are the nearest approach I have met with to those numbered (10) to (17) and the other expressions at the end of § 5, but (besides that an imaginary transformation is required to reduce them to these forms) they do not put in evidence the periodicity of the functions.

§ 14. It is perhaps desirable to place side by side, for convenience of comparison, all the different forms into which one of the functions, the cosine amplitude, has now been thrown Writing, as before,

$$x = \frac{\pi u}{2\mathrm{K}},\quad z = \frac{\pi u}{2\mathrm{K}'},\quad q = e^{-\frac{\pi\mathrm{K}'}{\mathrm{K}}} = e^{-\mu},\quad r = e^{-\frac{\pi\mathrm{K}}{\mathrm{K}'}} = e^{-\nu},$$

$$\cos \operatorname{am} u = \frac{2\pi}{kK}\left\{\frac{q^{\frac{1}{2}}}{1+q}\cos x + \frac{q^{\frac{3}{2}}}{1+q^3}\cos 3x + \frac{q^{\frac{5}{2}}}{1+q^5}\cos 5x + \&c.\right\}$$

$$= \frac{2\pi}{kK}\cos x\left\{\frac{q^{\frac{1}{2}}(1-q)}{1-2q\cos 2x+q^2} - \frac{q^{\frac{3}{2}}(1-q^3)}{1-2q^3\cos 2x+q^6} + \&c.\right\}$$

$$= \frac{\pi}{kK'}\left\{\frac{1}{r^{\frac{z}{\pi}}+r^{-\frac{z}{\pi}}} - \frac{1}{r^{\frac{z}{\pi}-1}+r^{-\left(\frac{z}{\pi}-1\right)}} - \frac{1}{r^{\frac{z}{\pi}+1}+r^{-\left(\frac{z}{\pi}+1\right)}} + \&c.\right\}$$

$$= \frac{\pi}{2kK'}\{\operatorname{sech} z - \operatorname{sech}(z-\nu) - \operatorname{sech}(z+\nu) + \&c.\}$$

$$= \frac{\pi}{2kK'}\left\{\operatorname{sech} z - 4\cosh z\left(\frac{\cosh\nu}{\cosh 2z+\cosh 2\nu} - \frac{\cosh 2\nu}{\cosh 2z+\cosh 4\nu} + \&c.\right)\right\}$$

$$= \frac{\pi}{2kK'}\left\{\operatorname{sech} z - \frac{4r}{1+r}\cosh z + \frac{4r^3}{1+r^3}\cosh 3z - \&c.\right\};$$

while x, z, μ, ν being any four quantities subject to the relations

$$\mu\nu = \pi^2,\ z = \frac{\pi x}{\mu}\left(\text{whence } x = \frac{\pi z}{\nu}\right),$$

the identities are:—

$$\operatorname{sech} x - \operatorname{sech}(x-\mu) - \operatorname{sech}(x+\mu) + \operatorname{sech}(x-2\mu) + \operatorname{sech}(x+2\mu) - \&c$$

$$= \operatorname{sech} x - 4\cosh x\left\{\frac{\cosh\mu}{\cosh 2x+\cosh 2\mu} - \frac{\cosh 2\mu}{\cosh 2x+\cosh 4\mu} + \&c.\right\}$$

$$= \operatorname{sech} x - \frac{4\cosh x}{e^{\mu}+1} + \frac{4\cosh 3x}{e^{3\mu}+1} - \&c.$$

$$= \frac{2\pi}{\mu}\left\{\frac{\cosh z}{\cosh\frac{1}{2}\nu} + \frac{\cosh 3z}{\cosh\frac{3}{2}\nu} + \frac{\cosh 5z}{\cosh\frac{5}{2}\nu} + \&c\right\}$$

$$= \frac{2\pi}{\mu}\cos z\left\{\frac{\sinh\frac{1}{2}\nu}{\sin^2 z+\sinh^2\frac{1}{2}\nu} - \frac{\sinh\frac{3}{2}\nu}{\sin^2 z+\sinh^2\frac{3}{2}\nu} + \&c.\right\}.$$

Another form will also be given in the next section It is scarcely necessary to observe that corresponding formulæ and identities exist for sin am u, Δ am u, cosec am u, $\frac{\sin \operatorname{am} u}{\Delta \operatorname{am} u}$, &c

§ 15. The identities (18) to (23) can also be proved by trigonometry in another distinct manner, by starting from the trigonometrical sides of the equations Thus, for (18), from the formula

$$\frac{\pi}{4}\operatorname{sech}\tfrac{1}{2}\pi\beta = \frac{1}{1^2+\beta^2} - \frac{3}{3^2+\beta^2} + \frac{5}{5^2+\beta^2} - \&c.,$$

we have (writing z for $\frac{\pi x}{\mu}$ for brevity)

$$\frac{\pi}{4}\operatorname{sech}\frac{\pi^2}{2\mu}\cos z = \frac{\mu^2}{\pi^2}\left\{\frac{\cos z}{\frac{\mu^2}{\pi^2}+1^2} - \frac{3\cos z}{\frac{3^2\mu^2}{\pi^2}+1^2} + \frac{5\cos z}{\frac{5^2\mu^2}{\pi^2}+1^2} - \&c.\right\}$$

$$\frac{\pi}{4}\operatorname{sech}\frac{3\pi^2}{2\mu}\cos 3z=\frac{\mu^2}{\pi^2}\left\{\frac{\cos 3z}{\frac{\mu^2}{\pi^2}+3^2}-\frac{3\cos 3z}{\frac{3^2\mu^2}{\pi^2}+3^2}+\frac{5\cos 3z}{\frac{5^2\mu^2}{\pi^2}+3^2}-\&c.\right\}$$

$$\frac{\pi}{4}\operatorname{sech}\frac{5\pi^2}{2\mu}\cos 5z=\frac{\mu^2}{\pi^2}\left\{\frac{\cos 5z}{\frac{\mu^2}{\pi^2}+5^2}-\frac{3\cos 5z}{\frac{3^2\mu^2}{\pi^2}+5^2}+\frac{5\cos 5z}{\frac{5^2\mu^2}{\pi^2}+5^2}-\&c.\right\}$$

$$\cdot \qquad \cdot \quad \cdot \qquad \cdot \qquad \cdot \quad \cdot \quad \cdot,$$

whence

$$\frac{2\pi}{\mu}\left\{\operatorname{sech}\frac{\pi^2}{2\mu}\cos z+\operatorname{sech}\frac{3\pi^2}{2\mu}\cos 3z+\operatorname{sech}\frac{5\pi^2}{2\mu}\cos 5z+\&c.\right\}$$

$$=\frac{8\mu}{\pi^2}\left\{\frac{\cos z}{\frac{\mu^2}{\pi^2}+1^2}+\frac{\cos 3z}{\frac{\mu^2}{\pi^2}+3^2}+\frac{\cos 5z}{\frac{\mu^2}{\pi^2}+5^2}+\&c\right\}$$

$$-\frac{8\mu}{\pi^2}\left\{\frac{3\cos z}{\frac{3^2\mu^2}{\pi^2}+1^2}+\frac{3\cos 3z}{\frac{3^2\mu^2}{\pi^2}+3^2}+\frac{3\cos 5z}{\frac{3^2\mu^2}{\pi^2}+5^2}+\&c\right\}$$

$$+ \qquad \cdot \qquad \cdot \qquad \cdot$$

$$=2\left\{\frac{\sinh(\frac{1}{2}\mu-x)}{\cosh\frac{1}{2}\mu}-\frac{\sinh 3(\frac{1}{2}\mu-x)}{\cosh\frac{3}{2}\mu}+\frac{\sinh 5(\frac{1}{2}\mu-x)}{\cosh\frac{5}{2}\mu}-\&c\right\}$$

$$=2\left\{\left(e^{-x}-\frac{2\cosh x}{1+e^{\mu}}\right)-\left(e^{-3x}-\frac{2\cosh 3x}{1+e^{3\mu}}\right)+\&c\right\}$$

$$=\frac{2}{e^x+e^{-x}}-\frac{4\cosh x}{1+e^{\mu}}+\frac{4\cosh 3x}{1+e^{3\mu}}-\&c,$$

which, as shown in § 4,

$$=\operatorname{sech}x-\operatorname{sech}(x-\mu)-\operatorname{sech}(x+\mu)+\&c.$$

We thus in the course of the proof obtain another form for $\operatorname{sech}x-\operatorname{sech}(x-\mu)-\operatorname{sech}(x+\mu)+\&c$, viz

$$2\left\{\frac{\sinh(\frac{1}{2}\mu-x)}{\cosh\frac{1}{2}\mu}-\frac{\sinh 3(\frac{1}{2}\mu-x)}{\cosh\frac{3}{2}\mu}+\frac{\sinh 5(\frac{1}{2}\mu-x)}{\cosh\frac{5}{2}\mu}-\&c.\right\}, \quad . \quad . \quad . \quad (46)$$

whence, in addition to the forms for $\cos\operatorname{am}u$ in § 14, we have

$$\cos\operatorname{am}u=\frac{\pi}{k\mathrm{K}'}\left\{\frac{\sinh(\frac{1}{2}\nu-z)}{\cosh\frac{1}{2}\nu}-\frac{\sinh 3(\frac{1}{2}\nu-z)}{\cosh\frac{3}{2}\nu}+\&c\right\}.$$

This method of proof is not so interesting as that of § 11, both because the formulæ required cannot be obtained in so elementary a manner, and also because the identities (18) to (23) are not so directly verified, as their right-hand members are shown to be equal to expressions such as (46), which themselves need some transformation before they assume the desired forms. The formula

$$\frac{\cos x}{1^2+\beta^2}+\frac{\cos 3x}{3^2+\beta^2}+\&c.=\frac{\pi}{4\beta}\,\frac{\sinh(\frac{1}{2}\beta\pi-\beta x)}{\cosh\frac{1}{2}\beta\pi},$$

which was required in the verification, is best obtained by deducing it from the well-known theorem

$$\frac{\cos x}{1^2+\beta^2}+\frac{\cos 2x}{2^2+\beta^2}+\frac{\cos 3x}{3^2+\beta^2}+\&c=\frac{\pi}{2\beta}\frac{\cosh(\beta\pi-\beta x)}{\sinh\beta\pi}-\frac{1}{2\beta^2}, \quad . \tag{47}$$

from which, by writing $\frac{1}{2}\beta$ for β and $2x$ for x, dividing the equation so obtained by 4, and subtracting it from (47), we find

$$\begin{aligned}\frac{\cos x}{1^2+\beta^2}+\frac{\cos 3x}{3^2+\beta^2}+\&c.&=\frac{\pi}{2\beta}\left\{\frac{\cosh(\beta\pi-\beta x)}{\sinh\beta\pi}-\frac{1}{2}\frac{\cosh(\frac{1}{2}\beta\pi-\beta x)}{\sinh\frac{1}{2}\beta\pi}\right\}\\ &=\frac{\pi}{2\beta}\left\{\frac{\cosh(\beta\pi-\beta x)-\cosh(\frac{1}{2}\beta\pi-\beta x)\cosh\frac{1}{2}\beta\pi}{\sinh\beta\pi}\right\}\\ &=\frac{\pi}{4\beta}\quad\frac{\sinh(\frac{1}{2}\beta\pi-\beta x)}{\cosh\frac{1}{2}\beta\pi}.\end{aligned}$$

It is to be noticed that (46) is only true if x lies between 0 and μ. This may be regarded as a consequence of the fact that (47) only holds good when x is positive and less than 2π, but the necessity for the condition is also evident from the process of verification by ordinary algebra. Thus the expression in (46)

$$\begin{aligned}&=2\left\{e^{-x}-e^{-3x}+e^{-5x}-\quad-\frac{2\cosh x}{1+e^{2\mu}}+\frac{2\cosh 3x}{1+e^{6\mu}}-\&c.\right\}\\ &=\frac{2e^{-x}}{1+e^{-2x}}-2(e^{x}+e^{-x})(e^{-\mu}-e^{-3\mu}+e^{-5\mu}-\&c.)+2(e^{3x}+e^{-3x})(e^{-3\mu}-e^{-6\mu}+e^{-9\mu}-\quad)-\&c\\ &=\frac{2}{e^{x}+e^{-x}}-\frac{2e^{x-\mu}}{1+e^{2(x-\mu)}}-\frac{2e^{x+\mu}}{1+e^{2(x+\mu)}}+\&c\\ &=\operatorname{sech}x-\operatorname{sech}(x-\mu)-\operatorname{sech}(x+\mu)+\&c,\end{aligned}$$

wherein we see that to justify the summations of $e^{-x}-e^{-3x}+\&c$, and $e^{x-\mu}-e^{3(x-\mu)}+\&c$ as ordinary geometrical progressions we must suppose x to be positive and less than μ. Also since $\operatorname{sech}x-\operatorname{sech}(x-\mu)-\operatorname{sech}(x+\mu)+\&c$ is periodic, while the expression in (46) is not so, we see that the equality will not hold good beyond these limits.

I have worked out the corresponding proofs of the other five identities (19) to (23) in the same way, but none of them call for any special remark. The process is not in all cases exactly similar, as, *ex gr*, in deducing (19) from

$$\frac{\pi}{\sinh\beta\pi}=\frac{1}{\beta}-\frac{2\beta}{\beta^2+1}+\frac{2\beta}{\beta^2+2^2}-\&c.,$$

$$\frac{\sin x}{\beta^2+1^2}+\frac{3\sin 3x}{\beta^2+3^2}+\frac{5\sin 5x}{\beta^2+5^2}+\&c.=\frac{\pi}{4}\frac{\cosh(\frac{1}{2}\beta\pi-\beta x)}{\cosh\frac{1}{2}\beta\pi},$$

we find

$$\begin{aligned}\frac{\sin z}{\sinh\frac{1}{2}\nu}+\frac{\sin 3z}{\sinh\frac{3}{2}\nu}+\&c.&=\frac{2\mu}{\pi^2}(\sin z+\tfrac{1}{3}\sin 3z+\tfrac{1}{5}\sin 5z+\&c.)\\ &\quad-\frac{\mu}{\pi}\left\{\frac{\cosh(\mu-2x)}{\cosh\mu}-\frac{\cosh 2(\mu-2x)}{\cosh 2\mu}+\&c\right\},\end{aligned}$$

whence, since the first series on the right hand side$=\frac{1}{4}\pi$, when x is positive and less than μ,

$$\frac{\sin z}{\sinh \frac{1}{2}\nu}+\frac{\sin 3z}{\sinh \frac{3}{2}\nu}+\&c.=\frac{\mu}{2\pi}\left\{1-2\frac{\cosh(\mu-2x)}{\cosh\mu}+2\frac{\cosh 2(\mu-2x)}{\cosh 2\mu}-\&c.\right\}$$

and

$$\tanh x-\tanh(x-\mu)-\tanh(x+\mu)+\&c=1-2\frac{\cosh(\mu-2x)}{\cosh\mu}+2\frac{\cosh 2(\mu-2x)}{\cosh 2\mu}-\&c.$$

The other transformations to which the method of this section leads are

$$\coth x-\coth(x-\mu)-\coth(x+\mu)+\&c.=1+2\frac{\cosh(\mu-2x)}{\cosh\mu}+2\frac{\cosh 2(\mu-2x)}{\cosh 2\mu}+\&c.,$$

$$\operatorname{cosech} x-\operatorname{cosech}(x-\mu)-\operatorname{cosech}(x+\mu)+\&c.=2\frac{\cosh(\frac{1}{2}\mu-x)}{\cosh\frac{1}{2}\mu}+2\frac{\cosh 3(\frac{1}{2}\mu-x)}{\cosh\frac{3}{2}\mu}+\&c.,$$

$$\operatorname{cosech} x+\operatorname{cosech}(x-\mu)+\operatorname{cosech}(x+\mu)+\&c=2\frac{\sinh(\frac{1}{2}\mu-x)}{\sinh\frac{1}{2}\mu}+2\frac{\sinh 3(\frac{1}{2}\mu-x)}{\sinh\frac{3}{2}\mu}+\&c.,$$

$$\operatorname{sech} x+\operatorname{sech}(x-\mu)+\operatorname{sech}(x+\mu)+\&c.=2\frac{\cosh(\frac{1}{2}\mu-x)}{\sinh\frac{1}{2}\mu}-2\frac{\cosh 3(\frac{1}{2}\mu-x)}{\sinh\frac{3}{2}\mu}+\&c.,$$

which can be readily verified by ordinary algebra in the manner explained above. In all these identities x must be positive and less than μ.

§ 16. It only remains to apply the methods of §§ 11 and 15 to the identities (33) and (36), which differ from the others by relating to non-periodic functions. Employing the method of § 11, we have

$$\tanh x=\frac{2x}{x^2+(\frac{1}{2}\pi)^2}+\frac{2x}{x^2+(\frac{3}{2}\pi)^2}+\frac{2x}{x^2+(\frac{5}{2}\pi)^2}+\&c,$$

$$\tanh(x-\mu)=\frac{2(x-\mu)}{(x-\mu)^2+(\frac{1}{2}\pi)^2}+\frac{2(x-\mu)}{(x-\mu)^2+(\frac{3}{2}\pi)^2}+\frac{2(x-\mu)}{(x-\mu)^2+(\frac{5}{2}\pi)^2}+\&c,$$

$$\tanh(x+\mu)=\frac{2(x+\mu)}{(x+\mu)^2+(\frac{1}{2}\pi)^2}+\frac{2(x+\mu)}{(x+\mu)^2+(\frac{3}{2}\pi)^2}+\frac{2(x+\mu)}{(x+\mu)^2+(\frac{5}{2}\pi)^2}+\&c.,$$

$$\cdots\cdots\cdots\cdots;$$

whence

$$\begin{aligned}\tanh x+\tanh(x-\mu)+\tanh(x+\mu)+\&c.&=\frac{4\pi}{\mu}\left\{e^{-\nu}\sin 2z+e^{-3\nu}\sin 4z+\&c.\right\}\\&+\frac{4\pi}{\mu}\left\{e^{-3\nu}\sin 2z+e^{-6\nu}\sin 4z+\&c.\right\}\\&+\frac{4\pi}{\mu}\left\{e^{-5\nu}\sin 2z+e^{-10\nu}\sin 4z+\&c.\right\}\\&+\cdots\cdots\cdots\\&=\frac{4\pi}{\mu}\left(\frac{e^{-\nu}}{1-e^{-2\nu}}\sin 2z+\frac{e^{-2\nu}}{1-e^{-4\nu}}\sin 4z+\&c.\right)\\&=\frac{2\pi}{\mu}\left(\frac{\sin 2z}{\sinh\nu}+\frac{\sin 4z}{\sinh 2\nu}+\&c.\right);\quad\ldots\quad(48)\end{aligned}$$

whereas the true equation is

$$\tanh x+\tanh(x-\mu)+\tanh(x+\mu)+\&c.=\frac{2x}{\mu}+\frac{2\pi}{\mu}\left(\frac{\sin 2z}{\sinh\nu}+\frac{\sin 4z}{\sinh 2\nu}+\&c\right) \qquad (49)$$

It is well known that if an infinite system of series be summed by rows and by columns, the results need not necessarily be the same, but the above is a striking instance of such a disagreement We should be prepared for some ambiguity from the observation that although the value of the left-hand side is liable to a change of a unit according as the number of terms retained is even or uneven, yet in the process of transformation no condition whatever with regard to the number of terms in the columns is, or can be, imposed, but we should scarcely expect to obtain an absolutely erroneous result by an apparently definite process

If the same method be applied to the hyperbolic cotangent, we have

$$\coth x=\frac{1}{x}+\frac{2x}{x^2+\pi^2}+\frac{2x}{x^2+(2\pi)^2}+\&c,$$

and finally

$$\coth x+\coth(x-\mu)+\coth(x+\mu)+\&c=\frac{\pi}{\mu}\coth z+\frac{4\pi}{\mu}\left(\frac{\sin 2z}{e^{2\nu}-1}+\frac{\sin 4z}{e^{4\nu}-1}+\&c\right), \qquad (50)$$

which is also erroneous, the term $\frac{2x}{\mu}$ being omitted on the right-hand side

The method of § 15, however, yields correct results, for

$$\frac{2\pi}{\mu}\operatorname{cosech}\nu\sin 2z=\frac{2}{\mu}\left\{\frac{\pi}{\nu}-\frac{2\frac{\nu}{\pi}}{1^2+\frac{\nu^2}{\pi^2}}+\frac{2\frac{\nu}{\pi}}{2^2+\frac{\nu^2}{\pi^2}}-\&c\right\}\sin 2z,$$

$$\frac{2\pi}{\mu}\operatorname{cosech}2\nu\sin 4z=\frac{2}{\mu}\left\{\frac{\pi}{2\nu}-\frac{2\cdot\frac{2\nu}{\pi}}{1^2+\frac{2^2\nu^2}{\pi^2}}+\frac{2\frac{2\nu}{\pi}}{2^2+\frac{2^2\nu^2}{\pi^2}}-\&c\right\}\sin 4z,$$

$$\cdots\qquad\cdots\qquad\cdots$$

whence

$$\frac{2\pi}{\mu}(\operatorname{cosech}\nu\sin 2z+\operatorname{cosech}2\nu\sin 4z+\&c.)$$

$$=\frac{2}{\pi}\left(\sin 2z-\frac{2\sin 2z}{\frac{\mu^2}{\pi^2}+1^2}+\frac{2\sin 2z}{\frac{2^2\mu^2}{\pi^2}+1^2}-\&c.\right)$$

$$+\frac{2}{\pi}\left(\frac{\sin 4z}{2}-\frac{4\sin 4z}{\frac{\mu^2}{\pi^2}+2^2}+\frac{4\sin 4z}{\frac{2^2\mu^2}{\pi^2}+2^2}-\&c\right)$$

$$+\qquad\cdots\qquad\cdots$$

$$=\frac{2}{\pi}\left\{\frac{\pi}{2}-z-\pi\frac{\sinh\frac{\mu}{\pi}(\pi-2z)}{\sinh\mu}+\pi\frac{\sinh\frac{2\mu}{\pi}(\pi-2z)}{\sinh 2\mu}-\&c.\right\}$$

(by use of the formula $\sin\theta+\frac{1}{2}\sin 2\theta+\frac{1}{3}\sin 3\theta+\&c.=\frac{1}{2}\pi-\frac{1}{2}\theta$)

$$=1-\frac{2x}{\mu}-2\frac{\sinh(\mu-2x)}{\sinh\mu}+2\frac{\sinh 2(\mu-2x)}{\sinh 2\mu}-\&c.,$$

and therefore

$$\frac{2x}{\mu}+\frac{2\pi}{\mu}\left(\frac{\sin 2z}{\sinh\nu}+\frac{\sin 4z}{\sinh 2\nu}+\&c\right)=1-2\frac{\sinh(\mu-2x)}{\sinh\mu}+2\frac{\sinh 2(\mu-2x)}{\sinh 2\mu}-\&c$$

$$=1-2e^{-2x}+2e^{-4x}-\&c.+2\left(\frac{e^{2x}-e^{-2x}}{e^{2\mu}-1}-\frac{e^{4x}-e^{-4x}}{e^{4\mu}-1}+\&c\right)$$

$$=1-\frac{2}{e^{2x}+1}+2\left\{(e^{2x}-e^{-2x})(e^{-2\mu}+e^{-4\mu}+\&c.)-(e^{4x}-e^{-4x})(e^{-4\mu}+e^{-8\mu}+\&c.)+\&c\right\}$$

$$=\tanh x+2\left\{\frac{e^{2(x-\mu)}}{1+e^{2(x-\mu)}}-\frac{e^{-2(x+\mu)}}{1+e^{-2(x+\mu)}}+\&c.\right\}$$

$$=\tanh x+\tanh(x-\mu)+\tanh(x+\mu)+\&c,$$

which is the true formula

In the same way, since

$$\frac{2\pi}{\mu}\coth\nu\sin 2z=\frac{2}{\mu}\left\{\frac{\pi}{\nu}+\frac{2\frac{\nu}{\pi}}{1^2+\frac{\nu^2}{\pi^2}}+\frac{2\frac{\nu}{\pi}}{2^2+\frac{\nu^2}{\pi^2}}+\&c\right\}\sin 2z,$$

we find that

$$\frac{2\pi}{\mu}(\coth\nu\sin 2z+\coth 2\nu\sin 4z+\&c)=-\frac{2x}{\mu}+1+2\frac{\sinh(\mu-2x)}{\sinh\mu}+2\frac{\sinh 2(\mu-2x)}{\sinh 2\mu}+\&c$$

$$=-\frac{2x}{\mu}+\coth x+\coth(x-\mu)+\coth(x+\mu)+\&c.,$$

which is correct, and agrees with (36)

It is of course easy to assure one's self that (48) cannot be true, for, taking $\mu=\pi$ for simplicity, and differentiating with regard to x or z,

$$\frac{4}{(e^x+e^{-x})^2}+\frac{4}{(e^{x-\pi}+e^{-(x-\pi)})^2}+\frac{4}{(e^{x+\pi}+e^{-(x+\pi)})^2}+\&c=8\left\{\frac{\cos 2x}{e^{\pi}-e^{-\pi}}+\frac{2\cos 4x}{e^{2\pi}-e^{-2\pi}}+\frac{3\cos 6x}{e^{3\pi}-e^{-3\pi}}+\&c.\right\},$$

and it is evident that if we take $x>\frac{1}{4}\pi$ and $<\frac{3}{4}\pi$ we should have a positive quantity equated to a negative quantity

I thought it of interest to actually verify numerically the truth of the formulæ (33) and (36) in one or two cases. Working with seven-figure logarithms, and taking $\mu=2$, $x=\frac{1}{2}$, I found that each side of (33) was $=0{\cdot}545188$, and for $\mu=2$, $x=\frac{1}{4}$ that each side was $=0{\cdot}282281$, while for $x=\frac{1}{2}$, $\mu=2$ each side of (36) was $=2{\cdot}07112$, and for $x=\frac{1}{4}$, $\mu=2$ each side was $=4{\cdot}04247$, placing beyond doubt the correctness of (33) and (36)

It is a characteristic property of the identities noticed in this paper that in all cases the series on both sides are convergent whatever may be the values of x and μ. For the actual calculation of the elliptic functions the formulæ (10) to (17) would be preferable to (1) to (8) if the angle of the modulus was very near to 90°, so that q was

nearly equal to unity, but as probably the theta functions (or their transformations as in (28)) would always afford the best means of actually calculating the elliptic functions, I have not investigated whether (10) to (17) would present any advantages over the formulæ which result directly from the change of modulus from k to k', as *ex. gr.* the formula at the beginning of § 4, viz.

$$\cos \operatorname{am} u = \frac{\pi}{k\mathrm{K}'}\left\{\frac{1}{e^x+e^{-x}} - \frac{r}{1+r}(e^x+e^{-x}) + \frac{r^3}{1+r^3}(e^{3x}+e^{-3x}) - \&c.\right\}$$

§ 17. There are two well-marked classes of identities that are derived from the theory of elliptic functions, viz. pure algebraical identities, in which only one single letter is involved, as *ex. gr.*

$$(1-2q+2q^4-\&c.)^4+(2q^{\frac{1}{4}}+2q^{\frac{9}{4}}+\&c.)^4=(1+2q+2q^4+\&c.)^4,$$

and what may for the sake of distinction be called transcendental identities, viz. in which a function of μ is equated to a function of $\frac{\pi^2}{\mu}$. To this latter class belong the chief identities discussed in this memoir, and if special values be assigned to x such that the left-hand member of the equation is of the same function of μ that the right-hand member is of $\frac{\pi^2}{\mu}$, or, in other words, if the identity is of the form $\phi(\mu)=\phi(\nu)$, where $\mu\nu=\pi^2$, such a result is usually very interesting. The best known identity of this class is

$$\sqrt[4]{\log\frac{1}{q}}\,(\tfrac{1}{2}+q+q^4+q^9+\&c.)=\sqrt[4]{\log\frac{1}{r}}\,(\tfrac{1}{2}+r+r^4+r^9+\&c.), \qquad (51)$$

but there is another elegant formula of the same kind to which ABEL has drawn attention (Œuvres, t. i. p. 307), viz.

$$\frac{1}{\sqrt[24]{q}}(1+q)(1+q^3)(1+q^5)\ldots=\frac{1}{\sqrt[24]{r}}(1+r)(1+r^3)(1+r^5)\ldots, \qquad (52)$$

the relation between q and r being of course

$$\log q \,.\, \log r=\pi^2.$$

It seems probable that all the transcendental formulæ of this latter class can be deduced from the trigonometrical identities in § 11 and at the beginning of § 12 by elementary methods, without the introduction of elliptic-function formulæ, and it is of some interest to verify (52) in this way.

Starting from the formula (23), which may be written

$$-\tfrac{1}{2}\operatorname{cosec} x+\frac{\sin x}{1+e^{\mu}}+\frac{\sin 3x}{1+e^{3\mu}}+\&c.=-\frac{\pi}{2\mu}\left\{\frac{1}{e^{s}-e^{-s}}-\frac{1}{e^{s-\nu}-e^{-(s-\nu)}}-\frac{1}{e^{s+\nu}-e^{-(s+\nu)}}+\&c.\right\},$$

we have, on differentiation with regard to x,

$$\tfrac{1}{2}\frac{\cos x}{\sin^2 x}+\frac{\cos x}{1+e^{\mu}}+\frac{3\cos 3x}{1+e^{3\mu}}+\&c.=\frac{\pi^2}{2\mu^2}\left\{\frac{e^{s}+e^{-s}}{(e^{s}-e^{-s})^2}-\frac{e^{s-\nu}+e^{-(s-\nu)}}{(e^{s-\nu}-e^{-(s-\nu)})^2}-\frac{e^{s+\nu}+e^{-(s+\nu)}}{(e^{s+\nu}-e^{-(s+\nu)})^2}+\&c.\right\}$$

Put $x=0$, and

$$\tfrac{1}{4}\frac{\cos x}{\sin^2 x}=\tfrac{1}{4}\cdot\frac{1-\frac{1}{2}x^2}{x^2(1-\frac{1}{3}x^2)}=\frac{1}{4x^2}(1-\tfrac{1}{2}x^2+\tfrac{1}{3}x^2)$$

$$=\frac{1}{4x^2}-\frac{1}{24},$$

while

$$\frac{\pi^2}{2\mu^2}\frac{e^x+e^{-x}}{(e^x-e^{-x})^2}=\frac{1}{4x^2}\left(1+\tfrac{1}{2}\frac{\pi^2x^2}{\mu^2}-\tfrac{1}{3}\frac{\pi^2x^2}{\mu^2}\right)$$

$$=\frac{1}{4x^2}+\frac{1}{24}\frac{\pi^2}{\mu^2},$$

so that

$$-\frac{1}{24}+\frac{1}{1+e^{\mu}}+\frac{3}{1+e^{3\mu}}+\&c=\frac{1}{24}\frac{\pi^2}{\mu^2}-\frac{\pi^2}{\mu^2}\left\{\frac{e^{\nu}+e^{-\nu}}{(e^{\nu}-e^{-\nu})^2}-\frac{e^{2\nu}+e^{-2\nu}}{(e^{2\nu}-e^{-2\nu})^2}+\&c\right\}$$

$$=\frac{1}{24}\frac{\pi^2}{\mu^2}-\frac{\pi^2}{\mu^2}\Big\{(e^{-\nu}+e^{-3\nu})(1+2e^{-2\nu}+3e^{-4\nu}+4e^{-6\nu}+\&c.)$$

$$-(e^{-2\nu}+e^{-6\nu})(1+2e^{-4\nu}+3e^{-8\nu}+4e^{-12\nu}+\&c.)$$

$$+\&c.\Big\}$$

$$=\frac{1}{24}\frac{\pi^2}{\mu^2}-\frac{\pi^2}{\mu^2}\left\{\frac{e^{-\nu}}{1+e^{-\nu}}+\frac{3e^{-3\nu}}{1+e^{-3\nu}}+\frac{5e^{-5\nu}}{1+e^{-5\nu}}+\&c.\right\}$$

$$=\frac{1}{24}\frac{\pi^2}{\mu^2}-\frac{\pi^2}{\mu^2}\left\{\frac{1}{1+e^{\frac{\pi^2}{\mu}}}+\frac{3}{1+e^{\frac{3\pi^2}{\mu}}}+\&c.\right\},$$

whence, on integration with regard to μ,

$$-\frac{\mu}{24}-\log(1+e^{-\mu})-\log(1+e^{-3\mu})-\&c.=-\frac{1}{24}\frac{\pi^2}{\mu}-\log(1+e^{-\frac{\pi^2}{\mu}})-\log(1+e^{-\frac{3\pi^2}{\mu}})-\&c.$$

$$+\text{const},$$

viz.

$$e^{\frac{\mu}{24}}(1+e^{-\mu})(1+e^{-3\mu})\ldots=C\,e^{\frac{\pi^2}{24\mu}}(1+e^{-\frac{\pi^2}{\mu}})(1+e^{-\frac{3\pi^2}{\mu}})\ldots,$$

and $C=1$, as is seen by putting $\mu=\pi$, so that (52) is established.

The other identity (51), or rather the generalization of it,

$$e^{-x^2}+e^{-(x-\mu)^2}+e^{-(x+\mu)^2}+\&c.=\frac{\sqrt{\pi}}{\mu}\left\{1+2e^{-\frac{\pi^2}{\mu^2}}\cos\frac{2\pi x}{\mu}+2e^{-\frac{4\pi^2}{\mu^2}}\cos\frac{4\pi x}{\mu}+\&c\right\}\quad(53)$$

(which is much more difficult to prove by elementary methods than any of the identities discussed in this paper), I deduced by algebraical processes from the equation in § 12, viz. from

$$\frac{a}{x^2+a^2}+\frac{a}{(x-\mu)^2+a^2}+\frac{a}{(x+\mu)^2+a^2}+\&c=\frac{\pi}{\mu}\left\{1+2e^{-\frac{2\pi a}{\mu}}\cos\frac{2\pi x}{\mu}+2e^{-\frac{4\pi a}{\mu}}\cos\frac{4\pi x}{\mu}+\&c.\right\},\quad(54)$$

in the Philosophical Magazine for June 1874 (ser. 4, vol. xlvii. p. 437 *et seq.*); but it perhaps is worth while to note here what is the most natural way of obtaining it from

(54), viz. by help of the theorems

$$\left[e^{-n^2\frac{d^2}{da^2}}\frac{a}{x^2+a^2}\right]_{a=0}=\frac{\sqrt{\pi}}{2n}e^{-\frac{x^2}{4n^2}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (55)$$

$$\left[e^{-n^2\frac{d^2}{da^2}}e^{-ax}\right]_{a=0}=e^{-n^2x^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (56)$$

whence, operating on (54) with $e^{-n^2\frac{d^2}{da^2}}$, and making $a=0$, we have at once

$$\frac{\sqrt{\pi}}{2n}\left\{e^{-\frac{x^2}{4n^2}}+e^{-\frac{(x-\mu)^2}{4n^2}}+e^{-\frac{(x+\mu)^2}{4n^2}}+\&c\right\}=\frac{\pi}{\mu}\left\{1+2e^{-\frac{4n^2\pi^2}{\mu^2}}\cos\frac{2\pi x}{\mu}+2e^{-\frac{16n^2\pi^2}{\mu^2}}\cos\frac{4\pi x}{\mu}+\&c\right\},$$

which is (53) if we take $n=\frac{1}{2}$

Of the two lemmas (55) and (56) the truth of the second is seen at once, for

$$\begin{aligned}e^{-n^2\frac{d^2}{da^2}}e^{-ax}&=\left(1-n^2\frac{d^2}{da^2}+\frac{1}{1.2}n^4\frac{d^4}{da^4}-\&c\right)e^{-ax}\\&=\left(1-n^2x^2+\frac{1}{1.2}n^4x^4-\&c\right)e^{-ax}\\&=e^{-n^2x^2-ax},\end{aligned}$$

and (55) is easily established, since a being put $=0$ after the performance of the differentiations,

$$\begin{aligned}e^{-n^2\frac{d^2}{da^2}}\frac{a}{x^2+a^2}=e^{-n^2\frac{d^2}{da^2}}\int_0^\infty e^{-au}\cos xu\,du&=\int_0^\infty e^{-n^2u^2}\cos xu\,du\\&=\frac{\sqrt{\pi}}{2n}e^{-\frac{x^2}{4n^2}}\end{aligned}$$

But the investigation is not elementary, and if we assume a knowledge of the integral

$$\int_0^\infty e^{-a^2x^2}\cos 2bx\,dx=\frac{\sqrt{\pi}}{2a}e^{-\frac{b^2}{a^2}},$$

we may as well apply it directly to prove (53) by FOURIER's theorem as explained in § 7, or employ it as SCHELLBACH has done ('Die Lehre von den elliptischen Integralen &c.,' 1864, p 30) It does not seem to be easy to establish (55) without the aid of an integral, for, expanding in ascending powers of x, we have to show that when $a=0$,

$$e^{-n^2\frac{d^2}{da^2}}\left(\frac{1}{a}-\frac{x^2}{a^3}+\frac{x^4}{a^5}-\&c\right)=\frac{\sqrt{\pi}}{2n}\left(1-\frac{x^2}{4n^2}+\frac{x^4}{32n^4}-\&c\right),$$

and, taking the first term only, although we see at once that

$$e^{-n^2\frac{d^2}{da^2}}\frac{1}{a}=e^{-n^2\frac{d^2}{da^2}}\int_0^\infty e^{-au}du=\int_0^\infty e^{-n^2u^2}du=\frac{\sqrt{\pi}}{2n},$$

yet

$$e^{-n^2\frac{d^2}{da^2}}\frac{1}{a}=\frac{1}{a}-\frac{1.2.n^2}{a^3}+\frac{1.2.3.4n^4}{a^5}-\&c,$$

which is divergent, and cannot apparently by any simple method be so transformed that its value when $a=0$ may be evident, without the intervention of an integral. Thus the method depending upon (55), though more direct, is not so elementary as that described in the Philosophical Magazine

It is curious that all the formulæ of the form

$$\phi x \pm \phi(x-\mu) \pm \phi(x+\mu) + \&c = \text{series of sines or cosines}$$

which can be obtained by definite integrals, and which possess any interest, should be in reality elliptic-function identities. Of course every result that can be derived from these identities by differentiation, by multiplication by a factor and integration, &c., can as a rule be obtained directly from an integral, which integral itself would arise from a similar treatment of the original integral. This is true of the identities in the Philosophical Magazine, ser. 4, vol. xlii. pp. 422 *et seq.* (December 1871), and, for example, such an integral as

$$\int_0^\infty \frac{e^{-x^2}\cos 2bx}{a^2+x^2}\,dx = \frac{\sqrt{\pi}}{2a}\,e^{a^2}\{e^{-2ab}\operatorname{erfc}(a-b)+e^{2ab}\operatorname{erfc}(a+b)\} \tag{57}$$

(where $\operatorname{erfc} x = \int_x^\infty e^{-x^2}dx$) would give rise to identities which, however, could be deduced from (28) and (53) by a similar process to that by which (57) can be derived from

$$\int_0^\infty e^{-x^2}\cos 2bx\,dx = \frac{\sqrt{\pi}}{2}\,e^{-b^2}.$$

XVIII *On Repulsion resulting from Radiation.*—Part II.

By WILLIAM CROOKES, *F R S &c.*

Received March 20,—Read April 22, 1875

CONTENTS.

81 THE present paper is in continuation of one which I had the honour of reading before the Royal Society, December 11th, 1873, and which was published in the Philosophical Transactions, vol clxiv. part 2, page 501. In that paper I described various pieces of apparatus, chiefly in the form of delicate balances suspended in glass tubes, by means of which I was enabled to show attraction or repulsion when radiation acted on a mass at one end of the beam, according as the glass tube contained air at the normal pressure, or was perfectly exhausted At an intermediate internal pressure the action of radiation appeared *nil* Towards the end of the paper I said (70), "I have arranged apparatus for obtaining the movements of repulsion and attraction in a horizontal instead of a vertical plane Instead of supporting the beams on needle-points, so that they could only move up and down, I suspend them by the centre to a long fibre of cocoon-silk in such a manner that the movements would be in a horizontal plane With apparatus of this kind, using very varied materials for the index, enclosing them in tubes and bulbs of different sizes, and experimenting in air and gases of different densities up to Sprengel and chemical vacua, I have carried out a large series of experiments, and have obtained results which, whilst they entirely corroborate those already described, carry the investigation some steps further in other directions."

82. I have introduced two important improvements into the Sprengel pump* which

* Philosophical Transactions, 1873, vol. clxiii. p 295, 1874, vol clxiv pp 509, 516. Phil. Mag, Aug. 1874

enable me to work with more convenience and accuracy. Instead of trusting to the comparison between the barometric gauge and the barometer to give the internal rarefaction of my apparatus, I have joined a mercurial siphon-gauge to one arm of the pump. This is useful for measuring very high rarefactions in experiments where a difference of pressure equal to a tenth of a millimetre of mercury is important. By its side is an indicator for still higher rarefactions; it is simply a small tube having platinum wires sealed in, and intended to be attached to an induction-coil. This is more convenient than the plan formerly adopted (51), of having a separate vacuum-tube forming an integral part of each apparatus At exhaustions beyond the indications of the siphon-gauge I can still get valuable indications of the nearness to a perfect vacuum by the electrical resistance of this tube. I have frequently carried exhaustions to such a point that an induction-spark will prefer to strike its full distance in air rather than pass across the $\frac{1}{4}$ inch separating the points of the wires in the vacuum-tube. A pump having these pieces of apparatus attached to it was exhibited in action by the writer before the Physical Society, June 20th, 1874.

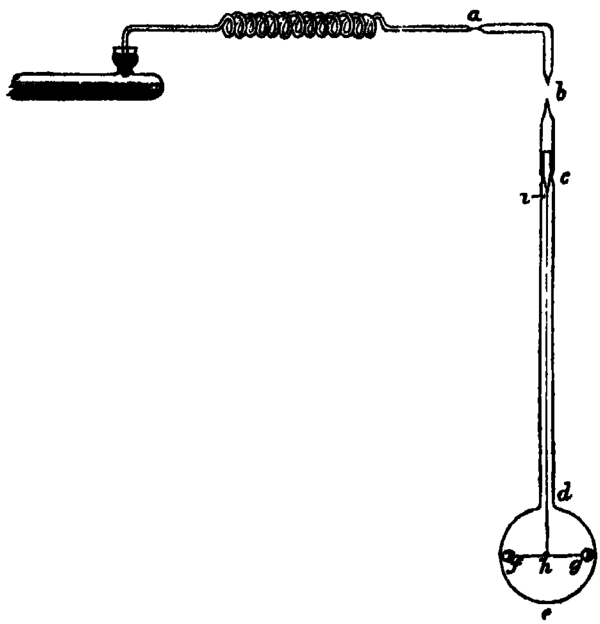
83 The cement which I have found best for keeping a vacuum is made by fusing together 8 parts by weight of resin and 3 parts of bees-wax. For a few hours this seems perfect, but at the highest exhaustions it leaks in the course of a day or two Ordinary or vulcanized india-rubber joints are of no use in these experiments, as when the vacuum is high they allow oxygenized air to pass through as quickly as the pump will take it out. Whenever possible the glass tubes should be united by fusion, and where this is impracticable mercury joints should be used. The best way to make these is to have a well-made conical stopper, cut from plain india-rubber, fitting into the wide funnel-tube of the joint and perforated to carry the narrow tube. Before fitting the tubes in the india-rubber, the latter is to be heated in a spirit-flame until its surface is decomposed and very sticky, it is then fitted into its place, mercury is poured into the upper part of the wide tube so as to completely cover the india-rubber, and oil of vitriol is poured on the surface of the mercury. When well made this joint seems perfect, the only attention which it subsequently requires is to renew the oil of vitriol when it gets weakened by absorption of aqueous vapour. Cement has to be used when flat glass or crystal windows are to be cemented on to pieces of apparatus, as subsequently described (99, 102)

It would be of great service could I find a cement which is easily applied and removed, and will allow the joint to be subjected to the heat of boiling water for some hours without leaking under the highest rarefactions. Hitherto I have failed to find one which answers these requirements. I mention this in the hope that some one who happens to read this may be in possession of the recipe for such a cement, and will communicate it to me.

84. Before my first paper on this subject was read before the Royal Society I had discarded the balance form of apparatus there described, and commenced experimenting with bulbs and tubes in which quantitative results could be obtained. On

December 11th, 1873, when illustrating my paper, I exhibited to the Society many of these new forms of apparatus For the purposes of simple illustration, and for experiments where quantitative determinations are not required, I find a horizontal index suspended in a glass bulb the most convenient The apparatus, with its mode of attachment to the pump, are shown in fig 1

Fig 1.



a, *b*, *c*, *d* is originally a straight piece of soft lead-glass tubing 18 inches long, $\frac{5}{8}$ of an inch external and $\frac{3}{8}$ internal diameter. At one end is blown a bulb, *d e*, about 3 inches diameter. The part *a b* of the tube is drawn out to about half its original diameter, and bent at right angles The tube is slightly contracted at *c*, and very much contracted and thickened at *b*. At *a* it is also contracted and cemented by fusion to a narrower piece of tube bent in the form of a spiral, and fitting by a mercury-joint into the sulphuric-acid chamber of the pump. The object of the spiral is to secure ample flexibility for the purpose of levelling the apparatus, and at the same time having a fused joint. *f g* is a very fine stem of glass, drawn from glass tubing, and having a small loop (*h*) in the middle. At each end of the stem is a ball or disk, made of pith, cork, ivory, metal, or other substance. *h i* is a fine silk fibre made from split cocoon-silk, it is cemented by shellac at the upper end to a piece of glass rod a little smaller in diameter than the bore of the tube, and drawn out to a point, as shown. The contraction (*c*) in the tube is for the purpose of keeping this glass rod in its place, when properly adjusted it is secured in its place by a small piece of hot shellac, care being taken not to cement the rod all round, and so cut off the connexion between the air in the bulb and that in the upper part of the tube. The silk fibre is tied on to the loop of the glass stem at *h*. The length of the fibre is so adjusted that the stem and

disks will hang about $\frac{1}{8}$ of an inch below the centre of the bulb; that much having to be allowed for the contraction of the silk when the air is exhausted.

85. The bulb-tube is firmly clamped in a vertical position, so that the index hangs freely, and the pump is set to work, the bulb being surrounded with a vessel of water which is kept boiling all the time exhaustion goes on. The gauge soon rises to the barometric height; but the operation must be continued for several hours beyond this point, in order to get the best effects. If the bulb is not heated during the exhaustion, the index loses sensitiveness after it has been sealed up for a few days, probably owing to the evolution of vapour from the pith; when, however, the precaution is taken of heating the pith, the apparatus preserves its sensitiveness. On this account it is necessary to tie the silk on to the loop in the centre of the glass stem, instead of adopting the easier plan of cementing it with shellac. During the latter stages of the exhaustion, oil of vitriol (which has been boiled and cooled *in vacuo*) should gently leak into the pump through the funnel-stopper at the top of the fall-tube (44). This covers each globule of mercury, as it falls, with sulphuric acid, and stops mercury vapour from getting into the apparatus*. I cannot find that any vapour is evolved from oil of vitriol

When the exhaustion is carried to the desired degree, a spirit-flame is applied to the contracted part of the tube at *a* (fig. 1), and it is sealed off. The apparatus is then unclamped and the tube is again sealed off at *b*. This double operation is necessary to secure strength at the final sealing, which can only be got by holding the tube horizontally and rotating it in the flame, watching the glass to prevent it softening too suddenly.

86 The best material of which to form the index in these bulb-tubes is pith, either in the form of a needle or bar, or as disks at the end of a glass stem. On December 11th, 1873, and again on April 22nd, 1874, I exhibited before the Royal Society a glass bulb 4 inches in diameter, having suspended in it a bar of pith $3\frac{1}{2}\times\frac{1}{2}$ inches. It had been exhausted in the manner above described; and so sensitive was it to heat, that a touch with the finger on a part of the globe near one extremity of the pith would drive the bar round 90°, whilst it followed a piece of ice as a needle follows a magnet.

To get the greatest delicacy in these apparatus there is required large surface with a minimum of weight (75, 76). Thin disks of pith answer these requirements very satisfactorily; but I have also used disks cut from the wings of butterflies and dragonflies, dried and pressed rose-leaves, very thin split mica and selenite, iridescent films of blown glass, as well as the substances mentioned in my former paper (25). Quantitative experiments to prove this law were attempted; but the bulb-apparatus was found too imperfect for accurate measurements, so another form was devised which will be described further on (102), together with the experiments tried with it.

* By adopting this precaution it is not difficult to raise the mercury in the gauge higher than that in the very perfect barometer by its side, the latter being somewhat depressed by the tension of mercury vapour.

87. With a large bulb, very well exhausted and containing a suspended bar of pith, a somewhat striking effect is produced when a lighted candle or other radiant source is brought about 2 inches from the globe. The pith bar commences to oscillate to and fro, the swing gradually increasing in amplitude until the dead centre is passed over, and then several complete revolutions are made. The torsion of the suspending fibre now offers resistance to the revolutions, and the index commences to turn in the opposite direction. This movement is kept up with great energy and regularity as long as the candle burns—producing, in fact, perpetual motion, provided only the radiation falling on the pith be perpetual*. If the candle is brought closer to the bulb, the rotation of the pith becomes more rapid, if it is moved further away the pith ceases to pass the dead centre, and at a still further distance the index sets equatorially The explanation of the different movements of the pith index according to the distance the radiant body is off, is not difficult on the supposition that the movement is due to the direct impact of waves on the suspended body.

88. It is not at first sight obvious how ice, or a cold substance, can produce the opposite effect to heat, cold being simply negative heat (33) The law of exchanges, however, explains this perfectly. The pith index and the whole of the surrounding bodies are incessantly exchanging heat-rays, and under ordinary circumstances the income and expenditure of heat are in equilibrium. A piece of ice brought near one end of the index cuts off the influx of heat to it from that side, and therefore allows an excess of heat to fall upon it from the opposite side Attraction by a cold body is therefore seen to be only repulsion by the radiation from the opposite side of the room

Bearing the law of exchanges in mind, several apparent anomalies in the movements of these indices are cleared up, and it is also easy to foresee what the movement of a body will be when free to move in space under the influence of varying amounts of radiation.

The heat which all bodies radiate into space can have no influence in moving them, except there be something in the nature of a *recoil* in the act of emitting radiation And even should there be such a recoil, if the body radiates heat equally all round, the recoil will be uniform, and will not move the body in one direction more than in another I need therefore only consider the effect of the radiation *received* by a body. Here also the influx of radiation to a body free to move in space of a uniform temperature may be considered to be equal, and it will acquire the temperature of space without moving in any direction.

89. The case is, however, different if two bodies, each free to move, are near each other in space, and if they differ in temperature either from each other or from the limiting walls of the space. I will give here four typical cases, with experiments sufficient to prove the reasoning to be correct.

CASE I. Two hot bodies, A and B, in space of a lower temperature than themselves. The body A receives heat uniformly from space, except where the body B intervenes; and on this side A receives more heat, as B is hotter than the space behind it; A will

* This experiment was exhibited for the first time at the Royal Society's Soirée, April 22nd, 1874.

therefore move from B. In the same manner it can be shown that B will move from A. The result will therefore be *mutual repulsion.*

CASE II. Two cold bodies, A and B, in space of a higher temperature than themselves.

Fig 2 Case I

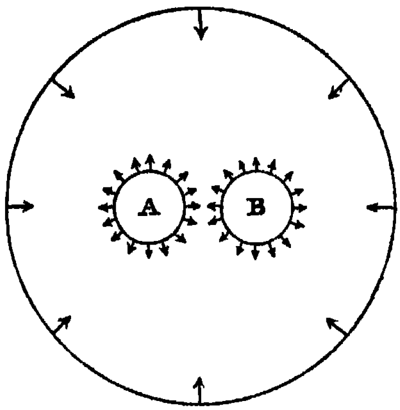
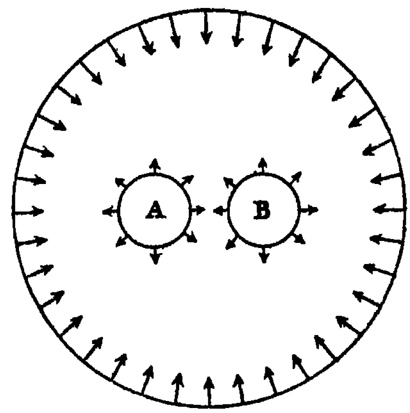


Fig 2 Case II



A will receive much heat from space, except where B cuts it off, and on that side it will only receive slight radiation from B A will therefore be driven towards B In the same manner it can be shown that B will be driven towards A, and the result will therefore be an *apparent mutual attraction*

CASE III Two bodies, A hot and B cold, in cold space. The body A receives heat uniformly from all sides, even from that opposite B (B being of the same temperature as space) A will therefore not move B receives heat uniformly from all sides, except from that opposite A, on which side the influx of heat is more intense The result will therefore be that A *remains stationary whilst* B *is repelled.*

Fig 2 Case III

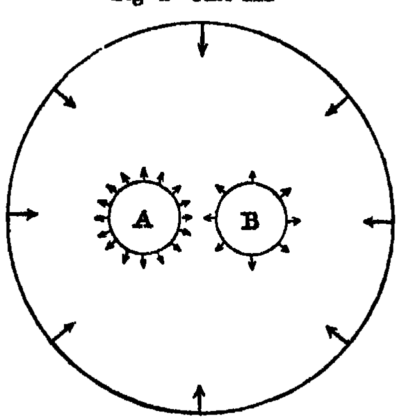
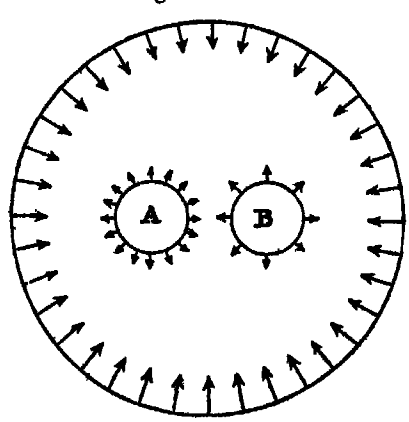


Fig. 2 Case IV.



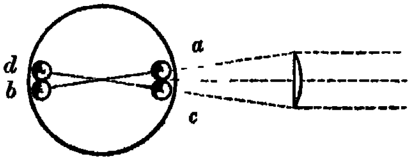
CASE IV. Two bodies, A hot and B cold, in hot space. The body A receives heat uniformly from all sides, except from that opposite B. Here the heat is less intense. A is therefore driven towards B by the extra influx of heat on the other side of A. B receives strong influx of heat from all sides, and just as much from the side opposite A

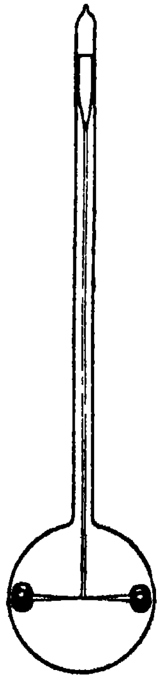
as from any other. B will therefore not move The result will be that A *will be apparently attracted towards* B, *whilst* B *will remain stationary.*

The force with which the bodies A and B in these four cases will be repelled, or apparently attracted, will vary with their distance from each other, being stronger when they are close and weaker when they are far apart. The diminution will not, however, follow the usual law of inverse squares, but a more complicated law.

90 Experiment proves the above reasoning to be correct. A bulb-tube was prepared in the manner already described (84), but in it were suspended, by separate silk fibres, two glass stems, each having pith balls at its extremity Fig 3 shows the elevation and plan of the apparatus The torsion of the silk fibres was so arranged that the pith balls *a b* hung freely about a millimetre from the balls *c d*. The glass stems were looped in the middle, and bent so that they did not touch each other. After complete exhaustion the following experiments were tried

Fig 3.





A beam of radiant heat was concentrated on to the two balls *a c* When applied momentarily and then removed the radiation simply drove the balls apart, and immediately allowed them to come together again. When, however, the beam was allowed to play upon the balls for about half a minute they became warm and widely separated, and upon now removing the beam of heat the balls did not fall together at once, but took several minutes to regain their original position This experiment therefore proves Case I.

The bulb and contents being of the ordinary temperature, a spirit-flame was rapidly passed round the bulb to warm it quickly on all sides The balls were thus in the condition imagined in Case II, being in a space warmer than themselves. They immediately came together, *a* touching *c*, and *d* touching *b*.

Many experiments were tried with the object of proving experimentally the propositions in Cases III. and IV.; but with this apparatus it was found impossible to warm one of the balls without at the same time producing repulsion of the ball by the beam of radiation concentrated upon it. There is, however, little doubt, from the experimental proof of Cases I. and II., that the reasoning is equally correct in the other cases.

91. With a highly exhausted bulb and light pith index, which was found to be exceedingly sensitive to radiation, numerous experiments were tried to see if there was any difference in action between the fingers and a tube of water of the same tempe-

rature. Many persons believe that there is a peculiar emanation or *aura* proceeding from the human hand, and Baron Von REICHENBACH* considered that he had proved this to be the case. Were this true it was not impossible that the emanation would affect the pith index. I have been unable, however, to detect the slightest action exerted by my own or any other person's hand which I could not entirely explain by an action of heat.

92. A similar series of experiments were tried with various large crystals, which were presented in different ways and with various precautions to the pith index. At first a decided action was observed, but in proportion as precautions were taken to eliminate the effect of heat, so was the action seen to diminish, until very little doubt was left in my mind that the slight residual action would have been entirely stopped had it been possible, with the apparatus then used, to altogether eliminate the action of heat.

93. Attempts were made to see if chemical action would attract or repel the index. I could not, however, produce chemical action close to the exhausted bulb, without at the same time liberating such an amount of heat as to mask any other action.

94. Although I most frequently speak of repulsion by *heat*, and in illustrating any of the results obtained I generally use either the fingers or the flame of a spirit-lamp as a convenient source of radiation, it must be clearly understood that these results are not confined to the heating-rays of the spectrum, but that any ray, from the ultra red to the ultra violet, will produce repulsion in a vacuum. I have already mentioned this fact in my first paper (58, 68) Experiments proving the similarity of action of all rays of the spectrum were shown before the Physical Society on June 20th, 1874†. They were, however, tried with a less perfect apparatus than the one I have since used for the same purpose, and need not be further alluded to till I describe the most recent results obtained with the spectrum (110, 111).

95. Some experiments were tried with the object of ascertaining whether the attraction by heat, which, commencing at the neutral point (30 *et seq.*), increased with the density of the enclosed air, would be continued in the same ratio if the apparatus were filled with air above the atmospheric pressure. Two bulbs containing ivory needles suspended by silk fibres were accordingly adjusted to show the same sensitiveness to a hot body. One was kept for comparison, and the other was attached to an apparatus whereby the internal air-pressure could be artificially increased by a column of mercury. A little increase of pressure was enough to show that the sensitiveness to radiation was greater; and under a pressure of $1\frac{1}{2}$ atmosphere the superior delicacy of the ivory in the dense air was very marked. Attempts to carry the pressure to higher points failed, owing to the bursting of the thin glass bulbs. With a little different arrangement no difficulty would be experienced in carrying the experiments to a much higher point; but hitherto the greater interest attending the vacuum experiments has prevented me from working further in this direction. My friend and pupil, Mr. C. H. GIMINGHAM,

* Researches on Magnetism &c., translated by Dr. GREGORY. London, 1850.

† Phil Mag., August 1874

succeeded in the very difficult feat of sealing up some of these tubes under an internal pressure of $1\frac{1}{2}$ atmosphere.

96. To carry this experiment a step further bulbs containing a suspended ivory or mica index were filled with carbonic acid gas, water, carbonic disulphide, ether, alcohol, and other liquids. The index in carbonic acid behaved as if it were in air of somewhat higher density than the atmosphere; movements were also obtained when the liquids were present, but they were so obviously due, in whole or in greater part, to currents, that they proved nothing of importance

97. Two other forms of the bulb-apparatus require mentioning. A thin glass bulb was blown $2\frac{1}{2}$ inches in diameter (fig. 4) Inside this another bulb was blown 2 inches in diameter, at the end of a glass tube 12 inches long In this a light glass index with pith terminals was suspended, and the whole was perfectly exhausted. Fig 4 shows the complete arrangement. In the space between the two bulbs various liquids were enclosed, such as water, solutions of sulphate of copper, alum, perchloride of iron, sulphate of iron, bichromate of potash, sulphate of nickel, &c. These were selected in the hope that amongst them one would be found which would sift out the heat-rays, and so allow me to obtain an action due to light They, however, only affect the dark or extreme red heat-rays, and do not affect the luminous rays which also have a heating-

Fig 4

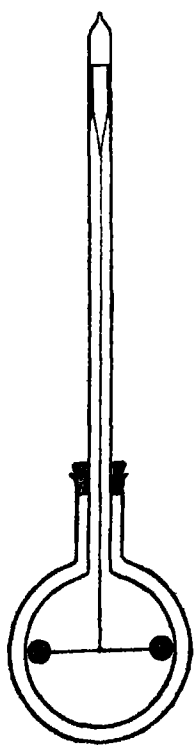
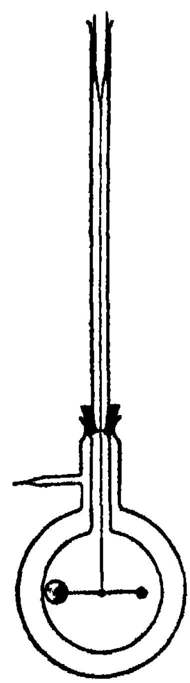


Fig 5



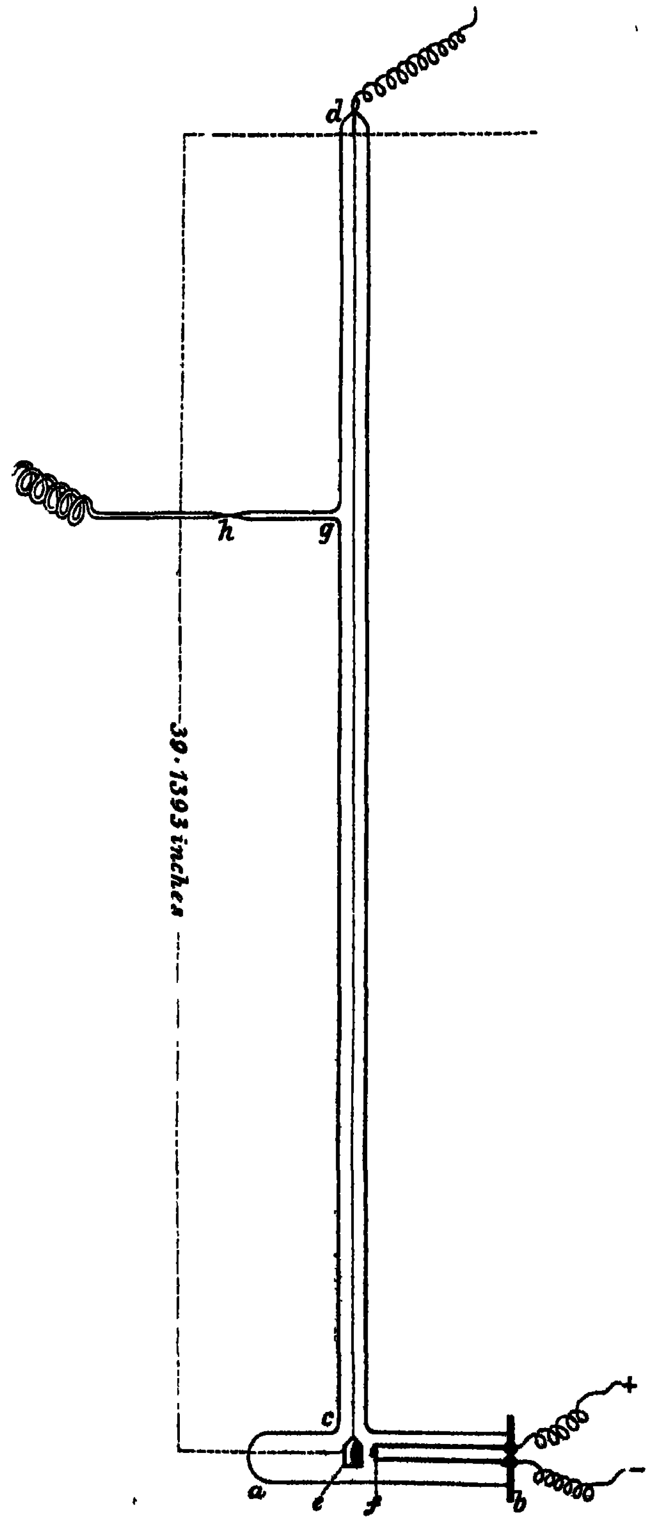
effect. By throwing a beam of sunlight on one of the pith disks powerful repulsion was obtained, whatever was the surrounding shell of liquid That all these liquids allowed

heat to pass through was proved with a thermopile. Solution of sulphate of copper was the most opaque to heat.

98. Another form of apparatus is shown in fig. 5. Two bulbs were blown one in the other, and they were fused together at the necks; to the neck a small tube was fused for connecting with the Sprengel pump The space between the two bulbs was then perfectly exhausted, and the small tube sealed up. I thus possessed what might be called a spherical shell of vacuum surrounding a bulb open to the air In this inner bulb was suspended a pith ball on the end of a glass arm balanced by a knob of glass on to the other end, the suspending fibre being protected by a glass tube fitting into the neck of the inner bulb with a cork. It was found that heat applied to any part of the outer bulb passed across the vacuum, and *attracted* the pith ball (suspended in air). The spherical shell of vacuum across which the heat passed, therefore, produced no change of action, but simply behaved like an extra thick glass bulb. This experiment bears upon the speculation in par. 81 of my former paper on this subject.

99. Having succeeded in proving the fact of repulsion resulting from radiation, I was desirous of getting some quantitative estimations of the forces under examination. A pendulum-apparatus was constructed as shown in fig 6. A wide glass tube (a b) has fused to it a narrower tube (c d), about 40 inches long, e is a

Fig. 6.



turned mass of magnesium, weighing 42 grains, suspended by a very fine platinum wire, the distance between the point of suspension and the centre of gravity of the magnesium bob being 39·139 inches, so that it forms a seconds' pendulum, *f* is a spiral made of platinum plate, fastened to two stout copper wires which pass through the thick plate of glass *b*, and thence pass to a contact-key and a battery. The plate *b* is cemented (83) to the end of the tube *a b*, which is ground flat. *g* is an arm fused into the upright tube for the purpose of connecting it to the glass spiral of the pump; it is contracted at *h* for convenience of sealing off. The fine platinum wire is fastened at its upper end to a thick wire which is sealed into the glass, and passes through to the outside for electrical purposes (120) The distance between the pendulum bob and the spiral is 7 millims. To ignite the spiral the current from two GROVE's cells was used; this brought it to a bright red heat in air, and to a white heat in vacuum

Three feet from the pendulum a telescope was firmly clamped to the bench, it was furnished with a micrometer-eyepiece, with movable spider-threads and graduated circle. The edge of the magnesium bob was brought into the same focus as the traversing cross wire. Observations were taken in the following manner:—The observer at the telescope brought the cross wire to zero, and then adjusted it to coincide with the edge of the pendulum bob An assistant, guided by a seconds' watch, pressed the contact-key down for one second, then broke contact for a second, next made contact for the third second, and so on, alternately making and breaking contact for either 10, 20, or 40 seconds, counting the seconds aloud. At each second the swing of the pendulum increased, and the milled head of the micrometer was kept turning so as to let the cross wire keep up to the furthest point to which the pendulum vibrated. At the end of the experiment the position of the cross wire was taken and its distance from zero recorded.

100. Experiments were first tried in air of normal density. The pump was then set to work, and observations were taken at different heights of the gauge. The difference between the height of the gauge and that of the barometer gave the tension of air in the apparatus in millimetres of mercury, this is recorded in the first column of the following Tables. The second column gives the greatest amplitude of the half oscillation of the pendulum in millimetres—the sign *plus* signifying attraction, and *minus* repulsion.

Near the centre of Table I., in the second column, are five observations to which I have affixed no sign. When trying the experiments I thought that either I had mistaken the direction of impulse, or my assistant had commenced to count the make-and-break seconds wrongly, as the movement *seemed* to be repulsion Never having had repulsion at such a pressure before, I was not prepared for it; and fearing there might be an error, left the sign queried. Another series of observations were taken to re-examine this point; they are given in Table II.

It is worthy of notice in these Tables that the attraction by the incandescent spiral is only moderate in air of ordinary density. The attraction diminishes to a

TABLE I.

Tension of enclosed air, in millims. of mercury Temp =16° C Bar =772 55 millims.	Amplitude of half oscillation, in millims., at end of 40" observation
772 55	+0 46
557·50	+0·54
472·00	+0·49
372·00	+0·39
322·00	+0 41
272 00	+0·28
242·00	+0 18
222 00	+0·15
201·00	+0 11
167·00	+0 12
140·00	0 07 ?
114 50	0 08 ?
89 50	0·12 ?
70 50	0·03 ?
54 00	0·02 ?
48·00	+0·12
37·00	+0 14
29·00	+0 14
20 00	+0 18
14·00	+0 30
9 15	+0 46
6·55	+0·66
4·65	+1·00
3·15	+1 40
2 25	+1·48
1·15	+1·72
0 75	+1·70
0 65	+1·46
0 55	+1·04
0 35	+0·64
0 25	−0 60
0 15	−1·16
−0 05	−5·90

minimum between a tension of 50 millims. and 150 millims, then rises as the pressure diminishes, until, at a tension of 1 15 millim, the attraction is nearly four times what it was in dense air. Above this exhaustion the attraction suddenly drops and changes to repulsion, which at the best vacuum I could get was nearly thirteen times stronger than the attraction in air

The last figure in the first column requires explanation All the others are obtained by subtracting the height of the gauge from that of the barometer, and are *positive* At the highest rarefactions, however, I get the gauge about 0·05 millim above the barometer (85, *note*); the sign, therefore, becomes *negative.*

Table II agrees in the main with Table I. The sign changes to repulsion at pressures corresponding to those queried in Table I, the repulsion, though slight, was unmistakable At 102 millims. pressure the observation has a positive sign. This looks like an error, but as it is so recorded in my notebook, and as I was at that time specially looking for repulsions, I do not feel justified in altering it. What I have called

TABLE II.

Tension of enclosed air, in millims. of mercury. Temp. =18° C. Bar. =772 millims.	Amplitude of half oscillation, in millims., at end of 40″ observation.
772·0	+0·460
770·0	+0·540
769·5	+0·570
769·0	+0·440
769·0	+0·520
769·0	+0·440
769·0	+0·450
565·0	+0·560
557·0	+0·540
472·0	+0·490
440·0	+0·550
369·0	+0·416
213·0	+0·233
207·0	+0·130
189·0	+0·180
173·0	+0·140
164·0	+0·100
162·0	−0·100
142·0	−0·120
132·0	−0·130
127·0	−0·090
105·0	−0·140
102·0	+0·083
73·0	−0·130
60·0	−0·123
56·0	−0·136
51·0	−0·030
41·0	+0·150
33·5	+0·170
32·0	+0·106
23·0	+0·110
22·0	+0·080
16·1	+0·170
16·0	+0·140
7·1	+0·380
6·0	+0·293
3·9	+0·610
1·9	+0·880
1·2	+0·755
0·9	+0·340
0·7	−0·740
0·6	−1·700
0·3	−3·800
0·2	−5·080
0·0	−5·680
−0·05	−6·320

the neutral point, or the point where attraction changes to repulsion, is in this series lower than in the former. There it occurred at a tension of about 0·3 millim. of mercury; here at about 0·8. Neither does the previous attraction attain such strength, although the ultimate repulsion is more intense. The agreement is, however, sufficiently satisfactory, considering the faulty method of measurement.

There are many errors almost inseparable from this form of apparatus. The making

and breaking contact by hand is not sufficiently certain, and hesitation for a fraction of a second would seriously affect the ultimate amplitude of arc. I tried making and breaking by clockwork, also by a seconds' pendulum, but there were difficulties in each plan.

Owing to the mode of suspension, there was uncertainty as to the length of the pendulum. I tried to make it the right length to beat seconds *in vacuo*. Assuming that I had succeeded in this, the pendulum would have executed fewer vibrations in the 40 seconds when oscillating in air, and consequently I should not have got the full benefit from the making and breaking contact, supposing these were accurately timed to seconds.

The battery-power varied, being stronger at the commencement, and gradually declining towards the end of the experiment, and even were the battery to remain constant, the spiral became much hotter, owing to the removal of the air from the apparatus, ranging from a bright red heat in air to a full white heat *in vacuo*.

Owing to the height of the centre of suspension of the pendulum from the stand of the apparatus, the slightest deviation from the perpendicular made an appreciable difference in the distance of the weight from the spiral, and thereby increased or diminished the effect of radiation. Thus the tread of a person across the floor of the laboratory, or the passage of a cart along the street, would cause the image of the edge of the magnesium weight apparently to move from the cross wires in the telescope.

Many of these sources of error could have been removed; but in the mean time having devised a form of apparatus which seemed capable of giving much more accurate results, I ceased experimenting with the pendulum.

Before proceeding to describe the apparatus subsequently employed, I may mention that a candle-flame brought within a few inches of the magnesium weight, or its image focused on the weight and alternately obscured and exposed by a piece of card at intervals of one second, will soon set the pendulum in vibration when the vacuum is very good. A ray of sunlight allowed to fall once on the pendulum immediately sets it swinging. The pendulum-apparatus above described was exhibited, and experiments shown with it, at the Royal Society, April 22nd, 1874, and also before the Physical Society*, June 20th, 1874

101. The difficulty which attended experiments with the balances and bulb-apparatus used at first was to bring the moving part accurately back to zero, and also to measure the deflection produced. I therefore tried several plans of giving a fixed zero-direction to the movable index. Thus a piece of magnetic oxide of iron was cemented to one end of the index, and a permanent magnet was brought near it. This answered pretty well, but was inconvenient, besides not being sufficiently accurate. A bifilar suspension from cocoon-fibres seemed likely to succeed better; but the difficulty of suspending the rod in this manner, so as to get exactly the same tension on each fibre, was very great, and unless this was done there was more tendency to move in one direction than in the

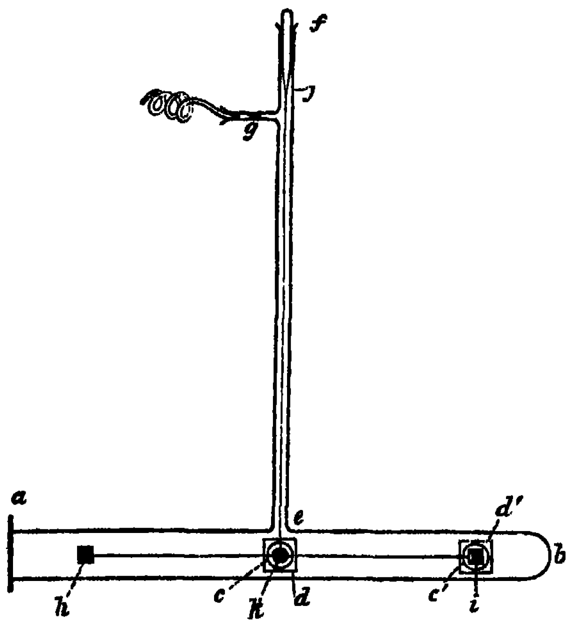
* Phil Mag., August 1874.

other. When I had succeeded in suspending the needle with an equal tension on each silk fibre, I found their elasticity to vary; and as soon as the vacuum was approached one was sure to contract more than the other, twisting the needle out of the axis of the tube, and sometimes causing it to touch the side. This method of suspension was therefore abandoned.

By increasing the length of the needle, and also of the fibre used to suspend it, it was possible to employ fibres with a considerable amount of torsion, and still preserve the delicacy of the apparatus. Fine platinum wire was first tried; but this was soon abandoned in favour of glass fibres, which were found to answer so perfectly that I have since used nothing else.

102. Fig. 7 shows the form of apparatus which I have finally adopted, as combining the greatest delicacy with facility of obtaining accurate observations, and therefore of getting quantitative as well as qualitative results It is a torsion-apparatus in which the beam moves in a horizontal plane, and may be called a horizontal torsion-balance. *a b* is a piece of thin glass tubing, sealed off at the end *b* and ground perfectly flat at the end *a* In the centre a circular hole, *c*, is blown, and another one, *c'*, at the end, the edges of these holes are ground quite flat. *a*, *c*, and *c'* can therefore be sealed up by cementing flat transparent pieces of plate glass, quartz, or rock-salt, *a*, *d*, and *d'* on to them (83). To the centre of *a b* an upright tube, *e f*, is sealed, having an arm, *g*, blown on to it for the purpose of attaching the apparatus to the pump

Fig 7



h i is a glass index, drawn from circular or square (22) glass tube, and as light as possible consistent with the needful strength A long piece of this tube is first drawn out before the blowpipe; and it is then calibrated with mercury until a piece is found having the same bore throughout, the necessary length is then cut from this portion *j k* is a very fine glass fibre, cemented at *j* to a piece of glass rod, and terminating at *k* with a stirrup, cut from aluminium foil, in which the glass index, *h i*, rests In front of the stirrup is a thin glass mirror, shown at *k*, silvered by LIEBIG's process, and either plane or concave as most convenient. At the ends of the glass index (*h i*) may be cemented any substance with which it is desired to experiment; for general observations I prefer to have these extremities of pith, as thin as possible, and exposing a surface of 10 millimetres square. The pith may be coated with lampblack or silver, or may retain its natural surface

103. The preparation of the suspending thread of glass requires some care. It should be drawn from flint glass, as this gives much tougher threads than foreign glass. The diameter varies with the amount of torsion required; it may be 0·001 inch or less. I select the piece best adapted for the special experiment in the following way:—Several threads of glass are first drawn out before the blowpipe, and a certain number selected as being likely to answer the purpose. These are then suspended, side by side, to a horizontal rod and equalized as to length. A piece of glass rod, about 2 inches long, which is always kept for this purpose, is then cemented by shellac on to the end of one of the threads. Air-currents are then cut off by a glass screen, and the thread being set in movement by a slight twist, the torsion is measured by timing the oscillations. This having been done with each thread in succession, one is selected and mounted in the apparatus. If it works properly, well and good; if not, it is easy to select a thread having the requisite amount of torsion, more or less, and substitute it for the one first used.

In fitting up one of these apparatus, threads were drawn out which were found to require respectively:—

	44 seconds,
	30 ,,
	28 ,,
	11 ,,
and	$3\frac{1}{2}$,,

for a half oscillation when the glass weight was hung on to their ends. The one oscillating in 30 seconds was first used, but was found to give insufficient torsion. The one making half an oscillation in 11 seconds was then used, and was found to answer well. Before I adopted this plan days were frequently wasted in the attempt to hit upon a glass thread of the requisite degree of fineness.

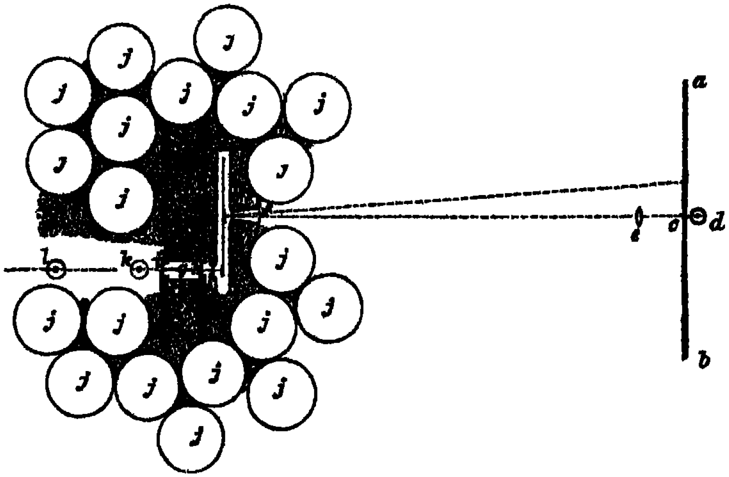
104. In taking accurate observations with an apparatus of this description, it is necessary to support it on a stand firmly fastened to a main wall. When resting on a bench, or connected in any other way to the floor, there is a constant oscillation which keeps the index from zero.

The apparatus being fastened firmly to its stand, accurately levelled, and sealed on to the pump, a divided scale, *a b* (fig. 8), is placed four feet from the small mirror; and immediately beneath the scale is a narrow brass slit, *c*, illuminated by a lamp, *d*. In front is a lens, *e*, which throws the image of the slit on to the mirror, where it is reflected back again on to the divided scale. Here the angular movement of the bright line of light shows the minutest attractive or repulsive force acting on the pith at the extremity of the movable index.

In order to keep the luminous index accurately at zero, except when experiments are being tried, extreme precautions must be taken to keep all extraneous radiation from acting on the apparatus. A slightly conical paper tube, *f*, about 6 inches long, and as narrow as the angular movement of the ray of light will admit of, is cemented on to the glass window in front of the mirror, and a similar tube, *g*, is cemented on to the

quartz window in front of the pith surface on which radiation is to act. The latter tube is furnished with card shutters, *h*, *i*, at each end, capable of easy movement up and down. The whole apparatus is then closely packed on all sides with a layer of cotton-wool, about 6 inches thick, and outside this is arranged a double row of Winchester quart bottles, *j*, *j*, filled with water and covered with brown paper, spaces being only left in front of the paper tubes. *k* and *l* represent the positions of the candle 140 and 280 millims. distant from the pith The whole arrangement has the appearance shown in fig. 8.

Fig 8



105. I will not discuss at present the phenomena presented when the apparatus is full of air, or when the vacuum is imperfect, but will proceed to the effects observed when the exhaustion has been pushed to the highest attainable degree. However much the results may vary when the vacuum is imperfect, or when the apparatus is full of air, I always find them agree amongst themselves when the residual gas is reduced to the minimum possible, and I have also ascertained that it is of no consequence what this residual gas is. Thus I have started with the apparatus filled with various vapours and gases, such as air, carbonic acid, water, iodine, hydrogen, or ammonia, and at the highest rarefaction I find no difference in the results which can be traced to the residual vapour, assuming any to be present. A hydrogen vacuum seems neither more nor less favourable to the phenomena than does a water or an iodine vacuum. If moisture be present to begin with, it is necessary to allow the vapour to be absorbed by the sulphuric acid of the pump, and to continue the exhaustion with repeated warming of the apparatus until the aqueous vapour is removed, then only do I get the best results. When pith surfaces are used at the extremities of the glass beam, they should be perfectly dry, and they are more sensitive if the apparatus has held a vacuum for some weeks, as the residual moisture in the pith will then have been absorbed by the sulphuric acid in the pump.

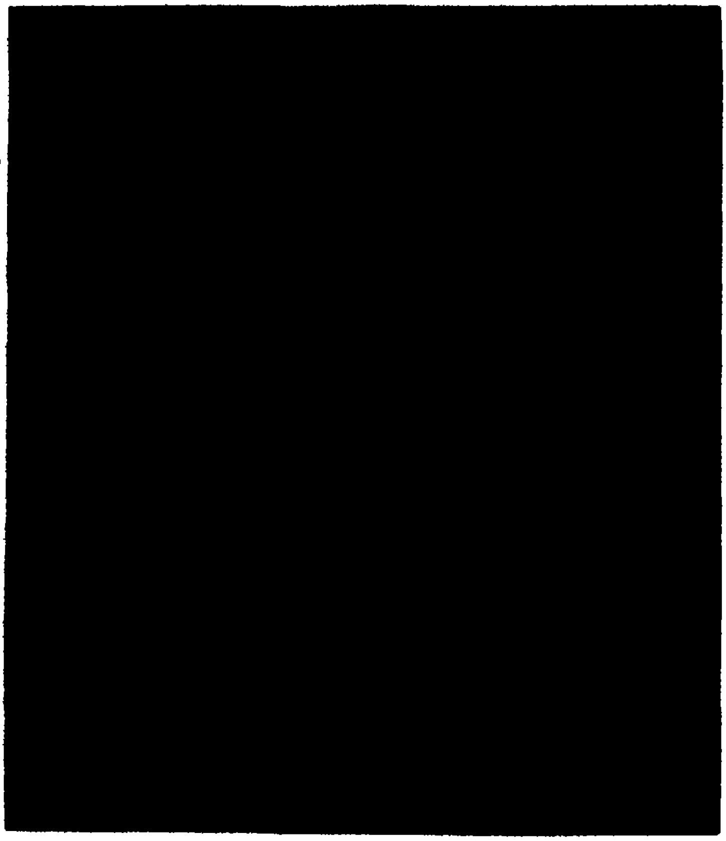
106. It was found that when a source of light and heat is suddenly allowed to shine on the pith surface and not removed, a deflection rapidly takes place, attaining its

maximum in about 11 seconds; the spot of light now returns a few degrees, and then proceeds in the first direction to a greater extent than at first. So it goes on, by alternate steps, advancing a little each oscillation, until, if the light be feeble, the index takes up a nearly fixed position; if, however, the light be strong, the beam is driven against the side of the tube. In illustration of this I select the following series of observations from a large number recorded in my note-book. The horizontal figures represent

Fig. 9.

Degrees on scale, representing repulsion.

Scale of seconds representing the duration of the experiment.



the degrees on the scale, starting from zero, where the spot of light normally rests. The vertical figures represent the seconds during which the experiment lasted. The zigzag line represents the oscillations of the spot of light, and shows the movement of the pith surface under the influence of a uniform source of radiation. The time was recorded by a chronograph. Starting from zero the spot of light is seen to have travelled to 97° in 11 5 seconds; at the end of 11 more seconds, or 22 5 seconds altogether, it had come back to 50°; at the end of 34 seconds the light had advanced again to 109°, and so on. The movements are tolerably uniform as to time, taking about 11·5 seconds for the half oscillation, but the amplitude of vibration is continually diminishing.

107. If, however, the light is only allowed to shine on the pith surface for 11·5 seconds (or for as long as the spot of light takes to perform its first half oscillation), and if it is then instantly cut off, the spot of light almost invariably returns to zero and stops there,

instead of swinging to the opposite side and only returning to rest after ten or a dozen oscillations, as is the case when the beam is set vibrating by mechanical means. This behaviour points to the return movement taking place under the influence of a force which remains active after the original radiation is cut off, and which is only gradually dissipated. This force is most probably from the heat which the pith has absorbed raising its temperature; and the steady return to zero seems to be due to the movement being controlled by the radiation of heat by the pith.

108 A series of observations taken with another apparatus, with the object of ascertaining the times of oscillation to and fro, showed that the first half, or the maximum deviation produced, whilst under the influence of radiation, occupied about the same time as the second half, or the return swing, when the source of radiation was cut off. The following are the observations. The source of radiation was a candle, the intensity of action being moderated by filtering the rays through glass screens.

Half oscillation, under influence of radiation	Whole oscillation, radiation being cut off during the return swing
8 seconds.	15 seconds.
7·5 „	15 „
7 5 „	14·5 „
7 5 „	15·5 „
7·5 „	14·5 „
7·25 „	15 „
7·5 „	15 „
7·5 „	15 „
7 „	14 „
7 „	14 „
6 75 „	14 „
7 „	14 „
7·25 „	15 „
7 „	14 „
7 „	13·25 „
8 „	16 „
8 „	16 „
7·5 „	15 „
7 „	15 „
8 „	15 „
8·5 „	15·5 „
7 5 „	15 „
8 „	15 „
8 „	15 „
7 „	14 „
Mean. .7·47 „	Mean .14·77 „

The average time of the first half oscillation is therefore 7·47 seconds*, and of the second half 7·3 seconds. This small difference is not unlikely to be due to errors of observation.

After a long series of experiments the zero gradually creeps up, showing that one side of the apparatus is becoming warmed. The conducting-power for heat and

* By referring to paragraphs 106 and 107 it will be seen that I have put the time of the first half oscillation as 11·5 seconds. This was with another apparatus, having a glass thread of different torsion.

condition of the surface (whether coated with lampblack or consisting of polished metal) of the body on which radiation falls materially influence the movements.

109. The accompanying Table gives the results of numerous experiments as to the effect of screens, tried with an exceedingly delicate apparatus, constructed as above

Interposed sceen	Magnesium wire, burnt for 7·5 seconds, distant 140 millims	Standard candle, distant 140 millims.	Standard candle, distant 280 millims	Copper ball, 400° C., distant 140 millims	Copper ball, 400° C., distant 280 millims	Copper ball, 100° C., distant 140 millims.
None	° —	° —	54°	° —	180°	9°
Rock-salt, 20 millims thick, not very clear	—	148	52	220	—	6
Rock-crystal, in two pieces, 42 millims thick altogether	—	88	32	115	—	1·5
Talc, clear but very dark, 1·25 millim thick	—	100	28	90	—	2
Plate glass, white, 2 millims thick, one piece	—	—	—	—	—	3·25
Ditto, two pieces	—	—	—	110	—	1·75
Ditto, three pieces	—	72	24	76	23	0·62
Ditto, two pieces, enclosing 8 millims water	—	—	—	0	0	0
Plate glass, of a greenish colour, 10·5 millims thick	—	55	17	—	20	0
Ditto, 20 millims thick	—	—	8	—	—	0
Alum, a clear plate, 5 millims thick	—	18·5	3	—	0	0
Plate glass, slightly greenish, 40½ millims, and clear alum plate, 8½ millims thick	30	0	0	0	0	0
Calc spar, 27 millims. thick	—	—	—	78	—	—
Very thin film of mica	—	—	off the scale	—	—	8
Ammonio-sulphate of copper, 8 millims thickness of solution, opaque to rays less refrangible than line F	72	7	—	0	0	0
Ditto, stronger solution, opaque below G	29	3	—	0	0	0

described, the window, c' (fig. 7), being of quartz. The candle used was the kind employed in gas photometry, and defined by Act of Parliament as a "sperm candle of 6 to the pound, burning at the rate of 120 grains per hour." The distances were taken from the front surface of the pith when the luminous index stood at zero. They were in the proportion of 1 to 2 (140 to 280 millims.), to enable me to see if the action would follow the law of inverse squares and be four times as great at the half distance. No such proportion can, however, be seen in the results, the radiant source possibly being too close to allow the rays to fall as if from a point. The figures given are the means of a great many fairly concordant observations. Where a dash rule is put I have tried no experiment. The cipher 0° shows that experiments were actually tried, but with no result.

The sensitiveness of my apparatus to heat-rays appears to be greater than that of any ordinary thermopile and galvanometer. Thus I can detect no current in the thermopile when obscure rays from copper at 100° C. fall on it through glass; and MELLONI gives a similar result.

110. An examination of this Table shows that the action is by no means confined to the rays usually called heat, *i.e.* to the extreme- and ultra-red of the spectrum. The strong action obtained when the light is filtered through greenish glass and alum, or through ammonio-sulphate of copper, shows that luminous rays produce a similar movement of repulsion.

Unfavourable weather has prevented me from obtaining good quantitative results with the different rays of the solar spectrum, but I have tried numerous qualitative experiments which leave no doubt on my mind that any ray, from the invisible ultra-red to the invisible ultra-violet, will produce repulsion in a vacuum. The following is an experiment tried with the electric light. The spectrum was formed with a complete quartz train, no glass whatever being in the path of the rays. The purity of the spectrum was evidenced by the fact of the lines being sharp when thallium, sodium, or lithium was put between the carbon poles The spectrum was so arranged that any desired ray could be thrown on to a lampblacked pith surface, screens being interposed to cut off the action when desired. The torsion-balance was similar to the one used in the last-named series of experiments (104), but was not quite so sensitive.

The extreme-red rays were first brought into position. On removing the screen the luminous index moved 9 divisions on the scale. The screen being replaced, the index returned to zero. A solution of iodine in disulphide of carbon was now interposed, and the screen again removed. The repulsion was almost as strong as before, showing that this liquid was transparent to the ultra-red rays.

The iodine solution was then replaced by a clear plate of alum 5 millims. thick, and the screen removed, a very slight movement only took place. The iodine solution was then put in front of the alum plate, so as to subject the extreme-red rays to a double process of sifting. No trace of action could be detected.

Whilst this double screen was in front of the pith disk, the spectrum was gradually passed along, so as to bring the rays, one after the other, into position. No effect, however, was produced, showing that alum and iodine solution practically obliterate the whole of the spectrum.

The alum plate and iodine-cell were now removed, and the green of the spectrum (the thallium line) was brought into position. The luminous index moved 6 divisions. The plate of alum cut off only a small amount of this action, but the iodine-cell brought the index to zero. This is a proof that the action in this case was not due to the heat-rays of the spectrum, for these are practically transmitted by iodine, and cut off by alum.

The indigo-rays were next brought into position. The spot of light moved three divisions on the graduated scale. Alum cut off only a very little of the action; but the iodine-cell was completely opaque to the rays, and brought the index to zero.

Finally, the invisible ultra-violet rays of the spectrum were brought into position. The train being of quartz these were abundant. Care was taken to keep any of the luminous rays away from the pith disk. I think I succeeded in this; but it was not

easy, owing to the fluorescence of the card and other surfaces on which stray rays fell. The spot of light moved two divisions, which were increased to five when the invisible rays were further concentrated by a quartz lens. The interposition of the iodine-cell cut off the whole of the action. The alum plate cut off about half of the action, but scarcely more than would have been cut off had a piece of colourless glass of the same thickness been interposed, and it must be remembered that the alum plate has glass and Canada balsam on each side

111. A similar experiment with the solar spectrum gave the following deflections, glass prisms being used:—

Ultra-red	2°
Extreme red	6
Orange	5
Green	4·5
Indigo	3·5
Ultra-violet	2

Although I give the number of divisions shown by the luminous index, I attach little importance to them as quantitative measurements. They are only single observations, and were taken before I had succeeded in getting any thing like the same sensitiveness I can now attain in the apparatus. As illustrations of the fact, however, that the more refrangible rays of the spectrum act as well as the lower rays, they may be taken as trustworthy *.

112. In my former paper on this subject I have already mentioned in detail that at a certain point of rarefaction there is neither attraction nor repulsion when radiation falls on the movable index (30, 43, 47, 66) I have long tried to ascertain the law governing the position of this neutral point. My results are not yet ready for publication, but they are shaping themselves in order, and will, I trust, lead to a true explanation of the cause of these phenomena.

The barometric position of the neutral point dividing attraction from repulsion varies according to circumstances, among these may be mentioned the density of the substance on which radiation falls, the ratio of its mass to its surface, its radiating- and conducting-power for heat, the physical condition of its surface, the kind of gas filling the apparatus, the intensity of radiation, and the temperature of the surrounding atmosphere.

When the surface exposed to radiation is pith, the neutral point is somewhat low. I have had it vary between 50 millims. and 7 millims (30) below a vacuum. It is, however, impossible to ascertain exactly, for a point of rarefaction can be obtained at which the warm fingers repel, and incandescent platinum attracts. With a heavy metal in the form of a sphere, so as to expose the smallest surface in proportion to the mass, I

* Every thing is ready to try a series of experiments with the solar spectrum, as soon as sunshine is available The results shall be communicated in a subsequent paper.

have not attained the neutral point until the exhaustion was within a very small fraction of a millimetre (43, 47); whilst if the metal is in the form of thin foil the neutral point may easily be got lower than with pith.

I am inclined to believe that the true action of radiation is repulsion at any pressure, and that the attraction observed when the rarefaction is below the neutral point is caused by some modifying circumstance connected with the surrounding gas, not necessarily of the nature of air-currents (80). As a proof of this I have not unfrequently obtained repulsion from radiation when the apparatus was full of air at the normal pressure.

113. The following experiments are too few in number, and have not been varied sufficiently as to conditions, to enable many inferences to be drawn from them. However, they afford glimpses of a law governing the position of the neutral point.

A torsion-apparatus was fitted up similar to the one described in paragraph 102. The beam was of glass, and at one extremity was fitted with a spring clip, also of glass, so that different bodies could be experimented with. Disks of platinum foil, 1 centimetre in diameter and weighing 1 28 grain each, were prepared, and they were fixed in the clip at the end of the torsion-beam, either singly or two, three, or four together, in such a manner that while the disk exposed was always 1 centim. in diameter, the weights should be in the proportion 1, 2, 3, 4. At the other end of the beam a movable counterpoise was arranged, so that the length of beam from the platinum disk to the centre was always the same.

The neutral points were as follows:—

No of disks	Barometer	Gauge	Diff = Neutral point	Differences.
1.	760	682	78	
				8
2.	760	690	70	
				16
3	760	706	54	
				24
4.	760	730	30	

114. Two pieces of platinum, *a* and *b*, were now cut from the same sheet, each having 1 square centim. of surface. *a* was left the full size, but *b* was carefully folded in four, so as to expose a surface of only a $\frac{1}{4}$ of a square centimetre, the weight remaining the same. The neutral points were then taken. The average of several observations (which, however, were not quite so concordant as could have been wished) were, below a vacuum,

a.	*b.*
136 millims.	70 millims.

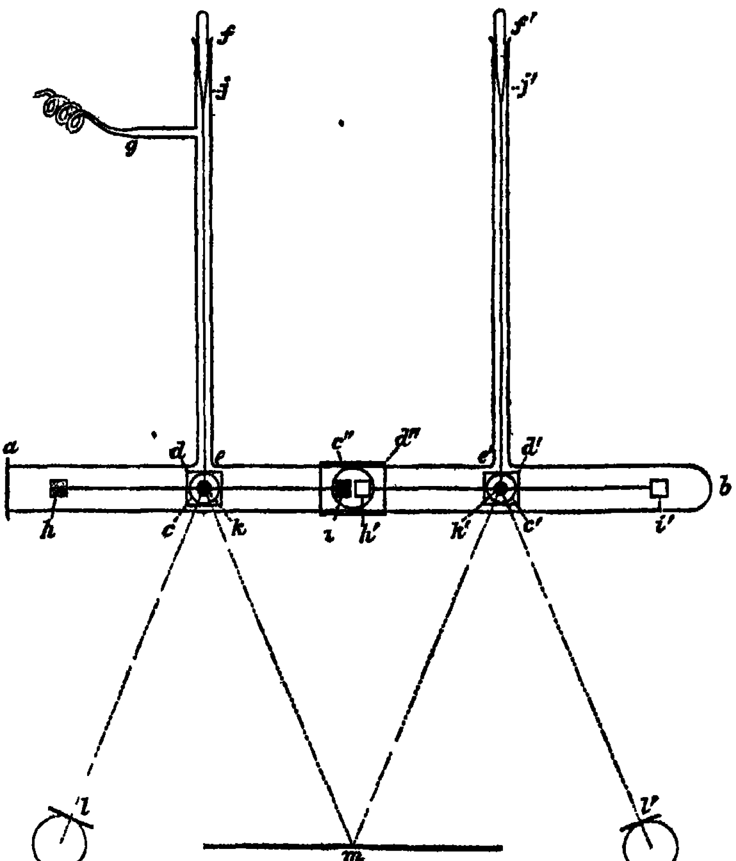
The pieces of foil were then coated with lampblack, and observations again taken. This time the neutral points came out—

a.	*b.*
66 millims.	124 millims.

An intimate connexion is thus shown to exist between the absorbing (and radiating) power of the surface on which radiation falls and the atmospheric tension at which the movement is reduced to a minimum. Further experiments on this subject are in progress.

115. It has already been said that when radiation falls on a thin surface of pith, the neutral point is low, whilst with a moderately thick piece of platinum it is generally high. I have constructed a double torsion-apparatus by means of which these actions can be easily studied. Fig. 10 shows the arrangement of apparatus. It consists of a torsion-

Fig 10.



apparatus constructed as the one shown in fig. 7 (102), with the exception of the arms being double. Similar parts in each drawing are shown by similar letters. *a b* is a piece of thin glass tubing, sealed off at the end *b*, and ground perfectly flat at the end *a*. In the centre a circular hole (c'') is blown, and two others are blown at the parts *c* and c', the edges of these holes are also ground perfectly flat. *a*, *c*, c', and c'' can therefore be sealed up by cementing flat transparent plates of glass, quartz, rock-salt, &c., *a*, *d*, d', d'' on to them. At right angles to *a b*, and at the parts *e*, e', upright

tubes, $f\ e, f'\ e'$, are sealed, one of them having an arm (g) blown into it for the purpose of attaching the apparatus to the pump. h, i, h', i' are glass beams made as light as possible consistent with the necessary stiffness. $j\ k$, $j'\ k'$ are glass fibres (103) cemented at j, j' to pieces of glass rod, and terminating at k, k' with a stirrup cut from aluminium foil, in which the glass beams h, i, h', i' rest. In front of these stirrups are thin glass mirrors (k, k'). At the ends of the beam (h, i) are cemented very thin pieces of blackened pith, each 1 centim. square; and at the ends of the other beam (h', i') are cemented pieces of platinum foil, also 1 centimetre square. At l and l' are narrow slits, with lamps behind them, so arranged that they send their rays of light respectively on to the mirrors (k, k'), whence they are reflected back to the divided scale m. When the torsion-beams are not acted on by any force, the rays of light both meet at zero (m), and there overlap, the slightest movement of either beam causing them to separate.

When the apparatus is full of air, a beam of radiation sufficiently wide to cover the whole window (c'') being thrown upon the plates i, h', the latter are instantly attracted, as shown by the movement of the reflected rays of light ($k\ m$, $k'\ m$). On exhausting the tube, and trying the effect of a hot body at the central window from time to time, it is seen that the movement of the pith surface (i) gradually diminishes, until at a certain point of exhaustion (in this apparatus at about 50 millims below a vacuum) the neutral point for pith is obtained. On increasing the rarefaction the pith is repelled by radiation, the platinum continuing to be attracted. On exhausting the air still further (to about 28 millims) the neutral point for the platinum surface is obtained, higher rarefactions producing repulsion of each when radiation falls on the pith and platinum surfaces (i, h')

At a rarefaction intermediate between the neutral point for pith (50 millims.) and the neutral point for platinum (28 millims.), the curious effect is produced of the same beam of radiation thrown into the window (c'') producing repulsion of the pith and attraction of the platinum, the two rays of light ($k\ m$, $k'\ m$) each moving to the right, and, if a piece of ice is presented to the central window, to the left. By adjusting the internal tension of the apparatus, a point may be reached (about 40 millims. below a vacuum) at which the repulsion of pith and the attraction of platinum are exactly equal, and then the two rays meeting at m do not separate, but together move to the right or left

116. A series of experiments have been tried with a view to ascertain what influence the state of surface of the substance submitted to radiation has on the amount or the direction of its movement. A torsion-apparatus was prepared similar to the one shown in fig. 7 (102), and having a thin disk of ivory at each end. One was coated with lamp-black, whilst the other retained its white polished surface. The average of a number of experiments showed that, under the influence of the same source of radiation acting for the same time (15 seconds), the white ivory was repelled so as to send the luminous index 41·5 divisions of the scale, whilst the blackened ivory caused the index to

move 46·8 divisions. These experiments were, however, tried in 1873*, when I had not succeeded in getting any thing like the delicacy I now obtain in the apparatus; and I propose to repeat them under varied conditions before employing the results to found any arguments upon.

117. In my former paper on this subject (74, 75, 76, 77, 78) I have discussed various explanations which may be given of attraction and repulsion resulting from radiation; and in a lecture delivered before the Physical Society† I entered more fully into the same arguments. The most obvious explanation is that the movements are due to the currents formed in the residual gas, which, theoretically, must be present to some extent even in those vacua which are most nearly absolute.

Another possible explanation is that the movements are due to electricity developed on the moving body, or on the glass apparatus, by the incident radiation.

A third explanation has been put forward by Professor OSBORNE REYNOLDS, in a paper which was read before the Royal Society on June 18th, 1874. Referring to the results of my experiments, Professor REYNOLDS says that it is the object of his paper to prove that these effects are the result of evaporation and condensation. In my exhausted tubes he assumes the presence of aqueous vapour, and then argues as follows:—"When the radiated heat from the lamp falls on the pith, its temperature will rise, and any moisture on it will begin to evaporate and to drive the pith from the lamp. The evaporation will be greatest on that ball which is nearest to the lamp; therefore this ball will be driven away until the force on the other becomes equal, after which the balls will come to rest, unless momentum carries them further. On the other hand, when a piece of ice is brought near, the temperature of the pith will be reduced, and it will condense the vapour and be drawn towards the ice."

It is not my intention to recapitulate the arguments I have already brought forward against these three explanations They are all fully given in my above-quoted lecture before the Physical Society. I shall, however, adduce a few experiments which have been devised specially with the view of putting one or other of these theories to the test In giving what I conceive to be reasonable arguments against the explanations which have already been proposed, I do not, however, wish to insist upon any theory of my own to take their place. *Any* theory will account for *some* facts; but only the true explanation will satisfy *all* the conditions of the problem, and this cannot be said of either of the theories I have already discussed.

118 The pendulum-apparatus, described and figured in paragraph 99, was specially devised to bear upon the air-current and the electrical theory. On referring to the description of the experiments tried with it (Tables I. & II), it is seen that in air the ignited spiral produced attraction, whilst in a vacuum the same source of radiation gave

* The torsion-apparatus with ivory terminals was exhibited in action at the Meeting of the Royal Society, Dec 11th, 1873

† June 20, 1874 (Phil Mag, August 1874).

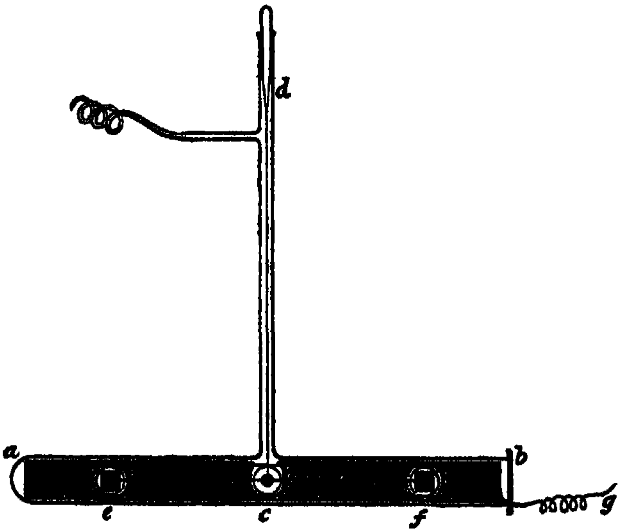
strong repulsion. Now the effect of raising a platinum spiral to whiteness in the air would be to rarefy the air all round, and the suddenness of its ignition would cause the air to be driven from it, as a centre, on all sides. Hence I was prepared to find that the pendulum would be mechanically blown on one side by what might be likened to a miniature explosion of heated gas But the action was always one of attraction, whilst, when there was no air at all present to be expanded and driven away by the hot platinum, the action was one of violent repulsion. A possible explanation of the attraction in air in this experiment may be given by assuming that the pendulum was driven inwards by the rush of cold air supplying the place of that rising upwards from the hot spiral; but it is not likely that this action should so completely overcome the effect of expansive action, and, moreover, it will only account for half the phenomenon (that in air), and leaves the still stronger action in a vacuum entirely unexplained.

119 I have tried special experiments to put the air-current theory to a decisive test. Bulb-tubes (84) and torsion-apparatus (102) have been prepared, containing terminals of metal, ivory, glass, mica, or pith, in the form of thin flat surfaces. These surfaces have been placed at an angle with the plane passing through the index and suspending-thread, in such a manner that the action of heat would be to cause currents and drive them round like the vane of a windmill. I, however, found the action of heat *in vacuo* to be repulsion, and in air to be attraction; and the latter was even strong enough to overcome the action of the air-currents, which could not fail to be developed under the circumstances of the experiment.

120. The pendulum-apparatus has also been used to show that electricity is not the cause of the attraction and repulsion On referring to the description (99), it is seen that the mass of magnesium forming the weight was in metallic contact with the platinum wire which supported it, and that the upper end of this platinum wire was fused into the glass tube, and passed thence to the outside With this I have tried numerous experiments bearing on the action of electricity I have connected the projecting end of the platinum wire with "earth," with either pole of an induction-coil (the other pole being more or less insulated), with either pole of a voltaic battery, and with a delicate electroscope, I have charged it with an electrophorus, and have submitted it to the most varied electrical conditions; and still, on allowing radiation to fall on the suspended mass, I invariably obtain attraction in air and repulsion in a vacuum The heat has been applied from the outside, so as to pass through the glass, and also inside by means of the ignited spiral; and the results show no difference in kind, but only in degree, under electrical excitement. I often obtain troublesome electrical interference with the usual phenomena, but never of such a character as would lead me to imagine that the normal results were due to electricity. I also obtain the normal actions whether the apparatus has been standing insulated in the air, or whether it has been completely immersed in water connected electrically with "earth" or surrounded with wet blotting-paper.

121. The following experiment was suggested by Mr. CROMWELL F. VARLEY, F.R.S., who informs me that he considers the results conclusive against the electrical theory. A torsion-apparatus was prepared, as shown in fig 11. The inside of the tube (a b) is lined with a cylinder of copper gauze, having holes cut in the centre (c) for the passage of the supporting-thread (d c) and the index ray of light, and holes at each end to admit of the plates (e, f) being experimented with A hole drilled in the plate (b) allows a wire to pass from the copper gauze to the outside, so as to give me electrical access to the gauze lining. Under the most diverse electrical conditions, whether insulated or connected with "earth," this apparatus behaves normally when heated, neither can I detect any electricity when the plate e or f is under the influence of radiation if I connect the wire g with a delicate electroscope. In experimenting with this apparatus I have also completely immersed it in liquids, such as water, solutions of metallic salts, ether, disulphide of carbon, &c. The heat has been applied in these cases by introducing a glass bulb containing water at different temperatures and a thermometer (28). Under all these varied circumstances the movements took place in the regular manner, and no electrical action whatever could be detected.

Fig 11.



122. I have already discussed Professor OSBORNE REYNOLDS'S theory of evaporation and condensation somewhat fully in the already quoted Physical Society paper*. I will, however, describe the following experiments, which I think prove that Professor REYNOLDS has not suggested a theory which accounts for all the facts of the case, and therefore has not hit upon the true explanation.

A thick and strong bulb was blown at the end of a piece of very difficultly fusible green glass, specially made for steam-boiler gauges. In it was supported a thin bar of aluminium at the end of a long platinum wire. The upper end of the wire was passed through the top of the tube and well sealed in, for electrical purposes (120). The apparatus was sealed by fusion to the Sprengel pump, and exhaustion was kept going on for two days, until an induction-spark refused to pass across the vacuum. During this time the bulb and its contents were several times raised to a dull red heat. At the end of two days' exhaustion the tube was found to behave in the same manner

* *Loc. cit.*; also Chemical News, July 17, 1874.

as, but in a stronger degree than, it would in a less perfectly exhausted apparatus, viz. it was repelled by light and heat of low intensity and attracted by cold.

A similar experiment was next tried, only water was placed in the bulb before exhaustion. The water was then boiled away *in vacuo*, and the exhaustion continued, with frequent heating of the apparatus to dull redness, for about forty-eight hours. At the end of this time the bar of aluminium was found to behave exactly the same as the one in the former experiment, being repelled by radiation.

Similar experiments, attended with similar results, were tried with a platinum and with a glass index, and instead of water, iodine has been put into the bulb to begin with.

It is impossible to conceive that in these experiments sufficient condensable gas or vapour was present to produce the effects Professor OSBORNE REYNOLDS ascribes to it After the repeated heating to redness at the highest attainable exhaustion, it is difficult to imagine that sufficient vapour or gas should condense on the movable index to be instantly driven off by a ray of light or even the warmth of the finger with recoil enough to drive backwards a heavy piece of metal

123 It seems to me that a strong argument against Professor REYNOLDS's theory (and also against the electrical and air-current theories) may be drawn from the fact that the repulsion in a vacuum is not confined to those red and ultra-red rays of the spectrum which mainly produce dilatation of mercury in a thermometer, excite an electrical current between antimony and bismuth couples, and cause a sensation of warmth when falling on the skin, but that any ray from the ultra-red to the ultra-violet will produce a similar effect. It cannot be reasonably argued that a ray of light, filtered through plates of glass and alum (109), can instantly vaporize a film of moisture or condensable gas from a surface on which it is caused to shine, or that it can produce air-currents in the almost perfect vacuum surrounding the surface shone upon, or that it will produce electrical excitement on such a surface

124. Facts tested and verified by numerous experiments, but scarcely more than touched upon in the present paper, are, I think, gradually shaping themselves in order, in my mind, and will, I hope, aid me in evolving a theory which will account for all the phenomena. But I wish to avoid giving any theory on the subject until I have accumulated a sufficient number of these facts. The facts will then tell their own tale, the conditions under which they invariably occur will give the laws, and the theory will follow without much difficulty. In the eloquent words of Sir HUMPHRY DAVY, "When I consider the variety of theories which may be formed on the slender foundation of one or two facts, I am convinced that it is the business of the true philosopher to avoid them altogether. It is more laborious to accumulate facts than to reason concerning them; but one good experiment is of more value than the ingenuity of a brain like NEWTON'S."

XIX. *On the Structure and Development of* Myriothela.

By Professor ALLMAN, *M.D*, *LL.D*, *F R S*, *President of the Linnean Society*

Received February 5,—Read February 11, 1875.

GENERAL DESCRIPTION.

MYRIOTHELA, of which we have as yet no satisfactory evidence of more than a single species being known, consists of a solitary attached hydranth, carrying near its proximal or attached end the blastostyles or appendages which give origin and support to the gonophores (Plate 55).

Full-sized specimens (fig 1) measure, when extended, nearly 2 inches in length. They are then cylindrical in form, with the mouth occupying the summit of a short conical hypostome, behind which the tentacles commence, and thence extend over somewhat more than one half the entire length of the body; while the proximal end of the body is bent at right angles to the rest, is invested with a chitinous perisarc, and gives origin to short sucker-like processes of attachment.

For some time after the animal has been removed from the sea and placed in the confinement of our jars, the tentacles will present the appearance of short papilliform processes (fig. 3) This condition, however, is that only of the tentacles in a state of contraction. When fully extended their form is very different, for they then attain a length of nearly half a line, and present a thin cylindrical stem, terminated by a large spherical capitulum, very well defined and distinct from the stem (fig 2). In this state the tentacles are kept in constant motion, the animal perpetually depressing them and elevating them with a peculiar jerking action.

The tentacles are very numerous; upwards of two hundred may be counted in a single hydranth. For the greater part of their extent they are set close to one another; but as they approach their proximal limit they not only become smaller, but are separated from one another by greater intervals. Almost every tentacle carries a small purple pigment spot on its summit.

The hydranth when contracted (fig. 3) becomes club-shaped or conical, and the tentacles then pass into the state of short, thick, imbricated papillæ

The contractility of the hydranth exists chiefly in the tentácular portion. In all that portion which carries the gonosome the contractility is much less marked In the tentacular region the contractility is shown not only in the great extent to which this part of the hydranth can elongate and shorten itself, but in the loops and contortions, like the writhings of a worm, into which, when fully extended, it frequently throws itself (fig. 11).

The blastostyles (a, a, a, a) spring from that portion of the hydranth which lies immediately below the tentacles. They form a dense group, surrounding the body on all sides. They are usually somewhat clavate, or of an elongated fusiform shape, but are very contractile and vary much in form. Towards their free extremity they carry several small scattered capitate tentacles; and at the proximal side of these are the gonophores (b, b, b), which belong to the type of simple sporosacs, and are large, of a globular form, and carried on very short peduncles, which spring without any regular arrangement from the sides of the blastostyle.

From the same part of the body there also spring numerous very extensile filiform organs resembling tentacles (c, c, c). These arise for the most part close to the base of a blastostyle, where they occur mostly in pairs, though sometimes singly. They terminate distally in a truncated sucker-like extremity. It will be afterwards seen that these organs, which have been hitherto entirely overlooked, perform an important function in the economy of the animal I shall designate them by the name of "claspers"

The section of the body from which the blastostyles and claspers spring is usually somewhat swollen, and is marked by close longitudinal shallow furrows After continuing naked for some distance beyond the proximal limit of the gonosome, the body bends at right angles to itself, becomes clothed with a chitinous perisarc (d), and fixes itself by the extremities of short truncated processes (e) to some solid support.

The general colour of the animal is a pale straw-colour. The tentacles are almost all tipped with a brownish-purple spot, the same colour sometimes extending over the greater part of the tentacle, and generally also spreading in clouds and streaks over the tentacula-bearing portion of the body. The gonophores are of a dull white, with their distal poles encircled by a ring of purple pigment dots

The genus *Myriothela* was instituted by SARS for an animal which he obtained off the coast of Norway, and described under the name of *Myriothela arctica* *. He has given an accurate, if not altogether adequate description of its external characters, and has correctly referred it to the Hydroida. Mr. W. STIMPSON, however, has pointed out † that the *Myriothela arctica* of SARS is identical with an animal which FABRICIUS, in his 'Fauna Grœnlandica,' has described under the name of *Lucernaria phrygia*, and for which DE BLAINVILLE afterwards constituted a new genus, to which he assigned the name of *Candelabrum*. DE BLAINVILLE, however, though he could have no difficulty in seeing that FABRICIUS's animal was not a *Lucernaria*, had notions of its affinities even less exact than those of the celebrated author of the 'Fauna Grœnlandica.' He could see no relations between it and the Cœlenterata, and asserts that its affinities are with *Sipunculus*.

If the laws of priority were rigidly enforced, SARS's name must yield to that proposed by DE BLAINVILLE; but as it is plain that DE BLAINVILLE knew nothing of the animal

* SARS, Zoolog. Reise i Lofoten og Finmarken, 1849.

† See AGASSIZ, Cont. Nat. Hist. U. S. vol. iv. p. 341, note.

and was totally mistaken as to its affinities, while SARS, evidently unaware that the animal had been previously noticed, had an accurate conception of its true zoological relations, the name of *Myriothela* may fairly be accepted without any violation of the spirit which ought to regulate biological nomenclature. And though no less an authority than Prof. LOUIS AGASSIZ has felt himself compelled to restore DE BLAINVILLE's name, I believe that further confusion will be avoided, and no injustice done, by adopting the later designation of the genus.

It is quite possible that the existing accounts of *Myriothela* include more than one species. At present, however, we have no evidence which would satisfy us in asserting that more than a single species has been observed; and the specific name assigned by FABRICIUS to the first known example of the genus must accordingly be accepted.

SARS's description is entirely confined to the external characters of the adult; and the first account which takes us beyond these is given by Mr. COCKS*, who describes the young locomotive stage which he saw developed from specimens obtained on the coast of Cornwall. Mr COCKS's observation has been confirmed by Mr ALDER, who, however, has left us no published account. Mr. HINCKS, from an observation of living specimens, has given us an excellent description of the external characters of the adult, and has correctly pointed out the true composition of the colony, maintaining the zooidal significance of the appendages which support the gonophores†.

The only other notices we possess are a short one by Mr. VIGORS‡, who, not aware of the previous descriptions by FABRICIUS and SARS, records the animal under the new generic and specific names of *Arum Cocksii*, and one by Mr. GOSSE §, who also describes it as a new genus and species, under the name of *Spadix purpurea*

The only published figures are one accompanying Mr. COCKS's description of the locomotive stage, a small woodcut outline by Mr. GOSSE, and a characteristic figure by Mr. HINCKS

The specimens which have afforded the material for the present memoir were obtained at Lulworth, on the coast of Dorsetshire. They were attached to the under surface of large stones, close to the low-water level of spring-tides.

ANATOMY.

THE TROPHOSOME.—STRUCTURE OF HYDRANTH.

1. *Endoderm.*

The character of the endoderm varies according to the region in which it is examined. Throughout the whole of the main cavity of the body it constitutes a thick layer, composed of many cells in depth (Plate 56. figs. 1 & 2, *a*). The cells which form the greater part of this endodermal layer consist of simple round masses of clear protoplasm, about $\frac{1}{500}$ of an inch in diameter, in which a nucleus is frequently visible, and in which are immersed

* Rep of Roy. Pol. Soc. Cornwall, 1853, p. 34.

† Brit. Zooph. 1868, p. 75.

‡ Rep. Roy. Pol. Soc. Cornwall, 1849.

§ Ann. Nat. Hist. 1853, and Man. of Marine Zoology, 1855.

numerous refringent corpuscles and a few brown granules. No boundary membrane was evident in any of these cell-bodies. At the inner or free surface the endoderm of the whole of the gastric cavity, except in the region immediately below the mouth, forms long conical processes, which project like villi into the cavity (figs. 1 & 2, *b*). These processes, like the more external parts of the endoderm, are mainly composed of large cells, formed of clear protoplasm, with nucleus and refringent corpuscles; but besides these there exist also towards the free ends of the processes numerous smaller spherical cells (fig. 2, *c*), loaded with dark-brown granules. These cells are most abundant in the villi-like processes which are developed towards the proximal end of the body. They form a much less coherent tissue than the large clearer cells, and may be easily isolated under the microscope. Indeed they are constantly being thrown off, and may be often seen to be voided through the mouth of the living animal.

Extending over the free surface of the endoderm is an exceedingly thin stratum of a clear homogeneous protoplasm (fig 2, *d*) This protoplasmic stratum is most obvious the villi-like processes, where it has the property of developing very minute, irregular, pseudopodial projections (*e e e*), which are constantly changing their shape, and may be seen under the microscope to be slowly protruded and withdrawn. The free surface of the endoderm carries also long, very slender vibratile cilia I believe that the thin layer of protoplasm which extends over the free surface of the endoderm is continuous with an interstitial undifferentiated protoplasm which exists in small quantity between the endodermal cells. Its occurrence, with its pseudopodial extensions, on the gastric surface of the animal is full of interest, and suggests a close analogy between the absorptive action of the gastric surface and amœboid reception of nutriment; more especially when we bear in mind that the cells between and over which the semifluid protoplasm is spread are destitute of membrane, and that their protoplasm must be in direct relation with that of the pseudopodial stratum.

The cilia are extremely fine and difficult of detection. They do not appear to be continuous over the whole gastric surface, but to exist only at intervals. They probably originate directly from the proper surface of the endodermal cells, in which case they must traverse the pseudopodial layer. They may, however, be direct processes of this layer. Indeed it is difficult in either case not to regard them as modified pseudopodia. True vibratile cilia, like pseudopodia, can originate only from the surface of membraneless protoplasm, which thus possesses, as one of its characteristic properties, the faculty of being able to develop two kinds of processes—the non-mutable vibratile cilium and the mutable pseudopodium

From the gastric cavity the endoderm is continued in an altered form into the cavity of the tentacles (fig. 2, *b*, & 3). Here its condition differs strikingly from that of the tentacular endoderm of other marine hydroid trophosomes; for instead of forming the clear septate core which is so very characteristic of these, it consists of a single layer of small round cells surrounding an open axile cavity, and so loaded with opaque granules that the axis of the extended tentacle appears nearly white under reflected light.

2. *Ectoderm.*

Under this head I shall include, not only the proper cellular ectoderm, but the hyaline lamella which forms its internal boundary, and is composed of a fibrillated or muscular stratum, with a supporting structureless membrane.

The proper *cellular ectoderm* (Plate 56. fig. 1, *c*, & fig. 2, *g*, *h*) forms a much thinner zone than the endoderm It is composed of two distinct strata—a superficial and a deep The superficial stratum (fig. 2, *g*) consists of small round cells, several in depth. These are destitute of membrane, and contain abundance of yellowish corpuscles; while on the summit of the tentacles (fig 3), and in irregular patches on other parts of the body, they contain dark brownish-purple pigment granules.

Lying irregularly among these ectodermal cells, and chiefly towards the free surface of the ectoderm, are the thread-cells (figs. 2 & 3) Two forms of thread-cells may be distinguished,—one oviform (fig. 4, *a*, *a'*), with the invaginated sheath occupying the axis; the other fusiform (fig. 4, *b*, *b'*), with a slightly curved axis, and having the invaginated sheath oblique. Both kinds of thread-cells are formed in the interior of certain cells belonging to the superficial layer of the ectoderm, and may be seen, some lying free among the true cells of this layer, others enclosed in their generating-cells, and either completely immersed in the granular matter of the cell or surrounded by a large clear vacuole (fig 5). No facts, however, have come to my knowledge tending to throw further light on the mode of origin of the thread-cells

The deep layer of the cellular ectoderm (fig 2, *h*) is formed by a very remarkable tissue, to which I shall refer under the designation of the *claviform tissue*. This is composed of cells consisting of a yellowish granular protoplasm, entirely destitute of membrane, and each drawn out into a long caudal process. They are frequently provided with an obvious nucleus By the union of their caudal processes groups of claviform cells (fig 6, *a*) are produced whose common stalk runs to the hyaline lamella, where it loses itself in the fibrillated stratum (*b*). The whole forms a very soft, pulpy, and somewhat glandular-looking tissue, easily broken down under the compressor.

Caudate cells, of apparently the same significance, were first made known by KLEINENBERG*, who discovered them in *Hydra*, where he believes that he has followed their caudal prolongations into direct continuity with the fibrillæ of the muscular lamella. He regards the body of the cell as destined for the reception of stimulus from without, and, looking upon the whole cell with its fibrilliform continuation as representing a combined nervous and muscular system, he gives it the name of "neuro-muscle-cell" According to this view *Hydra* would represent in the phylogenesis of animals a form in which the nervous and muscular tissues are as yet but imperfectly differentiated from one another.

I believe that we are quite justified, with our present data, in attributing to the claviform tissue the general function of a nervous system. Indeed I do not see what other place it is possible to assign to it in the economy of the animal. In *Myriothela*, however,

* *Hydra*, eine anatomisch-entwickelungsgeschichtliche Untersuchung. Leipzig, 1872.

I have never succeeded in tracing a direct continuity of the caudal processes of the cells with the fibrillæ of the muscular lamella There is no doubt that the stalks of the claviform tissue pass into the muscular layer and become intimately associated with it; but I do not believe that any more direct continuity with the individual fibrillæ can be here demonstrated.

KLEINENBERG has further described the bodies of the caudate cells in *Hydra* as united laterally with one another, and forming the outer surface of the body, while the spaces which must necessarily lie between their caudal prolongations are occupied by a tissue composed of small non-caudate cells, to which he gives the name of "interstitial tissue," and in which he maintains that the thread-cells and the generative elements are formed.

I can find nothing like this interstitial tissue in *Myriothela*, and I believe that its place is here taken by an undifferentiated protoplasm, through which the prolongations of the caudate cell-clusters make their way to the muscular layer.

If we except the case of the long transitory arms of the actinula or free locomotive stage, which will be afterwards described, the claviform tissue does not in *Myriothela* come to the surface of the body. Throughout the whole of the body of the adult it forms a deep zone, intervening between the hyaline lamella and the superficial layer of the ectoderm, and very distinct in sections made from specimens hardened in chromic acid.

The *hyaline lamella* (fig 2, *i*) forms the internal boundary of the ectoderm, and is found everywhere between the endoderm and the cellular ectoderm. It consists of two layers,—internally (fig. 6, *c*) a perfectly transparent, thin, structureless membrane, and externally (*b*) a layer of fibrillæ, which adheres closely to the structureless membrane.

Special attention was first called to the presence of the structureless membrane in other hydroids by REICHERT*, who named it "Stutzlamelle;" but he refused to admit the existence of a true fibrillated layer. The fibrillated layer, however, is extremely distinct in almost all hydroids. In *Myriothela* it can be separated, after a short maceration in water, from the underlying structureless membrane. It is here composed of longitudinal fibrillæ, which adhere to one another by their sides in a stratum of a single fibre in thickness, which forms a continuous lamella, even after detachment from the supporting structureless membrane. The fibrillæ are about $\frac{1}{15000}$ of an inch in diameter, soft, and compressible, very transparent, with a very minutely granular structure, but otherwise apparently homogeneous. They show a convex surface when seen in profile on the folded edge of the lamella. That they are contractile elements, forming by their union a muscular lamella, there would seem to be little reason to doubt. They do not, however, possess the character of true muscle-cells. So far as I was able to trace them, they retain a uniform diameter, and show no appearance of nuclei.

As already said, I have failed to find any direct continuity between the fibrillæ and

* Ueber die contractile Substanz &c. Berlin, 1867.

the caudal prolongations of the claviform tissue. These prolongations run to the surface of the muscular lamella, and become there intimately united with it, so that it is perhaps impossible to detach them without laceration; but I cannot affirm any thing further regarding the nature of this union. But though *Myriothela* does not seem to afford any evidence of the direct continuation of the muscular fibrillæ with the caudal prolongations of the claviform tissue, it cannot be regarded as in any way contradicting the hypothesis that this tissue is destined for the reception of external stimulus—in other words, that it represents a nervous system

The general structure of the ectoderm of the *Myriothela* hydranth is that which has been now described, in the globular capitula of the tentacles, however, we have a most singular modification of those structures which lie external to the hyaline lamèlla Here the place of the caudate cells is taken by a remarkable tissue, composed of closely appressed transparent prisms, or, to speak more exactly, of greatly elongated pyramids (fig. 3, *a*, & fig. 7), which are attached by their inner or apical ends to the hyaline lamella of the capitulum to which they are perpendicular, and thence radiating outwards terminate at some distance from the outer boundary of the capitulum in a curved surface, which occupies somewhat more than a hemisphere. The distal or basal extremity of each pyramid is formed by a curve of greater convexity than that of the general surface formed by their combined bases, and this surface thus acquires a minutely papillose appearance The whole organ thus constituted caps the hyaline membrane and endoderm of the summit of the tentacle In its structure it strongly suggests the rod-like tissue which in higher animals we know to be associated with special organs of sense.

Radiating from its convex surface are a multitude of slender filaments, which make their way among the cells of the ectoderm, and terminate distally at a short distance within the outer surface of the capitulum, where each carries on its summit an oviform, transparent, very thin membranous sac (fig. 3, *b*, *b* & fig 8) This sac bears, close to its distal end, a minute bristle-like process, and is completely filled by a firm refringent capsule, within which may be seen a transparent cylindrical cord wound in two or three coils. The capsule (fig 9) is easily liberated from its enveloping sac, and under slight pressure the contained cord may sometimes be ejected through its distal end (fig. 10). The whole assemblage of sacs, with their included capsules, forms a zone parallel to the surface of the capitulum and a little within it (fig. 3).

The close resemblance of the capsule, with its contained cord, to a thread-cell is abundantly obvious; and even the external sac, with its bristle-like process, has its parallel in the generating-cell of certain thread-cells. But besides the presence of the filiform peduncle there are other points in which these remarkable bodies differ from true thread-cells. The included cord does not, like the contents of an ordinary thread-cell, consist of a wider portion continuous with a narrower one, which during ejection becomes invaginated in the wider, but, on the contrary, possesses a uniform diameter considerably greater than that of the filament of an ordinary thread-cell; and instead of

presenting a vast multitude of coils rolled together into a complicated mass, as in the latter, it has only two or three such coils. Further, when ejected from the capsule (while it still holds on by one end to the point of exit) it does not, like the filament of a thread-cell, straighten itself and shoot across the field of the microscope, but immediately on becoming free coils itself again into a spiral (fig 10). Indeed I believe that the significance of these pedunculated capsules is something very different from any which has been hitherto assigned to the thread-cells, and it is scarcely possible not to recognize a special apparatus of sense in the whole structure just described, including the rod-like tissue in which the peduncles of the sacs have their roots, and which is plainly but a modification of the structure which forms the claviform or nervous tissue in other parts of the body. Indeed it is impossible to overlook the striking resemblance between these pedunculated sacs, with their enclosed capsule and cord, and the Pacinian bodies of the Vertebrata. If this be a correct view of the nature of the structures here described, we have now for the first time evidence which would justify us in assigning a special apparatus of sense to a hydroid trophosome.

But with all this the resemblance between these pedunculated capsules and true thread-cells cannot be ignored, and indeed makes us hesitate, even more than we may have hitherto done, in regarding the latter merely as urticating organs. It is possible that the pedunculated capsules may throw new light on the function and significance of thread-cells, but with no facts beyond those at present before us, we are scarcely in a position to speculate further on this subject.

The best display of the capsules, with their investing sacs and peduncles, was obtained from specimens which had been for twenty-four hours immersed in a solution of osmic acid of 0 1 per cent., and afterwards placed in a mixture of 100 parts of glycerine with 5 parts of acetic acid; while the most beautiful demonstration of the rod-like tissue was found in sections which had been simply macerated in water for twenty-four hours, and then examined, without further preparation, under the compressor. The more external tissues of the capitulum had been softened and disintegrated by the maceration, and were now easily separated by the simple action of the compressor; while the firm, rod-like tissue, offering more resistance to the decomposing action of the water, remained beautifully isolated, with its component rods looking almost like the radiating acicular crystals of certain forms of zeolite.

External to the zone of pedunculated capsules is a thin layer of ectoderm, which forms the most superficial portion of the capitulum (fig 3). This is composed of small round membraneless cells, containing refringent corpuscles, while the summit of the capitulum is almost always occupied by a group of small cells, containing dark brownish-purple pigment granules The two forms of true thread-cells already described are here developed in greater numbers than elsewhere, and may be seen scattered, without any definite order, among the more superficial cells of the ectoderm.

The Gonosome.

The gonosome of *Myriothela* (Plate 55) consists of blastostyles with their gonophores and of claspers.

The blastostyles (fig. 2, *a*, *a*, *a*) arise from the hydranth towards its proximal or attached extremity. They may be followed over a section occupying about one fifth of the entire length of the extended hydranth, and spring from this region on all sides without any very definite arrangement They are very contractile, somewhat fusiform in shape when extended, but more clavate in various states of contraction. Towards their free extremities they carry several scattered tentacles resembling those of the hydranth, but much smaller; and where the tentacles cease to be borne the gonophores (*b*, *b*, *b*) commence, and continue with an irregular scattered disposition to within a short distance of the attached end of the blastostyle

The structure of the blastostyles resembles, in all essential points, that of the hydranth, with the exception of their being entirely deprived of a mouth. Their gastric cavity communicates with that of the hydranth which bears them, the villi-like processes of the endoderm are extremely well developed, and the spherical cells, loaded with brown granules, which enter into the composition of these processes are very abundant (Plate 57. fig. 14, *a*) The muscular lamella is well developed, and the structure of the tentacles is quite the same as in the hydranth, the rod-like tissue and pedunculated capsules being similar in both*.

The claspers (Plate 55 fig 2, *c*, *c*, *c*, and Plate 57. fig. 14, *b*, *b*), as already mentioned, are long tentacle-like organs of a cylindrical form, slightly enlarged towards their distal extremity, where they terminate in a sucker-like disk. They spring, like the blastostyles, from the body of the hydranth, and mostly in pairs from two points close to the base of a blastostyle They have, however, no definite arrangement; many blastostyles have no claspers at their base, and solitary claspers occur, not only at the base of a blastostyle, but here and there at some distance from it on the body of the hydranth

The claspers are very contractile Their structure differs considerably from that of the blastostyle The endoderm (Plate 56. fig 11, *a*) is composed of an external layer of closely applied large cells with clear contents, and an internal looser layer of small round cells filled with brown granules, this internal layer surrounding a very narrow axile cavity. There are no villi-like processes. The ectoderm, except in the terminal enlargement, essentially resembles that of the blastostyles and hydranth. The muscular

* Before I had an opportunity of examining specimens of *Myriothela*, I regarded the appendages which carry the gonophores not as true members of a zooidal colony, and therefore not as proper blastostyles, but as mere peduncular organs like those which carry the gonophores in *Tubularia* (Gymnoblastic Hydroids, p. 383). In thus viewing them I differed from Mr. Hincks, who looked upon them as true zooids, having a reproductive function, and forming with the hydranth from which they spring a compound colony (Hincks, Brit. Hydroid Zoophytes, p. 76). I must now abandon my former view and declare my entire agreement with Mr. Hincks as to the true zooidal significance of these bodies.

lamella (b) is very well developed, and is succeeded externally by a zone of claviform tissue (c) overlaid by a zone composed of small round cells with nearly colourless granular contents, and lying two or three in depth (d). Among these the oviform and fusiform thread-cells are scattered in considerable abundance.

The terminal enlargement (e) of the clasper differs from its narrower portion chiefly in the great development of the claviform tissue which constitutes the principal mass of its substance. The caudal prolongations of the cells (fig. 12) composing this tissue are very long, and do not unite with one another, so as to constitute botryliform groups to the same extent as in the corresponding tissue in the ectoderm of other parts of the hydroid; they radiate from the hyaline lamella, and possess a considerable resemblance to the constituent elements of the rod-like tissue in the tentacles On the summit of the clasper, where this organ exercises a special function of adhesion, the thread-cells so well developed in other parts of the ectoderm are deficient.

The function of the claspers, as we shall see more particularly under the head of development, is that of seizing, on its escape from the gonophore, the plasma mass which is to become developed into an embryo

The gonophores (Plate 55. fig. 2, b, b, b, and Plate 57. fig. 14, c, c, d) show nothing like a medusal conformation They are simple sporosacs of a spherical form, supported on very short peduncles, which spring without any definite arrangement from the sides of the blastostyles. They show no definite order of arriving at maturity, the more mature gonophores being sometimes at the distal side of the younger ones, sometimes at their proximal side, and sometimes scattered among them. Their law of maturation is thus strikingly different from that of the gonophores of most other hydroids, in which we find either a constant centripetal or a constant centrifugal order in the periods of their first appearance and of their arrival at maturity.

Myriothela is also extremely exceptional in carrying on the same hydranth, and even on the same blastostyle, both male and female gonophores. So far, however, as my observations extend, the male gonophores are borne at the distal side of the female ones No external difference between the two can be detected beyond the fact that the mature males are much smaller than the mature females

In the walls of the mature gonophores (Plate 57. figs. 7, 10, 12), whether male or female, several distinct structures may be demonstrated. Most externally is a zone of spherical cells (fig 12, a), which for the most part contain clear colourless granules, but towards the summit of the gonophore some of these cells are filled with purplish pigment granules, and form a coloured circle surrounding the distal pole of the gonophore (fig. 14, c, c, d). Passing from without inwards, this is followed by a zone of clavate tissue (fig. 12, b), and this by the structureless lamella (c) overlaid by muscular fibrillæ These three zones are direct continuations of the corresponding elements in the ectoderm of the blastostyle.

Lying immediately within the hyaline lamella is another cellular layer (fig. 12, d). In its thickness this layer corresponds to the depth of a single cell. Most of the cells

composing it contain only clear colourless protoplasm, with some clear granules; but towards the distal pole of the gonophore the cells increase slightly in size, and contain purple granules, which form a coloured ring internal and parallel to that belonging to the outer layer (figs. 7 & 10, *a*). In the centre of this internal ring the layer now under consideration is perforated by a narrow aperture, which thus lies immediately under the distal pole of the gonophore, which is itself quite imperforate.

The last described layer encloses the mass of the generative elements (figs. 7, 10, & 12), from which, however, it is separated by a very thin structureless membrane (fig. 12, *e*), by which the whole generative mass is surrounded, and which becomes reflected over the spadix where this is plunged into the midst of the mass of ova or spermatozoa.

DEVELOPMENT.

The first appearance of the gonophore shows itself in a minute offset of the gastric cavity of the blastostyle. This pushes itself outwards into the ectoderm of the blastostyle, carrying with it the endoderm, which continues to form its immediate boundary, separated from the cellular ectoderm by the hyaline lamella; but no well-defined external projection has yet become apparent.

The endoderm (Plate 57. fig. 1, *a*), which lies over the distal end of this gastric diverticulum, soon becomes excavated by a cavity of a nearly spherical shape (*b*). This cavity, which I shall speak of as the *gonogenetic chamber*, is separated from that of the diverticulum (*c*) by a considerable thickness of the endodermal layer; but the endoderm, which bounds it distally, forms a cellular membrane of only a single cell in thickness. The cavity, which as yet appears quite closed, is filled with clear contents, in which no formed matter beyond minute granules can be detected.

In the next stage the diverticulum from the cavity of the blastostyle has increased in size, and continuing to press the endoderm and ectoderm before it, the whole has begun to form a well-defined hernial projection from the side of the blastostyle, while the floor of the gonogenetic chamber has become convex, and the chamber, which has at the same time increased in size, presents in longitudinal section a crescentic shape. A minute orifice has now become visible in the summit of the chamber; and the endodermal cells, which immediately surround the orifice, have become somewhat larger, and are seen to be filled with brown pigment granules The ectoderm continues imperforate, the orifice being entirely confined to the thin layer of endoderm which forms the immediate roof of the gonogenetic chamber.

Up to this point there is nothing by which the male and female gonophores may be distinguished from one another. We soon, however, observe a differentiation of the contents of the gonogenetic chamber In the female gonophore a layer of more consistent protoplasm has accumulated on the free surface of the walls of this chamber (fig. 2, *b*), more especially on its proximal wall or floor. Minute, clear, nucleus-like bodies may be seen scattered through the protoplasm, and a few similar bodies float free in the more liquid contents which still occupy the centre of the chamber.

Following now the female gonophore in its development, we find that in the next stage (fig. 3) both it and its included gonogenetic cavity have increased in volume, while the floor of the cavity projects further into its interior in the form of a hollow conical core. This is easily recognized as the spadix, on the free surface of the cavity of the spadix (*c*) villi-like processes similar to those which occur in the general cavity are abundantly developed. The gonogenetic cavity has now become uniformly filled with a plasmatic mass (*b*), which is seen to consist of a multitude of nuclei (fig. 5) about $\frac{1}{3500}$ of an inch in size, each enclosing a minute nucleolus, and immersed in a minutely granular protoplasm. An extremely delicate structureless hyaline membrane (fig. 3, *d*) can now be traced over the whole surface of the generative mass, which it thus separates from the proper endodermal walls of the gonogenetic chamber.

As yet no distinct cell-boundaries can be detected in the contents of the gonogenetic chamber, and the nucleolated nuclei afford the only evidence of cell-differentiation. With the enlarging gonophore, however, the protoplasm which surrounds the nuclei increases in volume, and we soon begin to discover in it manifest cell-boundaries (fig. 4). Every nucleus is now surrounded by a differentiated mass of protoplasm, and the cavity of the gonophore has thus become filled with bodies which possess all the characteristic features of true ova, each with its well-defined germinal vesicle and germinal spot and its surrounding vitelline protoplasm.

These ova-like bodies continue to increase in size with the growth of the gonophore. They remain for some time closely pressed against one another, having thus acquired a polyhedral form (fig. 6), but they gradually become looser, assume an oval shape (figs. 7 & 12, *f*), and may be easily isolated by the needle or by the mere action of the compressor. Their germinal vesicle is now very large and distinct, and within the large germinal spot a well-defined spherule or nucleolina may be easily detected. Though their subsequent history differs in some points from the characteristic development of the ovum such as is met with in other animals, we should yet be scarcely justified in denying to them the significance of true ova.

They have no sooner attained their complete independence and acquired their full size in the sporosac, than they begin to present a very remarkable phenomenon. They lose their independent existence, and begin to undergo a fusion into one another; and when the contents of the sporosac are now liberated by rupture under the microscope, many of these nucleolated protoplasm masses may be seen united to one another by irregular pseudopodia-like extensions of their substance (fig. 8). By the gradual shortening and thickening of these processes the little masses which they connect are drawn closer to one another, and end by becoming completely fused together into a common protoplasmic mass (fig. 9). In this mass the cell-boundaries are completely lost, but numerous nucleolated nuclei are scattered through its substance. These are almost certainly the nuclei with their included nucleoli of the original independent protoplasm masses or ova.

The fusion commences among the ova which lie in the immediate vicinity of the spadix, to which the masses formed by their union continue for some time to adhere by

a considerable extent of their surface (fig. 7); while those ova which lie more towards the periphery of the cavity continue longer distinct, but ultimately follow the same course as the others by coalescing into compound masses.

Several such masses (fig. 10), eight or more, will thus be formed from the coalesced ova. They detach themselves more and more from the spadix. They are now of an oval form; and some of them may still be seen to be connected with the spadix by a narrow easily ruptured protoplasmic prolongation They do not, however, entirely fill the cavity of the gonophore, and the narrow intervals between them, as well as the small space which separates them from the walls of the gonophore, is occupied by a matter which appears to consist chiefly of free nuclei and of dwindled and degraded ova, all apparently undergoing a process of liquefaction, and doubtless an unused residuum of the bodies by the coalescence of which the compound masses had been formed.

If in this stage the gonophore be laid open, and the protoplasm masses, whose formation we have been tracing, be liberated under the microscope, we shall often succeed in witnessing very minute bristle-like processes of clear protoplasm which have become developed over their surface (fig. 11). These little processes, however, are not permanent structures, and they will often become entirely withdrawn while the object is under examination. They are, in fact, true pseudopodia, and are probably employed in the nutrition of the masses from which they arise

The contents of the gonophore, however, are intended to undergo further changes before the period of their liberation has arrived. The separate protoplasm masses increase in size, the residual matter which had surrounded them disappears, having probably afforded material for their nutrition, they begin to coalesce with one another, and there is ultimately formed a single large plasmodium, which entirely fills the cavity of the gonophore. When this plasmodium is examined under the compressor, the same nucleolated nuclei which had hitherto characterized the products of the coalescence of the ova are seen to be scattered in great numbers through its substance (fig. 13). These nuclei, however, have already begun to suffer a change; for while in some the nucleolus is still distinct, in others it has quite disappeared, and while in some the contents consist of a minutely granular matter, in others they are quite homogeneous.

When the separate protoplasm masses have all united with one another, but generally a little before they have become so completely fused together as to have their original distinctness entirely lost, the time has arrived when the contents of the gonophore are to be expelled. The walls of the gonophore now begin to contract on these contents; and here the use of the muscular layer, which is well developed in them, becomes at once apparent. The contained plasmodium is thus gradually forced out through the summit of the gonophore (fig. 14, *d*).

The orifice in the endodermal wall of the gonogenetic chamber is ready to aid in giving exit to the plasmodium, but the ectoderm has been hitherto imperforate. This, however, appears to have been becoming gradually thinner on the point immediately over the endodermal orifice, and it is now easily ruptured at this spot by the pressure

from within. By the continued contraction of the gonophore-walls the plasmodium is at last entirely expelled, completely enveloped, however, in a transparent structureless membrane. This is apparently the membrane which at a very early stage had shown itself lining the gonogenetic chamber; it is at first of great tenuity, but it soon acquires considerable consistence. The empty gonophore may now be seen retracted in the form of a shallow thick-walled cup with everted edges upon the summit of its short peduncle (fig. 14, *e*)*.

The liberated plasmodium closely enveloped in its delicate structureless capsule is of a nearly spherical form, and now lies upon the retracted gonophore, where it is usually retained by the spadix plunged for a short distance into its mass (fig. 14, *f*). It does not, however, continue long in this position, for the function of the claspers is soon brought into play. These curious organs now stretch themselves out towards the liberated plasmodium; and as soon as they reach it they attach themselves (*f*) by their sucker-like extremities to its capsule, and then by contracting pull it entirely away (*g*) from the remains of the gonophore.

Sometimes the plasmodium will be seized by only one clasper; very often, however, two or even three will fasten on it (Plate 55. fig. 2); and the plasmodium will sometimes be seen more or less distorted by the tension thus exerted on it at the same time in different directions.

Leaving for a while the further history of the female elements, we may now trace the development of the male. The male gonophore resembles the female in all points except in being about half the size of the latter; and I could detect no difference as to origin between the matter which in one case is to become differentiated into ova, and that which in the other is destined for the formation of spermatozoa. In every young gonophore I have examined, the first appearance of the matter in which sexual elements are afterwards to show themselves is within the gonogenetic chamber which has become excavated in the substance of the endoderm; and it is only when the ovarian nuclei become differentiated in the one case, and the spermatic cells in the other, that we obtain any decided indication of the sex of the gonophore.

As we have already seen, the primitive plasma which fills the gonogenetic chamber in the female presents after a time scattered nuclei-like bodies, which are to become the germinal vesicles of the ova. In the male, on the other hand, such nuclei never make their appearance, and the primitive protoplasm becomes changed into minute cell-like bodies, which entirely fill the chamber (Plate 57. fig. 15) These little bodies are the vesicles within which the spermatozoa originate; but in what way the latter are produced from them I have not succeeded in discovering. After a time the vesicles have disappeared, and are replaced by mature spermatozoa, which now fill the cavity of the gonophore, and which may be liberated by rupture of the latter. When thus set free they are seen to consist of a very minute oval head, with a vibratile tail of extreme tenuity (fig. 16).

* In a single instance a gonophore with two such plasmodia ready to escape from it came under my observation.

They are more minute than the spermatozoa of any other hydroid with which I am acquainted.

By what means the spermatozoa naturally escape from the gonophore I have not been able to determine with certainty. I could find no external orifice, nor could I detect a thinning of the summit of the gonophore like that which in the female precedes the escape of its contents; and when the mature male gonophore was subjected to pressure it was always by the rupture of the spadix and the escape of the spermatozoa through the peduncle, which would thus carry them into the cavity of the blastostyle, that the gonophore became emptied. It is not improbable, as we shall afterwards see, that this is their natural mode of escape.

Returning now to the contents of the female gonophore which, just after their escape, we had left in the grasp of the claspers, we find that by this time the coalescence of the separate plasma masses into a single spherical plasmodium has been completed; and it is probable that fecundation now takes place. Hitherto we have seen nothing which can be compared to any phenomena which we would be justified in regarding as the immediate consequence of the action of the male element on the female, but soon after the liberation of the plasmodium and its seizure by the claspers, we find that the whole has become broken up into a multitude of small round or irregularly shaped masses (Plate 57. fig 17) Some of these may be seen still connected to one another by narrow isthmuses of their substance, while others are quite free, and can be isolated under the microscope They all consist of a granular protoplasmic matter without any distinct boundary membrane, and with numerous nucleus-like bodies immersed in their substance. The common external structureless membrane is distinct, but it is still thin and weak.

I must regard this breaking up of the plasmodium into separate masses as representing a true segmentation, such as in the simple ovum occurs as the immediate result of fecundation. I have not, however, succeeded in witnessing its earlier stages, and I cannot say whether it proceeds in accordance with the ordinary binary law of vitelline segmentation.

How far this breaking up of the plasmodium is continued before a true histological differentiation becomes apparent, I am unable to say, for the next stage which showed itself (Plate 58. fig. 1) presented a marked advance on the previous ones. The segmented condition had now entirely disappeared, and the developing mass had acquired a true cellular structure, while it had become further differentiated into two distinct layers—an external (*a*) layer, ectoderm, in which the cell-boundaries were with some difficulty made out, and an internal (*b*), endoderm, composed of very obvious cells larger than those of the ectoderm, and each with a clear nucleus and granular protoplasm. This internal layer formed the boundary of a cavity (*c*) produced apparently by liquefaction of the more central parts of the mass.

The developmental stage to which we have now arrived is thus represented by a hollow spherical body, whose walls are formed by two layers, an ectoderm and an endo-

derm, and which plainly corresponds to the planula of other hydroids. It is, however, entirely destitute of cilia, and is still confined within its external structureless capsule (*d*), which has now acquired considerable thickness.

We next find that the planula presents numerous minute pits distributed without any definite arrangement over its surface (fig. 2, *b*, *b*). These are points where the walls of the planula have begun to invaginate themselves, and if at this time a section be made of the planula (fig 3), its cavity will be found to be occupied by numerous hollow conical projections (*b*, *b*), which radiate into it on all sides from the inner surface of its walls. These projections are simple involutions of the walls, and are therefore composed, like the walls themselves, of an ectoderm and an endoderm, but in an inverted order.

If an uninjured planula in this stage be dissected out of its external structureless capsule, which now lies loosely over it, and be subjected to carefully moderated pressure, the internal projections will become suddenly evaginated, and will shoot out in all directions over the outer surface in the form of hollow cylindrical arms.

The evagination which has thus been effected by artificial pressure takes place naturally in the progress of development, and in the next stage (fig 4) we find that the arms which had been formed internally by a process of involution have become external, the embryo being still enclosed within its capsule The ectoderm had already, by the multiplication of its cells and the development in it of the clavate tissue, increased considerably in thickness, and the hyaline lamella may now be seen on its inner boundary.

Up to this period the embryo had retained its nearly spherical form; but it now begins to elongate itself, and assumes an oval shape (fig. 5). From its surface there project on all sides the tubular arms, which, from their original position within the cavity of the body, had become external by evagination, while at one extremity of the greater diameter the body has become truncated, and here numerous short papilliform processes (*a*) have become developed from its surface.

The arms continue to elongate themselves, and soon present a well-defined terminal capitulum. The papilliform processes, too, increase in number, and extend further back on the body of the embryo, which has become still more elongated. It is probably at this stage that the mouth is formed in the truncated end. The embryo is now ready to escape from its enclosing capsule, which has all along remained adherent to the extremity of the clasper, and which now becomes ruptured, and allows the little animal to enter on a free life in the surrounding water (Plate 55. fig. 2, *d d*).

The free embryo of *Myriothela* (Plate 58 fig. 6) is very contractile, and when fully extended is of nearly cylindrical form, about a quarter of an inch in length, slightly attenuated at one end so as to form a short conical hypostome (*a*), which carries the mouth on its summit, and more decidedly so at the opposite end, where it terminates in a little sucker-like disk (*b*). The papilliform processes (*c*) have now attained the form of the permanent tentacles, presenting a short stem with a terminal enlargement.

They commence just below the hypostome, and extend for some distance backwards on the body. Springing from between the short permanent tentacles, and from a considerable portion of the body which lies at their proximal side, are the long arms (*d*, *d*, *d*) which made their appearance at an early period of embryonic development, and which are destined to disappear entirely before the arrival of the animal at maturity They are about twenty in number, capable of great extension, and when stretched out to their utmost (fig. 6) are in the form of long straight filaments slightly tapering towards their distal extremities, where they terminate in a well-defined spherical capitulum. In complete retraction they are short, somewhat ovoid bodies marked by strong transverse rugæ

In accordance with the terminology I have already adopted in describing the early stage of *Tubularia**, I shall designate the free locomotive embryo of *Myriothela* by the name of actinula. It moves about by the aid of its long arms, whose terminal capitula are capable of being used as suckers of attachment, while the proximal end of the body, like that of a hydra, also admits of being temporarily attached by means of its little suctorial disk.

After the actinula has enjoyed for some days its free locomotive existence it begins to fix itself (fig. 7). This fixation is effected by means of the proximal sucker-like extremity (*b*) After it has thus become stationary it continues to manifest great contractility, becoming sometimes much extended, and at other times contracted into a nearly spherical mass. The long arms now undergo a rapid degradation (*d*, *d*, *d*); they lose their terminal capitula, become much shortened, and ultimately entirely disappear (fig 8)

In the mean time the short papilliform tentacles become more numerous, extending further backwards on the body The proximal extremity of the animal becomes bent at right angles to the rest of the body so as to form a sort of horizontal stolon-like foot, from which small fleshy processes with sucker-like extremities, and having a considerable resemblance to the claspers, are emitted. The function of these processes, however, is very different from that of the claspers; they serve to attach the animal permanently to some solid support, to which they fix themselves by their extremities Along with the stolon-like foot they become clothed in a chitinous perisarc, and the actinula has thus acquired all the essential characters of the adult trophosome.

The gonosome has not, however, as yet begun to develop itself, but it soon makes its appearance by the budding of the blastostyles and claspers from the hydranth at the proximal side of the tentacles From the blastostyles the gonophores are subsequently developed in the manner already described, and the animal thus attains its complete maturity (Plate 55).

In the histological structure of the actinula there are several points which deserve special consideration. In the very young animal, at the time when the arms are about to become changed from internal to external appendages, the endoderm and ectoderm can be everywhere followed without difficulty. The endoderm is composed of clear

* Gymnoblastic Hydroids, p 90.

cells, several in depth, the most internal presenting convex surfaces to the gastric cavity, but forming no villi-like projections. The ectoderm already consists of two zones besides the muscular lamella—a superficial zone composed of several layers of small round cells with clear granular contents, and a deeper zone of claviform tissue. The hyaline lamella with its muscular fibrillæ lies everywhere between the claviform tissue and the endoderm. All these elements can be followed from the walls of the body into those of the arms. In these the endoderm, composed of small, round, clear cells, surrounds a wide axial cavity.

When the arm has acquired its terminal capitulum, we find that the zone of claviform tissue, hitherto simply continued into the arm from the walls of the body, has become specially developed in the capitulum (Plate 56. fig. 13, *c*), and here envelops the endoderm in a nearly spherical cap, which takes exactly the place of the rod-like tissue in the permanent tentacles. The tissue composing this cap, moreover, is intermediate in its form between the ordinary clavate tissue and the rod-like tissue; for its component elements do not form branching groups as in the clavate tissue of other parts, but consist of radiating, simple, greatly elongated clavate cells, very similar to those already described as forming the claviform tissue in the distal extremity of the clasper, and thus affording further evidence that the rod-like tissue is only a modified claviform tissue.

The capitulum of the actinula arm further resembles that of the permanent tentacle in the presence of the pedunculated capsules These differ, however, in some points from the corresponding organs of the permanent tentacles; for they are not more than half their size, while the included cord is finer and longer, and is wound into closer and more numerous coils (Plate 56 fig. 14, *a*). Like the cord of the larger capsules, it continues after its emission to form a spiral, instead of straightening itself out in the field of the microscope like the filament of the true thread-cells. The spiral, however (fig 14, *b*), is more open and more elongated than that formed by the cord ejected from the stalked capsules of the permanent tentacles. The styliform process of the external sac is long and slender

When the transitory arms of the Actinula have attained their full growth, the ectoderm of their stem (fig. 13, *a*) no longer presents the two zones which were present in their younger stages It is the superficial zone which appears now to be wanting, so that the clavate tissue comes to the surface. In thus becoming superficial the distal ends of the cells composing this tissue have become wider, and lie more closely on one another, and very often contain a large vacuole excavated in the midst of their granular contents. Their caudal prolongations, moreover, do not seem to run into one another to such an extent as to give rise to the botrylliform condition which characterizes this tissue in other parts of the animal.

The endoderm of the arm (fig. 13, *b*) is formed externally by a tissue of large, clear, polygonal cells containing some minute granules, which are chiefly accumulated on the walls of the cells, while internally there is an irregular disconnected layer of small round cells filled with brown corpuscles. The increase of the endoderm in volume has nearly obliterated the axile canal of the arm.

The arm is very contractile, and, when in different states of contraction, the cells of the ectoderm may often be seen forming irregular projections of various length and thickness. These vary from time to time in shape and size, and look so exactly like pseudopodial processes that without careful observation they might easily be mistaken for them. They are, however, mainly the result of the contraction of the arm. When the arm is shortened by the action of its contractile elements, the hyaline lamella is thrown into irregular corrugations, and these are communicated to the superjacent cellular ectoderm. In macerated sections of the arm the cellular ectoderm will become disintegrated and broken down, and then the exposed hyaline lamella will often show nearly an exact repetition of the pseudopodia-like projections. I am, however, inclined to think that, after the contraction of the fibrillated layer has thus crumpled the hyaline lamella and overlying ectoderm, the protoplasm of the latter exerts a certain contractility which exaggerates the prominence of its projections, and thus to a certain extent brings them within the category of pseudopodia.

In the ectoderm of the body of the actinula we find not only the deep clavate tissue, but the more superficial layer of cells well developed. Here, during certain states of contraction, pseudopodia-like projections are also formed; and I believe that these are referable to the same cause here as in the ectoderm of the tentacles.

The proximal extremity of the actinula body is capable, as already said, of acting as a sucker of attachment; and here the ectoderm has acquired a considerable increase of thickness (Plate 56. fig 15). The increased thickness is mainly owing to the great development of the clavate tissue at this spot. This tissue forms here a hemispherical cap over the *cul-de-sac* of the gastric cavity, and the elements composing it are scarcely at all united to one another into ramified groups Its peculiar development here is probably connected with a special irritability with which this part of the walls would appear to be endowed. Over this cap the superficial ectodermal layer is continued, forming a zone of small, spherical, membranous cells with minutely granular contents. In the uninjured state a fine longitudinal striation may be witnessed in this part of the actinula (Plate 58. fig 6, *b*), it is caused by the appearance of the terminal mass of clavate tissue as seen through the overlying layer*.

The endoderm of the stem-like proximal portion of the actinula (Plate 56. fig. 15) closely resembles that of the transitory arms. It is composed of an external layer of large, clear, polygonal cells, with an internal one of small round cells filled with brown corpuscles.

GENERAL REMARKS

I believe we are justified in regarding the claspers as true zooids rather than as mere organs, and if so *Myriothela* may be compared with *Hydractinia* in the extent to which the polymorphism of the zooids is carried. We have here hydranths, blastostyles,

* A very similar appearance may be seen in the actinula of *Tubularia*, and I have now little hesitation in referring it to a similar cause.

gonophores, and claspers, all different forms of zooids, each endowed with its own special function in the physiological division of labour, and all associated into a compound colony which forms the proper zoological Individual, the logical element of the species*. In *Hydractinia* we have hydranths, blastostyles, gonophores, and "spiral zooids" similarly associated. In *Hydractinia*, however, there is a common cœnosarcal basis which gives origin to many hydranths, as well as to the blastostyles with their gonophores, and to the spiral zooids; while in *Myriothela* the hydranth is solitary, and the blastostyles and claspers are budded off from this.

It will be seen that the account here given of the development of *Myriothela* offers no support to the view that the generative elements originate in certain cells of the ectoderm—a view which has been defended by KLEINENBERG, who, in his excellent memoir on the structure and development of *Hydra*, maintains that both ova and spermatozoa have their origin in what he calls the "interstitial tissue" of the ectoderm. Neither does it support the view more recently put forward by ED. VAN BENEDEN in his valuable memoir on the origin of the testis and ovary†. According to the Belgian zoologist the ova in *Hydractinia* always originate in the endoderm, while the spermatozoa just as constantly have their origin in the ectoderm. To this observation M. ED. VAN BENEDEN attributes great significance, for by adopting the highly probable hypothesis enunciated many years ago by HUXLEY, that the ectoderm represents the outer layer of the blastoderm in the higher animals and the endoderm the inner layer, he generalizes the results of his observations on *Hydractinia*, and maintains that throughout the animal kingdom the female generative system is a product of the inner leaf of the blastoderm, and the male of the outer leaf.

From the observations on *Myriothela*, however, recorded above, it would seem to follow that both ova and spermatozoa originate in a special chamber which has become excavated in the substance of the endoderm, and that the ectoderm has nothing to do with either.

I believe this to be the legitimate conclusion to be drawn from the appearances presented. At the same time I admit that other observers may put a different interpretation on these appearances; for it may be asserted that the material which is to become developed either into spermatozoa or into ova is in one or both cases a product of the ectoderm, and that it has subsequently to its origin migrated into the endoderm; while in proof of this the orifice which exists in the roof of the endodermal chamber will probably be adduced and maintained to be the channel through which the generative elements have gained access to this chamber.

Knowing the memoir of M. E. VAN BENEDEN, in which he maintains that the spermatic

* The terms Zooid and Individual are used here with the significations originally proposed by HUXLEY. The former is the "Individual of the fifth order, Person" of HAECKEL, the latter the "Individual of the sixth order, Stock or Cormus" of HAECKEL.

† EDUARD VAN BENEDEN, "De la Distinction Originelle du Testicule et de l'Ovaire," Bull. de l'Acad. Roy. de Belgique, 2e série, tome xxxvii. no. 5, Mai 1874.

mass originates as a cellular bud from the inner surface of the ectodermal layer of the gonophore, and that this pushes itself into the endoderm and becomes afterwards cut off from its attachment to the ectoderm, I paid great attention to the gonophores of *Myriothela* from the earliest moment when they became recognizable, but entirely failed to detect any process resembling that described by the Belgian zoologist as taking place in *Hydractinia*. In the very earliest stages of the gonophore which I could find the gonogenetic cavity had been already formed and filled with the primitive generative matter, and I failed to meet with any thing which would lead me to believe that this had its origin in the ectoderm. It is true that in *Myriothela* a difficulty occurs in the observation which we do not meet with in *Hydractinia*; for while the complete separation of the sexes on different colonies in *Hydractinia* will enable us at all times to say, no matter how young may be the gonophore under examination, whether this be male or female, in *Myriothela* we have no certain sign by which to decide as to the sex of the gonophore in its youngest stages, gonophores of both sexes being here borne on the same blastostyle It is scarcely possible, however, that among the many cases of extremely young gonophores which I examined there were not both male and female examples, and in no case did I find any thing which would lead me to believe that the origin of the generative elements in one was different from what it was in another.

The facts here noted have thus led me to maintain that both male and female elements have their origin in the endoderm. Still, considering the difficulty of the observations, and the fact that the appearances lie possibly open to another interpretation, I do not desire to insist on the impossibility of the generative elements being in one or both sexes primarily introduced from the ectoderm into the endoderm, and I am willing to wait for the confirmation which may be expected from further investigations

As is well known, all the fixed hydroids pass through a free locomotive stage before finally attaching themselves. I have elsewhere* pointed out that this free stage shows itself under one or other of two forms, namely, that of a planula (as in the great majority of hydroids, *Campanula*, *Sertularia*, *Coryne*, &c.) and that of an actinula (as in *Tubularia*). The free hydroid planula is a closed sac in whose walls an endoderm and an ectoderm are differentiated, not by a process of invagination, but by one of dilamination, and in which an oral orifice is afterwards formed by a perforation of its walls, the planula thus becoming the *gastrula* of Haeckel. The external surface of the planula is almost always clothed with vibratile cilia.

The actinula represents a form more highly organized than either the planula or the gastrula; for not only is a mouth always present in it, but locomotion is effected not by vibratile cilia, but by means of external appendages in the form of tentacles or arms, which may be either transitory or permanent

It must not, however, be supposed that the planula stage does not exist in hydroids whose free phase is that of an actinula. It is, on the contrary, as truly a phase of their development as it is of that of the others: but the planula stage is then, if we

* Gymnoblastic Hydroids, p 85

except *Hydra*, entirely passed within the gonophore, and the planula in such cases is never ciliated or locomotive.

In *Tubularia* the planula is a non-ciliated compressed sac, developed directly out of the plasma mass which occupies the cavity of the gonophore; while still retained within the gonophore it develops tentacles by outgrowths from its sides, elongates itself, becomes perforated by a mouth, and then escapes as a free locomotive actinula destined to undergo further changes of shape before attaining the final form of the hydroid trophosome.

Just in the same way *Myriothela* passes through the non-ciliated planula stage before it attains the form of the free actinula In one important point, however, the actinula of *Myriothela* differs from that of *Tubularia*, namely, in the possession of embryonic transitory organs which take the form of long contractile arms, by which locomotion is aided, and which entirely disappear during the subsequent course of the development.

In *Hydra*, too, which never presents a permanently fixed trophosome, we find a true planula stage, the planula being here, as in the actinula-forming hydroids, destitute of cilia. It acquires a mouth by perforation, and develops itself by continuous growth and the emission of tentacles into the form of the adult without passing through any intermediate actinula stage.

Properly speaking, *Hydra* represents a permanent actinula. *Hydra* (if we except the somewhat obscure form described by GREEFF under the name of *Protohydra*) may thus be assumed as the lowest known hydroid, and, in accordance with the Descent Theory, would be the remotest ancestral form yet discovered of the Cœlenterata.

In all cases, however, it must be borne in mind that the planula is nothing more than the blastodermic sac after the two leaves of the blastoderm have become differentiated. In some few cases it never clothes itself with cilia, and then it almost always remains, as long as it continues a planula, included within the gonophore, while in the great majority of cases it develops cilia over its surface, and becomes free and locomotive.

KLEINENBERG, finding that in the adult *Hydra* the entire cellular ectoderm is composed of the caudate cells with an interstitial network of simple cells interposed between their proximal attenuated ends, while their wide distal ends form the outer surface of the animal, concludes that there is here no external epithelium or epidermis. *Hydra* would thus present an apparent anomaly, inasmuch as one of the most universal features in ontogenesis—the development of an epidermal layer from the outer germ-lamella (ectoderm)—would seem to be absent.

This anomaly, however, is brought into agreement with the established facts of development by KLEINENBERG, whose observations have led him to maintain that the so-called egg-shell of *Hydra* is really a transformed epidermis, but, being needed only as a protective investment for the embryo, is a transitory structure destined to be cast off in the later periods of development.

Though this may be a correct view of the state of things in *Hydra*, it is certain that in *Myriothela* we have a perfectly distinct and well-developed layer which lies external

to the claviform tissue, and forms the outer surface of the body. To this layer we must attribute the significance of a true epidermis It appears, however, to be absent from the stems of the transitory arms of the actinula after these have attained their full growth. In their early stages, while yet they are invaginated processes of the body walls, and even for some time after their complete evagination, it is present as elsewhere; but during the growth of the actinula it is gradually absorbed, and then allows the claviform tissue to come to the surface In the capitulum of the arm, however, it never disappears, being here needed as a protective envelope for the specially and more highly developed sensitive structures of this part.

It is thus obvious that *Myriothela* offers no exception to the ontogenetic law, which derives both the central nervous system and the epidermis from the outer layer of the blastoderm

One of the most remarkable features in *Myriothela* consists in the presence of the bodies to which I have here given the name of claspers. These, as we have seen, are tentacle-like zooids endowed with great contractility, and no sooner is the plasma mass, which is to become developed into the actinula, set free from the gonophore which had hitherto confined it, than one or more claspers direct themselves towards it, and fixing themselves to it by their sucker-like ends, hold it tenaciously during certain subsequent periods of its development The manner in which the claspers thus seize upon the liberated plasmodium forcibly reminds us of the way in which the Fallopian tubes are supposed to seize the mammalian ovum at the moment of its liberation from the Graafian follicle.

There is something very surprising in the selective faculty thus apparently exercised by the claspers, for it is, as a rule, to the liberated plasma mass alone that they become attached, while no reason whatever can be assigned why they should not seize upon some of the neighbouring parts which are just as easily within their reach Once or twice I have seen a clasper fixed to some other part of the hydroid, but this occurrence is so rare that it cannot in any way be regarded as a manifestation of its normal function

We have at present no data which will enable us to arrive at an absolute conclusion as to the object gained by the seizure of the plasmodium by the claspers. It is not improbable, however, that it is connected with fecundation We must remember that in *Myriothela* we have the very exceptional condition of one and the same blastostyle carrying both the male and the female gonophores, and, further, that the spermatozoa of this hydroid are remarkable for their extreme minuteness; they are smaller, indeed, than in any other hydroid with which I am acquainted. Now I have never seen the spermatozoa escape spontaneously as in other hydroids from the gonophore, and when one of the *Myriothela* gonophores containing mature active spermatozoa is subjected to slight pressure, it is not through any breach of continuity in the thick external walls of the gonophore that the spermatozoa are ejected, but through the walls of the spadix, which appear to be easily ruptured. In this way they pass directly into the gastric cavity of the blastostyle, and through this may be easily conducted to the base of a clasper, and

thence carried through its narrow axial channel to its summit, where this has become attached to the plasmodium just liberated from the female gonophore. When once arrived there the spermatozoa may make their way through the terminal tissue of the clasper, and be thus brought into immediate relation with the plasmodium, whose investing membrane is at this time exceedingly thin and weak, a process which will be obviously facilitated by the exceptional minuteness of the spermatozoa.

We should further bear in mind that it is not until after the seizure of the plasmodium by the claspers that we have any evidence of the phenomenon of segmentation—a fact which renders it highly probable that the act of fecundation also takes place subsequently to the seizure. Spermatozoa, if searched for in the cavity of the clasper, would probably be found there, but, short of their detection in this situation, we have a combination of facts about as strong as could be desired, all tending to the conclusion that the function of the claspers is that here suggested, and offering a case in many respects parallel with that of the hectocotyle in the Cephalopoda, or with certain phenomena of fertilization among the Algæ

EXPLANATION OF THE PLATES.

PLATE 55.

Fig. 1. *Myriothela phrygia* A group, natural size, attached to a stone; some of the individuals contracted, others extended

Fig. 2. Magnified view of an individual extended.

a, a, a, a. Blastostyles, *b, b, b, b* Gonophores, *c, c, c, c.* Claspers; *d.* Basal portion of the hydranth invested with its perisarc, *e, e.* Processes of attachment.

Fig. 3 Magnified view of an individual contracted

PLATE 56.

Fig 1. Transverse section of the hydranth at some distance behind the mouth. Magnified.

a Endoderm; *b* Villi-like processes of endoderm projecting into gastric cavity; *c.* Ectoderm, *d, d, d.* Tentacles.

Fig. 2. Portion of transverse section of hydranth, still more magnified.

a. Endoderm, *b* Villi-like processes from the free surface of endoderm; *c, c* Small spherical cells loaded with coloured granules, and terminating the villi, *d* Thin stratum of homogeneous protoplasm extending over the free surface of the endoderm, *e, e. e.* Pseudopodial processes emitted from the protoplasmic stratum, along with which fine vibratile cilia are also seen extending into the gastric cavity, *f* Base of a tentacle; *g.* External layer of cellular ectoderm, *h.* Internal layer of same (clavate tissue); *i.* Hyaline lamella

Fig. 3. Longitudinal section through summit of tentacle, much magnified.

a. Rod-like tissue; *b*. Pedunculated capsules.

Fig. 4. Thread-cells.

a. Oviform thread-cell in its quiescent state; *a'* Same, with the filament ejected; *b*. Fusiform thread-cell in its quiescent state; *b'*. Same, with the filament ejected.

Fig. 5. Cells of ectoderm of tentacle liberated at the commencement of putrescent histolysis. In each of the two larger cells may be seen a thread-cell

Fig. 6. A portion of the hyaline lamella with its attached clavate tissue, from the body of the hydranth.

a. Clavate tissue; *b*. Fibrillated layer of the hyaline lamella, *c* Delicate structureless layer of the same lamella.

Fig. 7. Some of the rods of the bacillar tissue of tentacle, greatly magnified.

Fig. 8. One of the pedunculated sacs, with its contents, from the tentacle isolated

Fig. 9. The capsule, with its contained cord liberated from the pedunculated sac

Fig 10 The capsule after the ejection of the cord, which is still attached by one end to its summit.

Fig 11. Distal extremity of a clasper.

a. Endoderm; *b*. Hyaline lamella, *c* Clavate tissue, *d*. External layer of ectoderm; *e* Extension of ectoderm with its clavate tissue greatly developed over the distal end of the clasper.

Fig. 12. Isolated cells of the clavate tissue from the distal extremity of a clasper

Fig. 13. Distal extremity of one of the transitory arms of the actinula.

a. Modified claviform tissue, which here forms the whole thickness of the ectoderm; *b*. Endoderm with axial cavity, *c*. Capitulum.

Fig. 14 Pedunculated sac from the capitulum of one of the transitory arms of the actinula.

a. The pedunculated sac with its contents still undisturbed, *b*. The capsule liberated from the sac and with its spiral cord ejected.

Fig. 15. Distal extremity of actinula, showing the peculiar development of the clavate tissue at the extreme end (*a*), which acts as a sucker of adhesion.

PLATE 57.

Fig 1. Very early stage in the development of the gonophore.

a. Offset from the endoderm of the blastostyle which has pushed itself into the ectoderm; *b*. Gonogenetic chamber filled with a granular plasma, *c* Diverticulum from the cavity of the blastostyle; *d*. Ectoderm of the blastostyle as yet scarcely raised above the general surface.

Fig. 2. More advanced stage (female); the gonophore has formed a very decided projection from the external surface of the blastostyle, and the gonogenetic chamber has begun to show a differentiation in its contents.

b. Gonogenetic chamber, in which the contents have become accumulated on the walls and show imbedded nucleus-like bodies; *c*. Diverticulum from the cavity of the blastostyle; *d*. Orifice in the endoderm forming the roof of the gonogenetic chamber.

Fig. 3. A still more advanced stage of the female gonophore.

b. Gonogenetic chamber filled with a granular plasma, in which a great number of nuclei have become developed; *c*. Diverticulum from the cavity of the blastostyle, which with its endodermal walls now projects as a spadix into the gonogenetic chamber, *d*. Very delicate structureless membrane, which separates the generative mass from the endodermal walls of the gonogenetic chamber.

Fig. 4. Stage still further advanced. Cell-boundaries have begun to show themselves in the plasma of the gonogenetic chamber, and the nuclei have become surrounded by differentiated masses of protoplasm.

Fig. 5. Nucleolated nuclei, isolated from the contents of the gonogenetic chamber in fig 3.

Fig. 6. Some of the cells forming the contents of the cavity of the gonophore in fig. 4.

Fig. 7. More advanced stage of the female gonophore. The ovarian tissue has become looser, and now consists for the most part of detached oval masses of protoplasm each with a nucleus and nucleolus. Towards the centre, where they are in contact with the spadix, some of these have coalesced into larger masses.

a. Cellular lining of the cavity of the gonophore, where at the summit of the gonophore its cells become loaded with coloured granules, forming a purplish ring which surrounds the orifice.

Fig. 8. Some of the nucleated oval masses of fig. 7 removed from the gonophore, and seen to have become united to one another by protoplasmic prolongations.

Fig. 9. A group of the same bodies. Between several of them the union has become closer.

Fig. 10. Gonophore still further advanced than fig. 7. Nearly all the free oval bodies have coalesced into a small number of large protoplasm masses.

a. As in fig. 7.

Fig. 11. Surface of one of the protoplasm masses of fig. 10, very much magnified, showing the presence of minute pseudopodial projections.

Fig. 12. Portion of the walls of a mature gonophore (fig. 7), very much magnified, showing details of structure.

a. External zone of spherical cells; *b*. Zone of clavate tissue, *c*. The fibrillated lamella; *d*. Cellular lining of the gonophore cavity; *e*. Very thin structureless membrane directly investing the generative elements; *f*. Generative elements.

Fig. 13. Structure of plasmodium formed by coalescence of the simple ova.

Fig. 14. Part of a blastostyle with gonophores, plasmodia, and claspers.

a. Blastostyle; *b, b.* Claspers; *c, c.* Young gonophores; *d.* A mature gonophore, with the plasmodium escaping through its summit, *e* Walls of gonophore retracted and everted after the liberation of the plasmodium, *f.* Plasmodium liberated from the cavity of the gonophore, but still held in in its place by the spadix, and already seized by a clasper. The plasmodia (*d* and *f*) present a lobed condition at the part turned towards the blastostyle, owing to the coalescence of their constituent plasma masses being here still incomplete, *g.* A plasmodium entirely withdrawn by a clasper from its original position on the summit of the gonophore peduncle

Fig. 15. A male gonophore filled with the generating vesicles of the spermatozoa.

Fig. 16. Mature free spermatozoa.

Fig 17. Structure of the plasmodium shortly after its seizure by the claspers.

PLATE 58

Fig. 1. Planula.

a. Ectoderm; *b.* Endoderm, *c* Cavity of planula, *d.* External structureless capsule

Fig. 2. Embryo after the walls of the planula had become invaginated to form the transitory arms

a. Body of the embryo; *b, b* Orifices of involution; *c* External structureless capsule.

Fig. 3 Section through the centre of the embryo represented in fig. 2.

a. Body of the embryo, *b, b.* Arms formed by involution of the walls of the embryo; *c* External structureless capsule

Fig. 4. Embryo after the arms have become external by evagination

b, b. The evaginated arms.

Fig. 5. Embryo after it has begun to elongate itself and acquire an oval form.

a. Commencement of permanent tentacles, *b, b, b.* Transitory arms.

Fig. 6. Embryo after its escape from its capsule when it enters on its free life in the surrounding water

a. Distal extremity; *b.* Proximal extremity, *c, c* Permanent tentacles, *d, d, d.* Long transitory arms fully developed.

Fig. 7. Embryo when it has begun to fix itself.

a. Distal extremity with mouth; *b.* Proximal extremity with disk of adhesion; *d, d, d.* Transitory arms in process of disappearance.

Fig. 8. The embryo has definitely fixed itself, and the transitory arms have entirely disappeared.

XX. *Spectroscopic Observations of the Sun*

By J. NORMAN LOCKYER, *F.R.S.*, *and* G. M. SEABROKE, *F.R.A.S.*

Received February 2,—Read March 19, 1874.

WE have the honour to communicate to the Royal Society the accompanying Spectroscopic Observations of the Chromosphere and of the Sun generally, made during the period between the 1st October, 1872, and the 31st December, 1873.

The London observations have been made in Alexandra Road, Finchley Road, N.W., the Rugby observations in the Temple Observatory at that place.

The following details are given of the instruments and methods of observation employed.

LONDON OBSERVATIONS.

A 6¼-inch refracting telescope by COOKE, of York, mounted equatorially, was employed, to which is attached the 7-prism spectroscope by BROWNING, of London, already described. A position-circle, made by COOKE, of York, was used for obtaining the position-angle of the prominences and of the various details of the chromosphere.

On the side towards the spectroscope the circle is provided with a pinion, which, acting on a circular rack, causes the graduated half of the circle to rotate, the vernier being on the fixed half attached to the telescope-body.

On the 16th of September, 1873, the prisms spectroscope was replaced by a diffraction-grating of speculum-metal containing 6121 lines to the inch, made by Mr. L. M. RUTHERFURD, of New York, by whom it was generously placed at Mr LOCKYER's disposal, the whole apparatus is only 15 inches in length, and weighs 3 lbs, while the 7-prism spectroscope, with its mounting, is 24 inches long, and weighs 10¾ lbs, the principal weight, moreover, being 18 inches from the end of the telescope. In dispersive power the 2nd order spectrum of the grating is equal to 7 prisms, while with equal dispersive power the grating gives much more light.

The positions of the prominences have been determined as follows.—

Standing with the back to the sun, and looking at the sun's image on the slit plate, the bottom of the image, being the image of the real North of the sun, is called North, the left-hand side of the image East, the right hand West, and the top South. The degrees are reckoned from North as zero through East to North again in the same direction as the hands of a watch, N., E., S., W. of the image on the plate being of course in the contrary direction to N., E., S., W. as seen directly on the sun. If,

then, the ring of chromosphere, as seen on the slit plate, be cut at North or 0° and straightened, we obtain a line with N. or 0° on the left hand, and extending to the right from N. 0° through E. 90°, S. 180°, West 270°, to N. 360°.

The adjustments for recording the positions of various parts of the chromosphere as observed with either the radial or tangential slit having been made, the telescope is clamped in R.A., the clock set going, and the spectroscope focused for the C line.

Should a prominence be observed, the telescope is moved in R.A. or Declination, until it appears in the middle of the field of the spectroscope, and the position-circle is then moved until the slit is either tangential or radial to the part of the limb where the prominence appears, this is determined, in the case of the tangential slit, by the narrow strip of continuous spectrum which flashes in the moment the limb of the sun overlaps the slit exactly

In the drawings executed in London, which accompany this paper, the positions of the prominences have been determined as follows, viz. the smaller ones, those from 2° to 3° wide, have had the central point of their base taken for the position, those wider than this have, in every case where possible, had the position-angle of each side determined, and very complicated groups have had, as far as possible, their principal components determined.

The height has been obtained by causing the slit to travel up the prominence, and estimating how many slits high above the limb it was—a process which is easy, as there are nearly always in the prominence details of structure which can be used as points for measurement.

The height of each prominence is set down in slits, and the width of the slit is measured at the end of the operation, and the true height in seconds calculated from the measurement.

The London observations and drawings have almost entirely been made by Mr. R. J. FRISWELL, Mr. LOCKYER's assistant, to whom great credit is due for the zealous and intelligent manner in which he has taken up this branch of the research.

RUGBY OBSERVATIONS.

The 8¼-inch equatorial by ALVAN CLARK, to which is attached the ring-slit arrangement, producing a virtual eclipse of the sun, described by us before this Society in January 1873, has been used for these observations. The spectroscope attached is constructed on the return principle, giving a dispersion of 8 prisms of 60°. The position of the prominences has been determined as follows:—Arranged radially round the disk, which cuts off the light from the body of the sun, are fine platinum wires at a distance of 10° from each other, and these being seen together with the ring of chromosphere serve to fix the position of the prominences, the shape and position-angle of which can be then easily drawn. There are four wires crossing the annulus 90° from each other that are rather thicker than the others, and these are made to coincide with the N., S., E., and W. points of the sun respectively by causing the upper or lower

limb of the sun's image to traverse the disk, and then turning the instrument round until the limb exactly passes from one wire to the opposite one; then, on bringing the sun's image concentric with the disk, the left-hand wire, as seen by looking on the disk with the back to the sun, corresponds to the East side of the sun as looked at directly, and is therefore at the position of 90°, and the right-hand one corresponds to the W. of the sun 270°; the lowest wire will then correspond to the North or 0°, and the upper to the South or 180°. The direction of reckoning the degrees is as usual N., E., S., W., or as looking directly at the sun in the contrary direction to the hands of a clock; but as looking on the disk with the back to the sun N., E., S., W will be in the same way as the hands of a clock, and if the ring of chromosphere, as it would appear to an observer looking at it in the annulus with the back to the sun, be cut at N. or 0° and straightened, the appearance would be that shown in the drawings; although the annulus of chromosphere is looked at with the spectroscope from the opposite side to that of the sun, the image is half inverted by a diagonal reflecting prism in the telescope of the spectroscope, so that its appearance is the same as if looked at in the annulus from the same side as the sun. The width of the annulus through which the light from the chromosphere passes is such that a prominence 100″ in height reaches across the annulus, so that the height of the prominences can be judged of with fair accuracy.

We have purposely refrained from any reduction of these observations, as we are of opinion that such reduction will be most usefully made when the observations of the Italian and other observers have been published, as it is hoped that the English and foreign observations may be in some cases so complementary of each other that long gaps may be avoided.

NOTES TO ACCOMPANY THE MAPS. (PLATES 59 TO 64.)

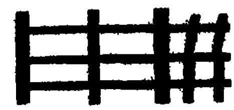
(LONDON OBSERVATIONS*.)

December 6th, 1872.—Chromosphere generally 10″ high.

January 1st, 1873.—Chromosphere about the usual height, except 150°–170°, where it was low.

March 8th, 1873.—Chromosphere very hair-like in its outline, about 12″ high Between 3.30 and 4.0 P.M. a large spot was observed between 240° and 250°, and close to the limb. Violent action was going on. C was intensely black over the spot, and, I think, slightly thickened; D was very thick, and bent towards the red. The magnesium lines did not seem to be affected; but the two lines of b, 5166·5$\left\{\begin{matrix}\text{Mg}\\ \text{Fe}\end{matrix}\right.$ and

* With these are included, in order of date, nine woodcuts of the more remarkable prominences, of the size of the original drawings, which were made, some at London by Mr. FRISWELL, some at Rugby by Mr SEABROKE. The locality, date, and position-angle, which are given in each case, will enable the reader to find the places of these prominences in the Maps.

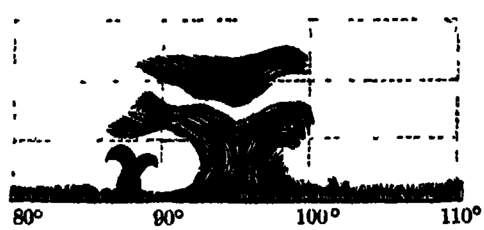


5168·5 $\left\{\begin{matrix}\text{Ni}\\ \text{Fe}\end{matrix}\right.$, violently bent towards the red. 4859·1 Fe intensely black and thick F very black, and bent in all directions over the region between the spots on the limb; it was perhaps rather thickened, but I could not be certain of this.

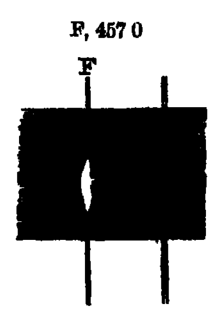
C and F presented the above appearance on the limb near the spot. Once the bright part of F filled up the space between 4859·1 Fe and the dark F line. This was probably only half the bright part, but I did not see it on the other side.

Another spot was close to the one in question, a little to the N. and E. of it.

London



March 8, 1873.

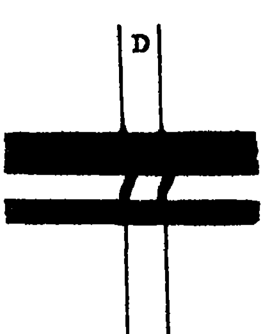


March 12th.—The bright line on the most refrangible side of *b* in the ordinary solar spectrum scarcely affected by the spot, *b* not thickened C gone on the edge of the spot; F like this.—

March 17th.—A group of spots, probably those seen on the limb on the 8th and on the sun on the 12th, the magnesium lines were not thickened. D is very thick, and C very black; but it is doubtful if it is thickened. The continuous absorption of the spot on either side of C is very small.

March 24th.—The prominence at 230° changed a good deal in form and brilliancy. Two spots were seen, but no satisfactory observations were obtained. One of them seemed to give a continuous absorption only.

March 25th.—Chromosphere like the edge of a grass plot, about 15″ high; a spot near N.E. limb. The following observations were made:—



Magnesium lines not much affected.
Calcium „ near D not much affected.
„ „ in red moderately thick, but certainly not in the same state of motion as the sodium; scarcely any, in fact.
Hydrogen thin and scarcely disturbed.

March 26th.—Chromosphere very hair-like, from 290° by 0° to 85°, except at 45°–55° At 20° the hairs inclined in all directions; at 290° inclined towards each other in two masses, one on each side of 290°; at 65° sharp inclination to the prominence at 60° The chromosphere was also hair-like at 110°–130°, 135°–150°, 215°–245°.

March 27th.—The chromosphere about usual height, generally hairy

March 28th.—From 90° to 180° no chromosphere seen, on account of mist and fog, from 180° by W. to 320° also misty, but observations made Chromosphere hair-like in N.E. quadrant, and about 8″ to 12″ high.

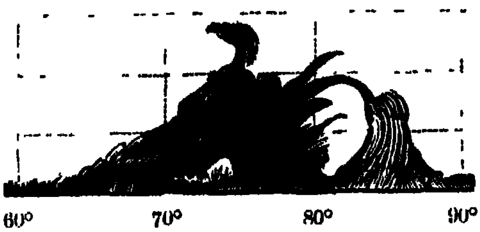
March 29th.—On a group of spots now in the centre of the disk the whole spectrum appeared full of narrow strips of absorption, as though the sun were mottled The Ca lines enormously thickened on the left-hand spot, as seen in the spectroscope, D formed nearly one line, and *b* also appeared joined into one Near F the absorption was so great and general that nothing could be seen.

April 1st —Hairy chromosphere near 10°, 30°, 70°, 90°, and 210°, at which latter place the hairs were sharply inclined towards the prominence at 205°.

April 2nd.—The group of prominences between 210° and 225° changing considerably Chromosphere generally low (5″?).

April 19th —Chromosphere hairy, and inclined to S. at 180°–190° and 270°–280°, to N at 240°–250°, and straight up near 120°

London.

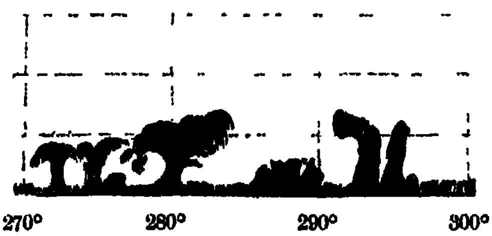


April 19, 1873

April 21st.—Chromosphere very hairy, high, and hairs straight up at 35° to 85°

May 1st.—Chromosphere generally hairy and rather low.

London



May 2, 1873

May 2nd and 9th.—Chromosphere rather low; on the 9th it was generally hairy, and the hairs straight up.

May 20th.—Chromosphere very low and regular, about 6″ high; at $212\frac{1}{2}$° a prominence of honeycomb structure (the note says, "looks like a coarse sponge"), the two northern quadrants not observed on account of mist.

May 22nd.—Chromosphere low.

May 23rd.—Chromosphere about 10″, inclined E. generally from 0° to 90°, and to E. at 105°–115°, 135°–145°, 155°–165°; straight up near 120°, 150°, 170°; a gap in it at 217°, and very low at 270°–275°.

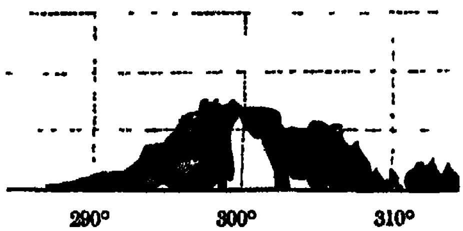
May 24th.—Chromosphere very low at 330°–10°, and sharply inclined to the W. at from 225°–240°.

May 31st.—Chromosphere 10″–12″ high at 220°; sharply inclined to prominence at 231°, and very hairy at 236°, so low as scarcely to be seen.

June 7th.—A spot observed. Calcium lines between C and D very thick; D and *b* very slightly or not at all affected.

Chromosphere undecided in character

Rugby



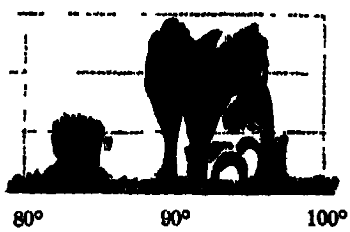
June 15, 1873

June 27th.—Chromosphere about 8″.

July 7th.—Chromosphere about 10″.

July 14th.—Chromosphere about 9″, undecided in character.

London



July 14, 1873

July 16th.—Chromosphere 8″–12″ high, hairy in S.E. quadrant, and inclined to the W ; high from 130°–140°, measured 12″ here at 110°; a jet overlapped the limb, but the prominence changed its form and it soon went off.

Much mist during observations of the two south quadrants.

July 21st.—180°, thin, very active, vertical hairs; 186°, the same; 190°, hairs increasing; 195°, chromosphere quieter; 200°, the same; 205°, vertical hairs; 210°, fuzzy, 215°, one hair longer than the rest; 220°, masses here and there; 225°, the same, 230°, fuzzy; 235°, more massive, 240°, nothing particular; 245°, chromosphere low; 250°, very low, not hairy; 255°, very faint.

July 22nd.—180°–185°, hairy, but massive; 195°–205°, tongues; 215°–225°, hairy, massive, 225°–235°, lumpy and low; a long cloud here connected with chromosphere by a very faint filament; 25°–35°, very spiky; 275°–295°, lumpy and very bright.

July 23rd.—175°–205°, spiky, and spikes inclined to S.; 205°–215°, very hairy, hairs straight up; 215°–225°, hairs inclined to S.; 225°–235°, same inclination, more

decided. In the N.W. quadrant the chromosphere lumpy, except near 0, where it is spiky.

July 24th.—S.W. quadrant, the chromosphere covered with fluffy hairs; in the N.E. quadrant it is very spiky.

July 25th.—Only the N.E. quadrant was observed, on account of the bad light; 0°–10°, spiky; 25°–35°, spiky; 40°–50°, lumpy, with hairs all turned to N.; 50°–60°, lumpy and low; 80°–90°, low and bright.

July 26th.—From 40°–90°, hairs inclined to S.; from 30°–40°, very sharply inclined S., the chromosphere very hairy; N W. quadrant, all the hairs inclined W., and high near 350°; S.E. quadrant, hairs to E, jets or splashes cover 3° at 110°, S.W. quadrant, spikes inclined to S., very decidedly at 190°.

July 28th.—Only the N.W. and N.E. quadrants were observed. In the former the chromosphere was hairy, and the hairs inclined to N. slightly, except at 330°, where they were divergent. In the N E. quadrant the hairs were generally straight up from 90° to 20°, where they were slightly inclined to W. From 10° to 0° they were straight again.

There was a large spot nearly in the centre of the disk; the C, D, *b* lines and the chromium lines near *b* were not affected, the iron lines 5190·5, 5191·7, 5226·0, and 5232·0 scarcely, if at all, affected. The Ca lines near D were slightly thicker. The spot is rather faint, and as the general darkening of the spectrum is considerable and the selective absorption almost *nil*, a cooling only would seem to be indicated.

July 30th.—In the S W. quadrant from 262° to 270° the chromosphere or a long low prominence was 25″ high, at 310° to 316° there was another prominence, a portion of which (about 310° to 313° or 314°) was like a coarse sponge in texture

July 31st.—The chromosphere as a rule is low, bright, and lumpy at from 120° to 140°, there were indications of an inclination towards a prominence at 140°, at from 308° to 320° there was a smoky appearance and a slight inclination to the W.

August 7th.—The chromosphere lumpy and low from 150° to 190°; from 105° to 115° there was a very sharp inclination to E. In the S.W. quadrant it was generally lumpy and any hairs straight up; in the N.E it inclined slightly to N., and was also lumpy. There was a spot close to the base of the prominence at 309°.

August 8th.—From 90° to 70° lumpy with straight hairs; at 48° a very low gap with a spike in the middle of it, there was scarcely a trace of chromosphere in the gap for 1° or 2°. From 70° to 30° the chromosphere was fumy or smoky, with hairs in the fume, at 23° it was very low again. From 20° to 10° it was fumy, but contained well-developed hairs. In the N.W. quadrant it was spiky, and high at 350° to 340°; 340° to 330° fumy, at 320° it was denser, and exhibited a slight inclination of its details to N. In the S.W. quadrant the chromosphere was fumy with spikes, which latter were well developed at 182°.

August 9th.—The chromosphere had generally a peculiar look, as though it was

viscous and had been drawn out into spikes. From 110° to 180° it was fumy; from 90° to 105° there was an inclination towards E. In the N.E. quadrant it was of both characters (spiky and fumy), and the spikes were straight up. In the N W. it was fumy; in the N.W. hairy from 270° to 340°, and from 340° to 0° rather fumy.

August 13th.—Observations at 5.20 to 6 P.M., when lowness of sun stopped them; S W. and N.W. with part of S.E. observed the chromosphere moderately spiky, but its inclination indeterminate except at from 230° to 250°.

August 16th.—90° to 115° spikes inclined E. 45°, 115° to 180° lumpy; spikes straight at 120° to 130° to E at 135°. In the N E the chromosphere was fumy and high, with hairs in the fume, the same in the S.E from 300° to 0°, from 285° to 295° the chromosphere and a prominence had a spotted or mottled appearance; about 275° spiky

London.  Aug 22, 1873

September 16th.—S E. chromosphere hairy, with a slight inclination to the E., S.W., and N.E., N.W smoky with indistinct hairs.

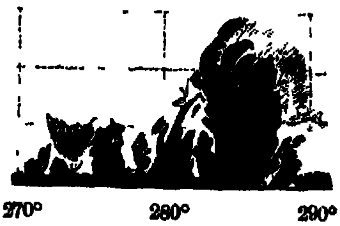
September 22nd —The chromosphere was generally smoky in appearance, at 154° there was an exceedingly faint high prominence scarcely visible; its height appeared to be 1′ 33″.

September 23rd —No particular details observable in chromosphere except at 115°, where it was lumpy; 253°, about, where it seemed composed of small flames, 307°, high and smoky. On the W side of the large prominence at 327° to 330° the chromosphere for 4° or 5° (222° to 227°) was hairy, and turned towards the prominence.

September 24th —Chromosphere in S E. and S.W. smoky and covered with irregular tongues, not hairs. A spot was observed in which the C line was distorted and not thickened; D distorted and thickened, Ca lines much thickened, but not much distorted. When the C line was distorted D was still, and *vice versâ.* *b* was distorted but not thickened A prominence at 304° was undergoing considerable change

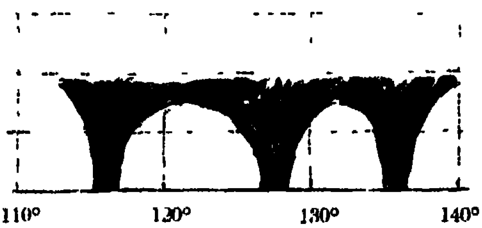
September 25th —No particular features in the chromosphere in the S E. and N.E. It was high round 270°. From 315° to 325° the chromosphere was high and covered with tongues, some 24″ high.

September 26th.—A brilliant point at 109° at 2.15 P M., at 3.15 P.M not a trace of it In the N W the chromosphere was fumy with a spiky edge; from 290° to 300° it was 15″ to 20″ high in the whole quadrant, and up to about 20° in the N.E. quadrant it was inclined in the same direction, *i. e.* to E. down to 0°, and to S. from 0° to 20°. The C line was seen broken over a spot.

London  Sept 26, 1873

September 27th.—S.E chromosphere smoky and covered with tongues, S.W. 180° to 200° fumy; 230° to 240° fumy, spikes inclined to E., 250° to 260° chromosphere 20″ high, spikes 30″ For 1° or 2° it was very low at 262°, and from 262° to 275° higher, and inclined to prominences at 280° &c.

October 2nd.—S E. and S.W. light very bad and chromoscope indistinct, but it appeared to be smoky, the same in N E In the N W. the chromosphere hairy and about 16″ high

Rugby  Oct 10, 1873

October 15th.—Chromosphere generally 12″, near 100° low (8″), and at 115° sharply inclined to N., at 145° it appeared to be squirting in all directions, as though from a hole, from 170° to 190° solid and spiky, very high and covered with tongues all through the S W quadrant, in the N.W generally fumy, but more hairy in the eastern part, in the N E lumpy from 80° to 90°, a tuft of hairs 16″ high and inclined to N. at 59°. A spot in this quadrant in which D, *b*, and the Ca lines are thick, but C unaltered

October 17th.—Chromosphere very level and low (8″), but light very bad.

October 28th.—S.E and S W lumpy and billowy with tongues, N E and N W. rather more hairy

October 30th.—Chromosphere about 8″, high, billowy, and smoky in N E., S.E., and S.W.; in N.W. the same, with a direction to W.?

November 1st.—No details observed, observation unsatisfactory

November 3rd —Only the S E quadrant observed, chromosphere about 16″

November 11th.—S.W., N W., and N.E the chromosphere smoky with hairs, at 136° a prominence nearly separated from the chromosphere, which was fumy.

Rugby  Nov 11, 1873

November 12th.—S E chromosphere billowy; S W. 200° to 220° fumy, direction W.; 250° to 254° very high, fumy chromosphere, 260° to 270° direction S.; N W. about 280° low and bright, direction N., at 290° to 300° straight up, about 340° fumy and flamy, with a W. direction, about 0° fumy tongues with a W direction

November 15th —S E. fumy, a few spikes straight, at 95° very billowy; about 140° S.W. fumy and billowy; N W. sharp inclination to N. at 280° to 290°, and hairy at that part, elsewhere fumy

December 9th.—Chromosphere rather spiky near 160°, from 185° to 210° fluffy; round 213° hairy hairs straight, remainder of the quadrant billowy, round 270° very brilliant, D_3 is very brilliant here between 265° and 272° in the lower parts of the chromosphere.

December 12th.—From 220° round by 0° to 90° light too bad for observation; rest of chromosphere regular.

December 17th.—From 58° to 90° hairs have a slight tendency towards 90°. Rest of chromosphere fumy, with a few tongues or billows

December 29th —Chromosphere near 20° hairy, then rather level, at 160° spiky, and inclined both to E and S., at 200° tongues, 215° hairy, which continues to 240° Prominences at 241° and 244° are like wreaths of smoke, 315° to 360° tongues inclined towards W

December 30th —From 90° round by 0° to 160° light too bad for observation, 90° to 160° chromosphere level, higher at 120° to 130°.

December 31st.—0° to 90° light not good, high at 85°, very faint cloud at 132°, at 1.55 P.M. great changes going on in the group of prominences between 252° and 265°; chromosphere generally level.

Spots were observed on the 23rd and 29th On the 23rd D and the Ca lines near it slightly thickened, and D a little distorted, C and *b* not affected. Absorption general rather than selective On the 29th general absorption again characteristic, C, D, *b* not affected.

It has been noted at Rugby that all the cyclones observed from the beginning of 1872 have, with one exception, had a motion of rotation, direct when in the northern hemisphere, and indirect when in the southern, corresponding, therefore, to our terrestrial cyclones.

In the Plates accompanying this paper the horizontal lines represent each one minute

XXI. *Tables of Temperatures of the Sea at different Depths beneath the Surface, reduced and collated from the various observations made between the years* 1749 *and* 1868, *discussed. With Map and Sections. By* JOSEPH PRESTWICH, *M.A., F.R.S., F.G.S.*

Received May 14,—Read June 18, 1874

CONTENTS.

§ I. *Introduction.*

THIS communication, the result of an inquiry having originally reference to the bearing of the subject on certain geological questions, was commenced more than twenty years ago, but abandoned for a time, partly owing to the pressure of other engagements, and partly waiting more accurate information of the range of life at depths*. The great impulse given to these questions by the more recent expeditions of the 'Lightning'

* A few of the geological questions were, however, noticed, and some of the early deep-sea temperature-observations given, in the author's Anniversary Address to the Geological Society of February 1871, Quart. Journ. Geol. Soc. vol. xxvii. pp. xliii–lxxv.

and 'Porcupine,' culminating in that of the 'Challenger,' has not only again directed attention to the subject of deep-sea temperatures, but has led to such improved methods of observation, that it may now seem late to bring forward the less accurate experiments of former observers. It might therefore seem almost a work of supererogation, now that the subject in connexion with these later voyages has been so ably and zealously taken up by my friend Dr CARPENTER, to introduce these more variable older elements into the discussion Still the older observations, though restricted to comparatively limited depths, have a wide range, and in the case of the Arctic voyages they were obtained under conditions of so much difficulty and danger, that it may be long before similar experiments are repeated, while many of the original opinions evidently deserve great consideration. It was, moreover, always my intention to complete the task I had begun when time and opportunity offered, and as Dr. CARPENTER's work commences with the observations made by him on board the 'Lightning' in 1868, it may not be out of place to have a record of all that was done in temperature-soundings* up to that time, even as supplementary to the more exact work of later voyages.

I may also notice that, notwithstanding the superiority of the more recent observations and the inaccuracy of many of the older ones, there are a certain number of the latter which were made with great care, and which may vie with recent experiments in exactness, while with respect to the others, the errors are such as may in most cases be computed and allowed for, or merely taking the old observations as they are, the *comparative* temperatures recorded at *corresponding* depths with the same or similar instruments have their own special value. The older observations are also so scattered through various narratives of voyages and in scientific periodicals, that no one can, without much difficulty of search, form an idea of their number and interest, or of the progress which the subject had made at the hands of the eminent men who had from time to time engaged in the inquiry on the Continent. I purpose, therefore, to show the state of the question at the time of the 'Lightning' expedition. For all that has been done since, I would refer to the exhaustive papers of Dr. CARPENTER†.

In former voyages the temperatures are variously noted in degrees of RÉAUMUR,

* The few old observations of life at depths will not now require notice

† "Preliminary Report," by Dr WILLIAM B CARPENTER, V P R.S, "of Dredging Operations in the Seas to the North of the British Islands, carried on in Her Majesty's Steam-vessel 'Lightning,' by Dr. CARPENTER and Dr WYVILLE THOMSON, Professor of Natural History in Queen's College, Belfast" Proc Roy. Soc vol xvii p. 168, Appendix, p. 197, 1868–69.

"On the Rhizopodal Fauna of the Deep Sea," by W. B. CARPENTER, M.D., V.P R.S. *Ibid* vol. xviii. p. 59.

"Preliminary Report of the Scientific Exploration of the Deep Sea in H.M. Surveying-vessel 'Porcupine,' during the Summer of 1869, conducted by Dr. CARPENTER, V P R.S., Mr J. GWYN JEFFREYS, F.R.S., and Prof. WYVILLE THOMSON, LL D, F R.S" *Ibid* vol. xviii p 397

"Report on Deep-sea Researches carried on during the Months of July, August, and September 1870, in H.M Surveying-ship 'Porcupine,'" by W B CARPENTER, M D, F R.S., and J. GWYN JEFFREYS, F.R.S. *Ibid.* vol. xix. p 146

"Report on Scientific Researches carried on during the Months of August, September, and October, 1871, in H.M Surveying-ship 'Shearwater,'" by WILLIAM B CARPENTER, LL.D., M.D., F.R.S. *Ibid.* vol. xx. p. 535.

FAHRENHEIT, and Centigrade, and the depths are recorded in feet, fathoms, the 'old French foot,' 'toise'*, 'brasse,' 'mètre,' and the 'yaden,' while the longitude is sometimes that of Greenwich, at others that of Paris. I have reduced these various measures to a common scale, adopting for temperatures that of FAHRENHEIT; for length, the English foot, and for longitude, the meridian of Greenwich As in these reductions some errors may have crept in, references are given to all the original readings.

In the Lists of Observations (pp 639–70) the degrees of temperature at depths stand as they are recorded by the several observers, without the correction adopted for the Sections. The place of each observation is laid down on a recent Admiralty Chart of the world (Plate 65), in accordance with the longitude and latitude given by each observer, without any attempt at correction, which, in some of the early observations, may possibly be necessary.

The observations thus reduced are tabulated in three groups. Table I gives the deep-sea temperatures in the Northern Hemisphere from the Equator to the Polar Circle, and in the same way Table II. gives those in the Southern Hemisphere. The observations in inland seas are given in a separate Table, No. III.

A list of temperature-soundings, made up to date, was given by PÉRON in 1816. It was limited to 4 of his own, and to 16 of FORSTER'S and IRVING'S† In 1832 D'URVILLE‡ gave a greatly extended list, embracing as many as 421 observations, which he arranged according to zones of depth, and in 1837 GEHLER§ published a list of 226 observations arranged according to latitude These, I believe, constitute all the general lists that have been published. The number of observations recorded in the present Tables amount to 1356

In the following pages I have given :—first, a notice of the many voyages on which soundings for deep-sea temperatures were taken, with an account, when possible, of the mode in which the observations were made; secondly, a summary of the opinions founded on these data, and thirdly, a statement of the results obtained and of the conclusions to be formed thereon

Besides the error due to pressure, which, as so many of the older soundings were made at small depths, is frequently unimportant, there is that arising from the angle of the line from the vertical caused by currents, and another due to the tension of the rope by strain and wet, which is sometimes not inconsiderable. I have, however, in drawing the sections, given the depths without correction for these causes, so as to place all the observations on the same footing, as it is but rarely, although there are exceptions, that these particular sources of error were noticed or mentioned ||.

* The Old Foot=12 79 inches, the Toise=76 68 inches, the Brasse=63·93 inches.

† Voyage de découverte aux Terres Australes, vol. ii p. 327

‡ Voyage de l'Astrolabe, vol. x Chap III. Physique

§ GEHLER'S Physikalisches Worterbuch. Sechster Band, Dritte Abtheilung, Mc–Mj, pp 1676–82.

|| The older deep-sea soundings have been found to be liable to serious error, arising from the difficulty in actual fixing the depth of sounding, but in these Tables there are few of that depth to involve this particular error; still some of the deeper ones must be looked upon with doubt.

Owing to the want of a reliable self-registering thermometer, three plans were resorted to by the earlier observers to ascertain the temperature of the sea at depths below the surface. The first and more common plan was to bring up water from a determined depth in sufficient quantity and with sufficient speed to prevent any material change of temperature, and then to try it with an ordinary thermometer, although sometimes the thermometer was suspended in and descended with the water-bucket. In the second place, the thermometer was surrounded with a non-conducting substance, and left down a length of time sufficient to acquire the temperature of the surrounding medium and then brought rapidly to the surface In the third place, the temperature was taken by means of mud or silt brought up from the bottom. On a few occasions metallic thermometers have been tried, but not with satisfactory results. These several plans continued in use from time to time up to a comparatively late period, until gradually superseded by self-registering thermometers.

As the error due to pressure in the use of the latter instrument has now been determined with sufficient accuracy, most of the older observations can readily be subjected to correction. Such correction has been applied to all the observations that have been used in constructing the Sections, Plates 66–68, but, as in the Tables themselves the original readings are given without correction, in order to obtain in any case, with a few exceptions named, an approximately true reading, the correction given at p. 612 must be applied. Where, from the use of proper precautions, the original readings are presumed to be correct, they are distinguished by being placed between brackets in the Sections.

§ II *Historical Narrative of Deep-sea Observations*, 1749–1868.

In this chapter I have enumerated in chronological order the various voyages on which I have found any record of deep-sea temperatures—stating generally the course gone over, the number of observations made, the depths attained, the methods employed At the end the correction for the errors attendant on these methods is determined. The particulars of the observations taken on each voyage will be found in the Tables by reference to Column VIII, under which is given the name of the officer in command, or of the scientific observer accompanying the expedition. The conclusions formed by them on these data are reserved to the next chapter.

It was about the middle of the last century that the subject of deep-sea temperatures first began to attract attention. In 1749 Captain Ellis, on the occasion of a voyage to the north-west coast of Africa, made two experiments at depths of 3900 and 5346 feet in lat. 25° 13′ N.*, with an instrument devised by Dr. Hales, and described by him in a paper to the Royal Society† It consisted of a bucket about the size of an ordinary pail, with valves at top and bottom, which remained open as the apparatus descended, and closed as it ascended. He obtained in both cases readings of 53°; and he rightly attributed this uniformity to the greater depth of water through which, in the deeper experiment, the instrument had to be hauled, and which caused a larger gain of heat.

* Phil. Trans. for 1751–1752, vol. xlvii. p. 214.

† *Ibid.* p. 213.

No further attempts of the kind seem to have been made until 1772, when COOK* went, with FORSTER† as naturalist, on his first voyage round the world. They each separately record three experiments made, at depths of 600 feet, between the equator and 64° South latitude, and they both recognized the decrease, within certain latitudes, of the temperature with depth. From some unexplained cause, the experiments were soon discontinued. No mention is made either in COOK's or FORSTER's narrative of the instruments used, except that the latter alludes (p. 45) to the use of thermometers, while PÉRON speaks (p. 318) of FORSTER's "cylindre à double soupape," so it may be presumed that he used HALE's apparatus with an ordinary thermometer enclosed in it. The apparatus was left at the bottom from 15 to 30 minutes.

In 1773, on the occasion of Captain PHIPPS's‡ voyage to Spitzbergen, he was furnished by the Royal Society with instructions how "to direct his inquiries." Sailing past Shetland and the Faroe Islands, to the west and north coasts of Spitzbergen, he reached 80° 48′ N. latitude. Dr IRVING, who accompanied the expedition, made nine observations at depths varying from 192 to 4098 feet, and extending from the German Ocean to the north of Spitzbergen. They first of all used thermometers contrived by Lord CHARLES CAVENDISH§ in 1757. They were on the principle of overflow thermometers, which registered the temperature by subtracting from a column of mercury of given length the portion which passed over into an attached receiving bulb, and comparing the instrument before and after with a standard thermometer, but, owing to its delicacy, difficulties of manipulation, and errors by compression, this instrument was soon abandoned. IRVING then devised a water-bottle with a coating of wool 3 inches thick, and shutting inside with a cone of lead when at the bottom. The temperature was taken when brought to the surface. For moderate depths the results, which are recorded in the Tables, seem to have been tolerably correct. Those obtained with CAVENDISH's thermometer are, on the contrary, so discordant‖ that I have not included

* Voyage towards the South Pole, 1772–1775. By Capt. COOK. 2nd edit. London, 1777, pp. 25, 29, 39.

† Voyage round the World, 1772–1775, in H.M.S. 'Resolution.' By GEORGE FORSTER, F.R.S. London, 1787, vol. i. pp. 48, 50, 51.

‡ A voyage towards the North Pole, undertaken by His Majesty's commands in 1773. London, 1774. Appendix, pp. 141–7.

§ Phil. Trans. vol. l. p. 308, and vol. liv. p. 261.

‖ I annex them here, for the purpose of record, with the correction for compression and unequal expansion of spirits afterwards introduced by CAVENDISH and applied by PHIPPS.

	North Latitude	East Longitude	Depth in feet	Temperature in degrees of Fahr.		
				By therm.	Corrected	Air
1773, June 20	67° 5′	0° 46′	4680	15°	26°	48° 5′
" " 30 A.M.	70 8	10 55	708	30	31	40 5
" " 30 P.M.	70 8	10 ?	690	33	33[1]	44 75
" Aug. 31	69 0	0 18	4038	22	32	59 5

[1] In this experiment the water brought up in IRVING's water-bottle gave a reading of 38° 5.

them in the Tables. The general conclusion PHIPPS and IRVING drew was that, except in Arctic seas, the temperature decreased with the depth.

In 1780 SAUSSURE made the two first observations on the temperature of the Mediterranean *—one off Genoa at a depth of 944 feet, and the other off Nice at a depth of 1918 feet Both the thermometers marked 55° 8, or, allowing his correction, about 55°·5, a singularly close approach to the more recent observations of AIMÉ and others. SAUSSURE used a spirit-thermometer of RÉAUMUR's with a large ball, which he surrounded with a mixture of wax, resin, and oil 3 inches thick, and the whole was then placed in an iron-wire cage. In both cases he sunk the thermometers at 7 o'clock in the evening, and left them down until 7 in the morning, so that they might acquire precisely the temperature of the surrounding water. The one sunk 1918 feet deep took twenty-four minutes to haul in, and he inferred that this would give the true temperature within a fraction (one fifth) of a degree The thermometer was specially made and graduated for the experiment; and he had previously ascertained that after lowering it to a temperature of 2° 3 R, and arranging so that by constant moving it traversed 1000 feet of water at 14° R in ten minutes' time, the instrument had only risen one tenth of a degree, or to 2° 4

In 1800–4 a voyage of circumnavigation was undertaken by command of the Emperor Napoleon. Monsieur F PÉRON† accompanied it as naturalist and physicist, but, owing to the indifference of the officers and ill-will of the men, he was unable to make more than 4 uncertain experiments, all in the tropical seas, and at depths only of from 320 to 2270 feet, the lowest temperature recorded being 45°·5 in lat 4° N M. PÉRON, not satisfied with former methods, employed a mercurial REAUMUR's thermometer, placed in a glass cylinder, with cotton-wool to protect it. This was enclosed in a wooden cylinder sufficiently large to allow of a packing between the two of powdered charcoal, and then put in a tin case, which was wrapped round with oil-cloth. The value of the results to be obtained by such protected instruments necessarily depends, as in the case of SAUSSURE's experiments, upon leaving the thermometer down for some hours; but in one case only was M PÉRON allowed to leave his apparatus down 1 hour 50 minutes, and once he had to haul it up after five minutes' submergence. PÉRON refers to and tabulates the experiments of his predecessors, and remarks on the same law of the temperature decreasing from the surface downwards.

In 1803 the 'Neva' sailed on a voyage of circumnavigation, under the command of Captain KRUSENSTERN. Touching at Falmouth, he passed round Cape Horn to the Sandwich Islands, Kamtschatka, Japan, and back by the Cape of Good Hope. KRUSENSTERN took out with him an apparatus made in St Petersburg on the model of HALES's; but this was abandoned for SIX's self-registering thermometer, which, although invented in 1782, was now for the first time employed at sea. Some thirty experiments were made by

* Voyages dans les Alpes. Neufchâtel, 1796, vol. iii. pp. 153 & 196.

† Voyage de Découvertes aux Terres Australes en 1800–4, rédigé par M. F. PÉRON, Nat. de l'Expéd. Paris, 1816, pp. 334–37.

him and Dr. HORNER in the tropical regions of the Pacific* and the Sea of Okhotsh. We have no description of his water-bucket, and are therefore without means of judging of the exact value of the results. The more numerous experiments made, on the other hand, by Dr. HORNER† with SIX's thermometer admit of correction.

A subject of so much interest did not escape the attention of SCORESBY; and he gives a Table of the twenty-four observations made by him in the seas around Spitzbergen, during his several voyages to the Arctic Ocean between 1810 and 1822, at depths varying from 78 to 4566 feet‡. He made use of an apparatus (no doubt based on that of HALES) consisting "of a cask capable of holding 10 gallons of water, composed of 2 inches of fir plank, as being a bad conductor of heat." Each end of the cask was furnished with a valve, these were connected with a wire so as to move simultaneously. They opened in descending and closed in ascending. The cask was allowed to remain down half an hour, and was hauled up briskly. A common thermometer was then used to ascertain the temperature of the water so brought up. This machine soon, however, got out of order, and he had one cast in brass, 14 inches in length by 5½ inches in diameter, which he called a marine diver. This he employed in all his experiments on and after the 1st May, 1811. A SIX's thermometer was enclosed, which could be read off through two glass sides in the "diver" on coming to the surface. The weight of the machine was 23 lbs. He recognized in these seas a uniform though slight increase of temperature from the surface to the greatest depth he attained, the temperature at the surface being generally 28° to 29°, and increasing in descending to 36° and even 38° (uncorrected). In a subsequent voyage he gives, however, an experiment made 7° or 8° further south, and off the coast of Greenland, in which the reverse held good, the surface-temperature being 34°, and at a depth of 678 feet 29° §.

Objections have been raised to SCORESBY's experiments, on the ground that they do not accord with those of MARTINS and BRAVAIS, which were made with more exact modern instruments. But these observers themselves accept SCORESBY's observations as true, subject to small corrections. The differences between them are, in fact, more apparent than real, and arise chiefly from the circumstance that their observations were made in the months of July and August, when the temperature of the air averaged from 35° to 45°, and that of the surface-water from 38° to 42°, whereas SCORESBY experimented in April and May, when these had temperatures respectively of 20° to 34° and of 28° to 30°, so that the relative differences between the surface and the deep waters are necessarily very different in the two cases. In the experiments at depths below 2000 feet there is little discordance after applying the corrections employed by MARTINS and BRAVAIS. The latter, however, took no depth exceeding 2854 feet, while SCORESBY gives

* Voyage round the World in the years 1803–6. English translation. London, 1813, vol. i. pp. 187 & 203.

† HORNER's observations are recorded by GEHLER (note, p. 589). They are given under his name, and not that of KRUSENSTERN, in the Tables. See also the original work of KRUSENSTERN.

‡ Account of the Arctic Regions. Edinburgh, 1820, p. 187.

§ Journal of a Voyage to the Northern Whale Fishery in the year 1822. Edinburgh, 1823, p. 237.

two exceeding 4000 feet; and these were made at some distance from those of the French observers, who experimented chiefly between Norway and lat. 76° N., whereas SCORESBY's observations were mostly north of that latitude, and in the sea west of Spitzbergen as far as 80° north.

It is easy to determine the depth at which, in inland seas like the Mediterranean, the effect of the diurnal variation of temperature ceases, but it is a much more difficult problem in Arctic seas. Exposed to the low temperatures of an Arctic climate, the surface-waters may continue to sink until their temperature is reduced to 25°·4, the point at which they attain their maximum of density. This, however, can only happen in a state of perfect calm or with waters of unusual saltness, as sea-water of the usual specific gravity freezes under ordinary conditions at 27°·4 F., though it has been shown that in a state of perfect rest it may be reduced to 20°, or even lower before freezing.

Under these conditions, and with the complicated action of warm currents from the south and of cold currents from the north, we must expect to find considerable variation in the temperature of the Arctic Ocean, down, at all events, to the depths hitherto reached of 4600 feet. Judging from the conditions prevailing in the Antarctic seas and the sea of Baffin Bay, it seems probable that more uniform readings will be obtained at greater depths, and that the anomalous readings in the upper strata are caused by the warmer waters which flow in from the south tending to take at and near the surface the temperature of the air at different seasons, while the deeper part of this mass of warm water remains unaffected, and in the deeper channels there may be, flowing from the north, the more permanent body of cold water produced by the winter refrigeration of the polar seas of still higher latitudes.

Subject to the corrections for the causes before named, SCORESBY's experiments command confidence. The effect of the corrections will be to reduce his readings where SIX's thermometer was used, while where the water-bucket alone was used a small addition may be generally needed.

In the mean time (1815–18) another Russian voyage of circumnavigation *, under the command of OTTO VON KOTZEBUE, was undertaken for scientific purposes. One hundred and sixteen carefully conducted experiments (often taken from day to day) were made in both the great oceans and amongst the islands of the Eastern Archipelago. These observations, many of them serial, taken at depths of from 24 to 2448 feet, were tabulated in the order of date. On this voyage KOTZEBUE used English-made (JONES) SIX's thermometers. They were protected by a wooden case closed with "wire grating," but not in any other way, and they were fastened on the sounding-line about 6 feet above the weight. KOTZEBUE considered that "seven or eight minutes suffice to give it the temperature of the surrounding water, and a quick or a slow pulling up has no effect on the observation" (vol. i. p. 89).

* Entdeckungs-Reise in die Sud-See und nach der Berings-Strasse zur Erforschung einer nordöstlichen Durchfahrt auf dem Schiffe Rurick (Weimar, 1821), dritter Band, von dem Naturforscher der Expedition Dr. CHAMISSO, Tables, p. 230; and Dr. HORNER's Report thereon, p. 233.

In 1816 Captain WAUCHOPE made two observations in the Atlantic, a few degrees north and south of the equator, at depths of 2880 and 6060 feet, and records temperatures at those depths of 51° and 42° *. The apparatus he used consisted of "a series of cases, one within the other, having valves opening up so as to allow the water to pass through in descending, but which closed in hauling the instrument up. The thermometer was enclosed in a glass tube in the centre of it." Elsewhere he mentions that the cases were $\frac{1}{4}$ of an inch apart, except the outer one, which was $\frac{1}{2}$ an inch, and that one was filled with tallow. This was enclosed in a case of wood 1 inch thick. The machine was 2 feet high by 10 inches in diameter. The time it took to haul up was from twenty minutes to one hour and twenty minutes After all, as SIX's thermometer was used, the correction to be applied is rather that due to pressure than to the change of medium. In measuring the depth, Captain WAUCHOPE allowed for the angle of the rope from the vertical.

In 1817, on the occasion of the voyage of the 'Alceste' to China, a few experiments were made by CLARKE ABEL † in the shallow waters of the Yellow Sea. No particulars of the methods he adopted are given.

In 1818 attention was again directed in this country to the Arctic seas, and the 'Isabella' and 'Alexander' were despatched to Baffin Bay, under the command of ROSS ‡ and PARRY, and the 'Dorothea' and 'Trent' to Spitzbergen, under BUCHAN and FRANKLIN §. As many as 72 valuable observations on deep-sea temperatures and soundings were made by the several commanders, assisted by SIR EDWARD SABINE, who accompanied Ross, and by BEECHEY and FISHER, who accompanied FRANKLIN. Some of these are recorded in the narratives of the several voyages, and the others are given by Dr. MARCET in his well-known paper "On the Specific Gravity and Temperature of Sea Waters" published in 1819 ||.

Sir JOHN ROSS adopted the plan of taking the temperature of a mass of mud or silt brought up from the bottom. For this purpose he contrived what he called a deep-sea clamm. It consisted of "a cast iron parallellogram" 18 inches high by 6 inches wide on the outside; inside 5×4 in. It weighed 1 cwt., and would bring up about 6 lbs. of mud. By this means, a bottom-temperature generally of 29°·5, and in one case, at the depth of 6000 feet, as low as 28°·75, was determined in Baffin Bay. This degree of cold was generally corroborated by a SIX's thermometer, both instruments apparently giving the same or nearly the same reading. It was on this occasion that the

* Mem Wernerian Nat. Hist. Soc. vol. IV p. 163

† THOMSON's Annals of Philosophy for 1819, vol. xiii. p. 314

‡ Ross's Voyage of Discovery to Baffin's Bay in 1818. 2nd edition. London, 1819. Appendix, xi pp. 234–236. Appendix, xiii. p. 250.

§ For a Table of the temperature of the Sea at various depths, taken during Capt. FRANKLIN's Voyage to Spitzbergen with Captain BUCHAN, see Edinburgh Phil. Journal for 1825, vol XII p 233.

|| Phil. Trans. for 1819, p. 161; FRANKLIN, table vi. p 203, BEECHEY, table VII p 203, FISHER, table VIII p. 203; PARRY, table x. p. 205, SABINE, table xi. p. 205 These are marked 'M' in the Tables.

remarkable low temperature of 25°·75 F. was recorded, at a depth of 4080 feet in Davis Straits, by Sir JOHN ROSS and Sir EDWARD SABINE*.

BUCHAN and FRANKLIN employed, on the suggestion of Mr. FISHER, a leaden box with two valves, which remained open in descending, and were closed in the ascent. No other particulars are given, but there is every probability that it was constructed on the model of those of HALES and SCORESBY. Their observations, with one or two exceptions, are, allowing for the difference of season (June and July), in tolerable agreement with those of SCORESBY; but they seem less carefully made, and to require, I suspect, a larger correction

In 1819–20 PARRY went out in command of the 'Hecla' and 'Griper'†, and penetrated the Arctic seas of North America as far as 113° W. long. He took several deep-sea temperatures on board the 'Hecla,' whilst Sir E. SABINE, on board the 'Griper,' made another series of observations. Mr. FISHER, who published an account‡ of the voyage, also notes some of those on board the 'Hecla.'

About this time Sir HUMPHRY DAVY suggested another contrivance for bringing up water from depths, which seems to have been used occasionally by ROSS and PARRY, but the observations with it are not specified. On the occasion of PARRY's voyage in 1819, Dr MARCET contrived his water-bottle, which PARRY appears to have occasionally employed, especially in 1821–23 (p. xvi), "in consequence of the failure of the thermometer when exposed to sudden changes," although elsewhere he says (p. xiii) that the temperature was taken, unless otherwise noticed, by SIX's thermometer Owing to the very small size (half a pint) of DAVY's and MARCET's water-bottles, and their being of metal, they were valueless for temperature-experiments§, although useful for obtaining small samples of deep-sea water, and they were consequently, with this exception, but little used for the former purpose.

In PARRY's second voyage of 1821–23|| he records a series of twenty-three experiments made in one of the inland seas of Arctic America, at depths of from 600 to 1200 feet. These show a temperature of from 29° to 31°·7 on the surface, and a like temperature,

* On reference to Sir E SABINE he informs me, from a note made at the time, that on bringing up the thermometer the index marked 25¾°, and that never having known it lower than 28°, he was very careful in examining the instrument, that both he and Captain Ross were on the spot, and that Captain Ross remarked, in drawing it out of the tin case, which was full of water, that the mercury was close up to the index. It fell instantly and rapidly, but Sir EDWARD had the same belief, that when he first looked it was close up to the index (See also Dr CARPENTER in Proc Roy Soc vol xvii p 187.)

† Voyage for the Discovery of a North-west Passage, 1819–20, in the 'Hecla' and 'Griper' By Captain PARRY, 2nd edit 1821, pp 4, 5, 6, 7, 45, 115, 261, 271, 272, 273, 289, 291, 292, 293, 294, 295, 307.

‡ Journal of a Voyage of Discovery in the Arctic Regions in H.M.S. 'Hecla' and 'Griper' in the years 1819–20 By A FISHER, Surgeon 3rd edit London, 1821.

§ FISHER, *op cit* p. 17.

|| Journal of a Second Voyage for the discovery of a North-west Passage from the Atlantic to the Pacific, performed in the years 1821–23 in H.M.S. 'Fury' and 'Hecla,' under the orders of Captain W. E. PARRY. London, 1824, p. 483.

or one only 0°·5 less, at the bottom. As, however, there is little doubt that all these observations in Lyons Inlet were made with MARCET's bottle, no reliance is to be placed on them *. In his third and last voyage of 1827 †, PARRY made as many as forty-five observations in the seas west and north of Spitzbergen, but none exceeded 700 feet in depth. With few exceptions, they show a lower reading than those of SCORESBY. On this occasion he reverted to the use of SIX's thermometers.

From PARRY's observing on his first voyage that his soundings were made with "SIX's self-registering thermometer confined in iron cases" ‡, and again, on his second voyage, "that he took out eight SIX thermometers with iron cases" §, I was led, in consequence of the low readings, to think that these cases might have been used for protection against pressure, but Sir EDWARD SABINE, who was with ROSS in 1818 and with PARRY in 1819, being in the latter expedition on board Captain CLAVERING's ship, the 'Griper,' informs me that all the observations were there made in concert between him and Captain CLAVERING, and that he had with him "half a dozen thermometers on SIX's construction, made expressly for him by the elder JONES, each of which fitted into (and was retained by an apparatus at top and bottom) a *tinned iron cylinder* pierced with holes in the top and bottom, through which the sea-water percolated freely. . . . The holes in the top and bottom of the cylinder were rather less in diameter than a seven-shilling piece, admitting a free current A weight attached to the rope at some little distance below the thermometer, caused the line to run out freely, and prevented the occurrence of 'kinks'" ||.

It is therefore to be presumed that the iron cases referred to by PARRY were merely to guard the instruments against accident, and not against pressure, and on comparing the observations made by him on board the 'Hecla,' often on the same day and nearly on the same spot, with those of Sir EDWARD SABINE in the 'Griper,' I find them in such close agreement as to satisfy me that such was doubtlessly the case At the same time

* Of the 23 readings recorded, ten give precisely the same temperature at depths of 600 to 1200 feet as was found on the surface, while the others in no instance show a difference of more than 1°, and generally of not more than 0°·5; whereas an inland sea in those latitudes might be expected to show extremely low temperatures at depths.

† Narrative of an attempt to reach the North Pole in the 'Hecla' in the year 1827. By Captain W E PARRY. London, 1828, Appendix vii

‡ *Op. cit* Introd p. xiii

§ *Op cit* Introd p. xvi

|| Sir EDWARD SABINE thus describes the mode of proceeding in making the temperature-soundings —"The cylinder, having the thermometer enclosed, was attached to the sounding-line, and was dropped into the sea from the extremity of a spar run out from the side of the ship, the line to which it was attached passing round a pulley near the end of the spar. In a similar way the cylinder when coming up from the bottom was waited for by a boat near the end of the spar, the cylinder released, and conveyed carefully by hand in an upright position to Capt. CLAVERING or myself at the gangway (or by ourselves), by whom the degree recorded by the index was immediately noted The record by the thermometer was then written down on the spot antecedently to any discussion or comment, the record being made either by Capt CLAVERING or myself. The spar from the end of which the thermometer case was dropt into the sea was always several feet distant from the side of the ship."

there is reason to believe that thermometers of stronger make than usual, and so better adapted to resist pressure, were used by Ross and Parry in their voyages of 1818–19*. The usual correction, therefore, cannot be applied to their observations of that date. Little or none may be needed.

In 1822 Sir Edward Sabine made an observation on the temperature of the Caribbean Sea at a depth of 6000 feet (the actual length of rope was 7380 feet, but of this 1380 feet were allowed for slack and drift), and a reading of 45°·5 F. was obtained†. On another occasion on this voyage, Sir Edward used a solid iron case to protect the thermometer against pressure, but it did not prove sufficiently close to exclude water.

In 1823–26 Kotzebue commanded another voyage of circumnavigation‡, and on this occasion he was accompanied by Emil. von Lenz, who subsequently published several important memoirs on the deep-sea temperatures and on the specific gravity of sea-water taken on this occasion§. His observations are remarkable from their being made at greater depths and their recording lower temperatures than any others made up to that time, or, in fact, until long subsequently, in tropical seas. One observation, in the Pacific, 21°·14 north latitude, indicated at a depth of 5835 feet, by his corrected reading, a temperature of 36° 4 F., and another, 6476 feet deep, in the Atlantic, 32°·20 north latitude, gave 35°·8 F.

Although only fifteen observations were made, they were mostly at considerable depths, and they were all taken with various precautions and subjected to careful corrections‖.

On his first voyage Kotzebue experienced so much trouble with the self-registering thermometer, owing to the mercury passing over the index and to the shifting of the index from jolts or shaking, that on this second voyage Lenz reverted to Hales's mode of taking deep-sea temperatures, using an improved apparatus arranged by Parrot, the Russian Academician¶. The apparatus, which he termed a bathometer, was 16 inches high by 11 inches in diameter, and held 27·49 kil (six gallons) of water It had valves at top and bottom opening upwards, and connected by a rod, to which was attached a mercurial thermometer made specially to bear pressure, with a ball 5 lines thick. The apparatus was covered over with four alternating layers of sheet iron and canvas, saturated with a mixture of boiling tallow and wax, and the whole enveloped in a cloth painted over several times. It was calculated to bear a pressure of 3000 toises (19,150 feet), and the practice was to leave it at the bottom 15 minutes. It was

* See also 'Depths of the Sea,' p 300.

† Phil. Trans. for 1823, p 206 See also his 'Pendulum and other Experiments.' London, 1825.

‡ Voyage round the World. English translation. London, 1830.

§ Annalen der Physik und Chemie, Band xx. 1830, pp 73–131; Edinb. Journ of Science, vol. vi. 1832, pp. 341–45; and St Petersburg Ac. Sc Bull. v 1847, col. 65–74

‖ Physikalische Beobachtungen angestellt auf einer Reise um die Welt unter dem Commando des Capitains von Kotzebue in den Jahren 1823–26. St. Petersburgh Acad. Sci. Mém. i. 1831, pp. 221–334.

¶ There are but few observations given in the English Translation of the Voyage (vol. i. pp. 24 & 29, and vol. ii. p. 4), and it is not stated whether or not they are corrected. To these the name of Kotzebue is attached in the Tables; the others made on this voyage are on the authority and in the name of Lenz.

found to leak slightly; but it was considered that the expansion of the water in coming to the surface would compensate for this loss.

This instrument was placed in water at 67° F. (19° 4 C.) until it acquired its temperature. It was then replaced with other water at 32°. Left in it for two hours, the temperature of the water in the bathometer fell to 52°·7, showing a difference, in that time, of 14°·3, which difference LENZ further estimated would have amounted only to 7° had the apparatus passed through water ranging from 32° to 67°, instead of being exposed to a constant temperature of 32°. Taking this as the rate of refrigeration at given temperatures and in given time, LENZ then employed BIOT's formula for ascertaining the gain or loss of heat of a body placed in a medium possessing a higher or lower temperature than itself, as the basis for calculating the correction required in each particular observation. Corrections were also made for the depths, by allowing on the one hand for the angle of the rope from the vertical, and on the other for the gain in length by tension under water*.

LENZ gives a Table of his observations as originally taken, and again repeats the Table with the corrected temperatures and depths. These two are combined in the following Table, in which it is shown that, even with uncorrected readings, LENZ obtained on three occasions a temperature below 4° C., while six corrected readings indicate a temperature below 3° Cent, or of from 36° to 37° Fahr.

Date	Lat	Long	Depth in toises	Angle of rope	Temperature.		Time employed in hauling up the instrument	Time of its remaining at the bottom	Corrected observations	
					At surface.	At depth			Depth	Temp
							min	min	toises	
1823 Oct 10	7° 20′ N	21° 59′ W	500	0° 0′	25°8 C	5° C	30	15	539	2° 20 C †
1824. May 18	21 14 „	196 1 „	139	10 0	26 4	16 7	6	15	140	16 36
„ „	„ „	„ „	399	0 0	„	5 1	17	10	413	3 18
„ „	„ „	„ „	649	10 0	„	4 9	32	10	665 1	2·92
„ „	„ „	„ „	979	25 0	„	4 6	56	15	914 9	2 44
1825. Feb 8	25 6 „	156 58 „	179	25 0	21 5	14	3	2	167	14·00
1825. Aug 31	32 6 „	136 48 „	89	10 0	21 45	13 54	4	15	89·8	13 35
„ „	„ „	„ „	229	25 0	„	7·06	8	15	214	6 51
„ „	„ „	„ „	479	25 0	„	4 75	15	15	450 2	3 75
„ „	„ „	„ „	579	10 0	„	3 56	19	15	592·6	2·21
1826. Mar 6	32 20 „	42 30 „	969	5 10	20·86	3·02	50	15	1014 8	2·24
1825 Aug. 24	41 12 „	141 58 „	199	10 0	19 2	5·9	10	15	205	5 16
„ „	„ „	„ „	525	20 0	„	3·4	25	15	512 1	2 14
1826. Mar 24 .	45 53 „	15 17 „	192	0 0	14·64	10 56	9	15	197 7	10·36
„ „	„ „	„ „	383	0 0	„	10·26	13	8	306 4	9·95

† This should probably be 3°·20

* POGGENDORFF's Annalen der Physik und Chemie, vol. xx. 1830, pp 78, 90, 106, and 'Bulletin Universel' for 1831, vol. i. p. 275.

In 1825 an important expedition*, under the command of Captain BEECHEY, was despatched by the Government round Cape Horn to the Pacific Ocean and Behring Strait. Aided by Mr COLLIE, the Surgeon, a large and valuable series of meteorological observations were made, including ninety-seven single and serial experiments on deep-sea temperatures in the North and South Atlantic and North and South Pacific, ranging from lat. 56° S to 70° N., and at depths from 30 to 5124 feet. These were arranged in Tables according to latitude for each ocean. No very low temperatures are recorded, but the decrease with the depth is persistent, SIX's thermometers were used, but no particulars are given of how they were used †.

The great voyage of Admiral FITZROY from 1826 to 1836, productive as it was of such valuable results in other branches of science, added little to our knowledge of deep-sea physics. Only two sets of observations, both serial, were made in the Indian Ocean at depths of from 30 to 2500 feet‡. SIX's thermometers are mentioned, but without any other particulars

In 1826–29 also another important surveying and exploring expedition § proceeded from France under the command of Captain DUMONT D'URVILLE, aided by a staff of scientific officers. He was instructed by ARAGO to pay particular attention to deep-sea soundings and temperatures, and informed of the precautions essential in making such observations D'URVILLE proceeded from Toulon through the Straits of Gibraltar to Teneriffe, across the Indian Ocean to Australia, New Zealand, the Eastern Archipelago, and back by the Mauritius, the Cape, and Ascension, making observations in all the seas he traversed, at depths varying from 50 to 6160 feet, and to the number altogether of 66, the lowest temperature recorded being 40°. These he tabulated according to zones of depth; and he incorporated also in his Tables the experiments of all preceding observers, beginning with COOK and FORSTER. D'URVILLE concluded from his observations that in the open ocean the temperature at and below 3198 feet (600 brasses) *is nearly constant between* 39° *and* 41°—*that it might be perhaps* 40° FAHR. He also supposed that *a belt of this uniform temperature* existed between the latitudes of 40° and 60°. D'URVILLE was evidently led to this hypothesis of a zone of uniform temperature from assuming the greatest density of sea-water to be, as with fresh water, between 39° and 40°. His observations in the Mediterranean confirmed those of SAUSSURE, viz. that the waters of that sea, below the depth of 1000 feet, had a uniform temperature of about 55°.

In 1828 GRAAH made a few observations in the North Atlantic ||, but no particulars are given of the instruments he used.

* Narrative of a Voyage to the Pacific and Behring Strait in H.M.S. 'Blossom' in 1825–28. London, 1831. Appendix, Table X. p 731.

† Sir EDWARD BELCHER, however, tells me that Captain BEECHEY's thermometers "were enclosed in copper cases with tow above and below," and that no protection against pressure was employed.

‡ Narrative of the Surveying Voyage of H.M S. 'Adventure' and 'Beagle.' Appendix to vol. ii. p. 361

§ Voyage de l'Astrolabe, vol v of Météorologie, Physique, et Hydrographie. Paris, 1833. Chapter III Physique, pp 51*–85*

|| Narrative of an Expedition to the East Coast of Greenland London, 1837, p. 21.

BÉRARD, in 1831–32*, made another series of observations in the Mediterranean, and ascertained that the temperature of about 55°, noted by SAUSSURE and D'URVILLE at depths of from 1000 to 3000, prevailed to the depth of 6400 feet.

I cannot ascertain precisely when protection against pressure on the thermometer was first used. PARROT and LENZ† made experiments on the effects of compression on thermometers in 1832, and found that in ordinary instruments they were excessive, but this did not apply to SIX's self-registering thermometer, which from its form of construction offers much greater resistance to compression.

It seems to me, however, that some form of protection must have been adopted by the French several years earlier. It is true that D'URVILLE merely says that he was provided with two of BUNTEN's instruments, and makes no mention of the mode in which they were used, but on comparing his observations in the Mediterranean, where the bathymetrical isotherms are at nearly constant levels, I find his results in such close agreement with those of AIMÉ, obtained with protected instruments, and so free from variation dependent on depth alone, that I can only conclude that D'URVILLE's thermometers were likewise protected. In the same way I infer that BÉRARD also used similar instruments‡. Thus their respective readings give:—

	1826 D'URVILLE.	1831 BÉRARD	1840 AIMÉ.
Soundings in the Mediterranean.	1062 ft. 54°·2 FAHR.	3189 ft. 55°·4 FAHR.	1148 ft. 54°·6 FAHR.
	1594 „ 54°·7 „	3829 „ 55°·7 „	
	3189 „ 54°·7 „	6377 „ 55°·4 „	

For this reason I think it not improbable that the ocean observations of D'URVILLE were made with the same precautions, and need little or no correction.

In 1839 Captain WAUCHOPE§ recorded two more experiments made by him in 1836 in tropical regions at depths of 1800 and 3918 feet, showing respectively temperatures of 52° and 43°. He also surmised that at a certain depth there might be a uniform temperature of about 40° in all seas.

But the most remarkable voyage‖ of the period was that of Captain ABEL DU PETIT-THOUARS between 1836 and 1839. This expedition sailed from Brest in December 1836, touched at Teneriffe, Rio Janeiro, sailed round Cape Horn along the South-American coast, thence to the Sandwich Islands, and back by New Zealand, Bourbon, and the Cape. Fifty-nine observations were made, but eleven failed owing to accidents with

* BÉRARD's observations are taken from AIMÉ's paper quoted further on.

† Expériences de forte compression sur divers corps. Mém. Acad. Sci. St. Pétersbourg, vol. ii. p. 595. F. MARCET and DE LA RIVE (Bibl. Univ. xxii. 1823, p. 265) had before this shown the influence of atmospheric pressure on the bulb of thermometers.

‡ On the 'Porcupine' expedition of 1869 a uniform temperature was noted of 54°·7 to 55°·5 in this area of the Mediterranean at and below a depth of 1000 to 1100 feet. (Proc. Roy. Soc. 1870, vol. xix. p. 221.)

§ Edinburgh New Phil. Journ. vol. xxvi. 1838–39, p. 399.

‖ Voyage autour du Monde sur la Frégate 'La Vénus,' Capitaine DU PETIT-THOUARS. Paris, 1844. Physique, par M. DE TESSAN, vol. ix. Tables, p. 385.

the instruments, and twenty gave wrong readings owing to the great pressure forcing water into the cylinder. Amongst the successful observations, two at a depth of 6600 feet in the Pacific, and of 6000 feet off the Cape, recorded temperatures of 36°·1 and 37°·4; a third in the North Atlantic, lat. 4°·23 and 6406 feet deep, gave 37°·8 F.; while another, at a depth of 12,271 feet near the equator in the Pacific (on which occasion the cylinder was crushed by the pressure and the instrument broken, and the index jammed and fixed), gave a reading of 34° 8 or 35° FAHR.

This was the first voyage in which precautions against pressure were systematically and professedly taken, instruments of special construction were provided. The form adopted was SIX's thermometer, modified by BUNTEN, of Paris. They were enclosed in strong brass cylinders* to protect them from pressure, and they were always left down for half an hour. After the return of the expedition the thermometers were tried with a standard instrument, and found to have a reading only $\frac{2}{10}$ to $\frac{3}{10}$ of a degree Cent. higher than on starting. It was found, however, that the cylinder would not bear a pressure of more than about 12,000 feet, and that at all depths it was occasionally filled with water In these latter cases DU PETIT-THOUARS used a correction of which we shall speak presently, and gives the corrected with the uncorrected reading. Corrections were also made for the angle the rope took with the vertical. There is therefore every reason to suppose that the deep-sea temperatures obtained on this voyage may be accepted as perfectly reliable.

The 'Bonite,' under the command of Captain VAILLANT†, was also despatched from France in 1836 to the Indian Ocean, Chinese seas, and the Pacific. Sixteen observations in the Atlantic and Indian Oceans are recorded at depths of from 244 to 8838 feet. The 'Bonite' was likewise provided with BUNTEN's thermometers. They were wrapped in wool and placed in a glass tube, which again was enclosed in a copper cylinder closed by a screw at each end, and left down 18 to 20 minutes. In the first deep sounding (700 brasses) recorded the cylinder is stated to have come up full of water This throws doubt on all the subsequent experiments; and as no reference at all is made to the state of the cylinder in the other soundings, and the readings are more concordant with the "full cylinder" ones of DU PETIT-THOUARS, I think a correction should be applied to all his deeper observations. A large number of surface-temperatures were taken, and it was remarked again that in the Pacific the sea is more frequently warmer than the air, *except* under the equator.

Another voyage‡ of research was undertaken by France in 1838 to the Arctic seas,

* DU PETIT-THOUARS gives no particulars of the construction of his instruments; but ARAGO, in his report of the results obtained on this voyage, speaks of the "thermométrographe de M. BUNTEN enfermé dans un étui cylindrique en laiton de 33·4 mill. de diamètre intérieur et de 15·6 mill. d'épaisseur," which I presume refers to DU PETIT-THOUARS's instruments —Comptes Rendus, 1840, vol. xi. p. 311.

† Voyage autour du Monde sur la Corvette 'La Bonite,' Capitaine VAILLANT. Géol. et Minér. par M. CHEVALIER, pp 232, 300–1, and Physique, par. M DARONDEAU, 'Observations Météorologiques.'

‡ Voyage en Scandinavie et au Spitzberg de la Corvette 'La Recherche.' Géographie et Physique, vol. ii. p. 279.

and a series of twenty-three interesting experiments were made by MM. MARTINS* and BRAVAIS between the North Cape and Spitzbergen, and off the west coast of that island, in depths of from 200 to 2460 feet.

The principle of overflow differential thermometers had been revived by WALFERDIN† in 1836—a maximum one for the purpose of taking the higher temperatures of deep wells and mines, and a minimum one for deep-sea soundings These instruments were free from the inconveniences of CAVENDISH'S, were of easy manipulation, and could bear jerks without affecting the registering column of mercury. To protect them against pressure they were enclosed in a tube of glass, of thickness proportional to the pressure to which it would be exposed, and hermetically sealed at both ends. M. WALFERDIN claimed for these thermometers greater accuracy and certainty than the ordinary self-registering thermometers‡.

These thermometers, termed "thermomètres à déversement," were used by MARTINS and BRAVAIS on their voyage to Spitzbergen, in conjunction with SIX'S thermometers (thermométrographes) modified by BUNTEN, of Paris. The former were enclosed in glass tubes exhausted as much as possible, and the latter in copper tubes, evidently not strong enough, as they "almost always came up full of water." To ensure accuracy, they employed in all these observations two instruments of each sort, and in some cases as many as four, and took the mean of each set When sunk to the bottom they were raised 1 mètre from it, and left there for an hour. Sometimes the thermometrographs were not protected, and in that case, or when the tubes were full of water, a correction was applied, of which we shall speak further on A correction was also used for the angle of the rope with the vertical. M. MARTINS states that he had much more confidence in WALFERDIN'S thermometers than in BUNTEN'S. I find, however, that, taking the 18 observations made with sets of the former, the average variation for each set amounted to 0°·45 Cent., or, averaging the variation of each of the 52 instruments employed, to 0°·16 C., while the 10 observations with 23 instruments of the latter give respectively 0°·18 C. and 0°·08 C.; but M. MARTINS shows that while the mean of the differences is 0°·19 C at depths not exceeding 131 metres, it is reduced to 0°·06 C. at depths of 640 to 870 metres The readings, on the whole, of WALFERDIN'S instruments are very slightly lower than those of BUNTEN'S; as they were more relied on by the observers, I have given them in the Tables in preference to the others.

But notwithstanding the successful use of WALFERDIN'S instruments on this voyage,

* Voyage de 'La Recherche,' Géogr et Phys vol. ii. (Memoire sur les Températures de la Mer Glaciale à la surface, à des grandes profondeurs, et dans le voisinage des glaciers du Spitzberg, par M CHARLES MARTINS) pp. 342–5. Tableau IV

† Bull Soc. Géol de France for 1836, vol. vii. pp 193 & 354.

‡ He instances a case of a well at Saint-André where, at a depth of 830 feet, two of his instruments gave 17°·96 C. and 17°·93 C respectively, whereas two self-registering instruments gave 19°·2 C and 16°·8. The latter were affected both from water getting into the case and from lowering of the index by shaking. In another case, two of his instruments both registered 23°·5 C., and two thermometrographs 23°·45 and 23°·50, while another of the latter had its index displaced by the shaking of the line.—Ibid. vol. ix. p. 255, vol xii p. 166, and vol. xiii. p. 113.

and the mention of them approvingly by POUILLET* and ARAGO†, I cannot find that they were again used, although a modified form contrived by M. AIMÉ was employed by him in his researches in the Mediterranean in 1840–44.

In 1838 a few observations were made in the Indian Ocean by the Rev. J H. PRATT‡. An American expedition made the round of the world in 1839–42 under the command of Captain WILKES, who gives twenty-eight § deep-sea temperatures at depths of from 60 to 5100 feet in the South Pacific and Southern Oceans, in one case recording a temperature of 27° 5 at a depth of 1420 feet in the latter sea. The subject was afterwards‖ further discussed by Captain WILKES in a separate paper, in which he expressed an opinion that there existed a zone of the uniform mean temperature of 39° 5 FAHR. It would appear that SIX's thermometers without protection were used.

In the same year (1839) a very important expedition was despatched from this country to the Antarctic seas under the command of Captain Sir JAMES C. ROSS. A special code of instructions was drawn up by a Committee of the Royal Society. Numerous results of great value were obtained, especially those relating to the soundings and sea-bed of the Antarctic Ocean As many as 161 deep-sea temperature-observations are recorded, chiefly in the Southern and Antarctic Oceans, with a few in the Atlantic and Indian Oceans¶. They vary in depth from 12 to 7200 feet, some of the soundings were much deeper.

These temperature-soundings claim particular notice in consequence of the undue weight which has been attached to them. In starting Sir JAMES ROSS took with him a supply of SIX's thermometers, but he gives no description of how they were used, or what precautions were adopted **. The observations also are not tabulated, but are scattered through the work without plan or order, and it is at times difficult to fix on their exact position, date, &c It would appear that, owing to want of protection and the great depths at which they were used, all the instruments he took with him from England were broken by the time he reached the Southern Ocean.

* Élements de Physique, 5th edit. vol. ii p. 653 † Œuvres complètes, vol. viii. p. 626.

‡ London and Edinb Phil Mag 1840, vol xvi p 176

§ United States Exploring Expedition, 1839–42 London, 1845, vol i. pp 137, 139, 230, 309, 310, vol ii. pp 290, 293, 299, 332, and vol iii Appendix I

‖ "On the Depth and Saltness of the Ocean," American Journal of Science and Arts for January 1848, p 41.

¶ A Voyage of Discovery and Research in the Southern and Antarctic Regions. London, 1847, vol i. pp 34, 130, 166, 167, 168, 170, 180, 200, 222, 231, 267, 280, 306, 309, 313, 317, 321, and vol. ii pp. 35, 52, 53, 55, 133, 138, 140, 141, 147, 156, 193, 195, 200, 214, 216, 227, 228, 282, 322, 351, 356, 358, 363, 369, 374, 379, 382, 384.

** Dr HOOKER, who accompanied the expedition, informs me that no precautions were taken against pressure, but that to prevent breakage "the thermometers were enclosed in a copper cylinder. Sometimes two thermometers were placed at different points of the same line (say 500 and 1000 fathoms), at others the line was drawn up and sunk again to a greater or less depth The first fathoms of the line were spun yarn, the next of 3 plies of the same, the rest whole line" It was hauled in by the whole ship's company. Dr HOOKER also says "that the average length of time, speaking entirely from memory, during which the thermometers were left at the depths reached was a quarter of an hour."

He then wrote to England for stronger registering thermometers, which were sent to him in Australia, but of which he gives no particulars further than stating that they were stronger. Before receiving these, he apparently renewed his observations with instruments obtained in Australia. Consequently it is probable that each of these sets of instruments were of different construction, and may require a different correction—those used during the first part of the voyage a larger correction than those used during the latter period. In the absence of sufficient information this cannot be attempted, and the general formula given further on has been applied to the correction of all his observations I have had occasion to use in the construction of the sections.

With regard to the observations themselves, they may be also sometimes open to objection in consequence of the great difficulties under which they were so constantly taken. The severe cold, the inclemency of the weather, and the tediousness of the operation are all elements of possible error to be taken into account The one cause may have led at times to the shifting of the index, and the other to some want of accuracy in the reading; for I cannot conceive it possible for any set of thermometers to have recorded, in the innumerable cases mentioned, the same one and uniform temperature of 39°·5 at and beyond a certain depth Even supposing a uniform temperature of that exact degree did exist at certain determined depths, it is in the highest degree improbable that any instruments would give the exact same record There is not only the risk of shifting of the index, but there is the certainty that the ordinary imperfection and variation of the instruments would most certainly prevent it. With the greatest care and with standard instruments especially selected, MM MARTINS and BRAVAIS, out of ten sets of observations each made with two, three, or four thermometers, only give one instance in which the readings of two of them agree. In the other cases they differ from 0°·1 to 1° FAHR

Nevertheless, apart from this point, and supposing them to be approximately correct, the observations of SIR JAMES ROSS are, from their number, depth, and position, very valuable, and, subject to correction, they furnish fairly available results, although, from the cause before mentioned, it may not be certain whether the correction applied gives the true reading in all cases within one, two degrees, or in some cases possibly more. Owing also to this use of unprotected instruments Ross came to the same conclusion as D'URVILLE with respect to the existence of a zone of a uniform surface-temperature in given latitudes, and likewise with respect to the persistence of the same uniform temperature of 39° 5 FAHR. at given depths in the great oceans. In this opinion he seems to have been biassed, similarly with his predecessors, by the belief that the density of sea-water was, like that of fresh water, greatest at that temperature.

In 1840–44 M AIMÉ made a series of important observations on the temperature of the air and sea between Marseilles and Algiers*. The experiments, which were carried on for a series of years, proved that the diurnal variations of temperature in the Medi-

* "Mémoire sur la température de la Méditerranée," Annales de Chimie et de Physique, 1845, 3me sér. vol. xv. p. 1, and Comptes Rendus for Sept 1844.

terranean ceased at a depth of 60 feet, and the annual variations at a depth of from 1150 to 1300 feet. At this point AIMÉ found a uniform temperature of 54°·7, and was of opinion, from the observations of BÉRARD, that no increase took place at greater depths. This degree he showed to be the average of the mean temperature between Toulon and Algiers, of the months of January, February, and March.

In order to determine whether the decrease of temperature was gradual, or whether the instrument passed through warmer strata, AIMÉ also used a thermometer which was let down upright and reversed at the bottom of the soundings. This he termed a "thermomètre à retournement" Besides these, AIMÉ employed the ordinary self-registering thermometer with an enlargement in one part of the tube to remedy the inconvenience of the quicksilver passing over the index. These several instruments were enclosed in copper cylinders strong enough to resist the pressure to which they were subjected. For moderate depths he preferred a glass tube hermetically sealed *

In 1845 Captain (now Admiral) SPRATT made 15 observations † from the surface to a depth of 1260 feet, in the Grecian archipelago, and obtained results in perfect accordance with those of SAUSSURE and AIMÉ in the Western Mediterranean. He afterwards made a more extended series of observations (34 in all) and to greater depths (7440 feet) in the eastern basin of the Mediterranean from Malta to Egypt ‡. Admiral SPRATT at first used SIX's thermometer; but finding that the index often moved, he resorted, in shallow seas of the archipelago, to the plan of taking the temperature of the mud brought up from the bottom by means of a sound formed of iron tubing. This plan, Admiral SPRATT considered, gave more reliable results than the other. In every case in FORBES's VII.th zone, or between 1080 and 1200 feet, the mud indicated a temperature of 55°·5, and he concluded that there was no reason to suppose the temperature to be lower than 55° at any depth under 1800 feet In the deeper waters he reverted to the use of SIX's thermometer

Captain (now Admiral Sir EDWARD) BELCHER gives a series of eight observations he made in mid-Atlantic when crossing the equator in 1843, at depths of from 1800 to 6000 feet § Sir EDWARD informs me that a much larger number were made, but that they were not published at the time and have been unfortunately lost, with the exception of the few others recorded by Sir JAMES ROSS ||. Sir EDWARD BELCHER also mentions that he had a water-bottle of great strength, with two enclosed thermometers specially made by CAREY, and that these instruments "were tested continuously between 1835 and 1846, and never found to vary from each other or from the standard which I [Sir E. B.] now possess, and which belonged to the Old Board of Longitude. They

* For a description of his instruments see *op cit.* Ann Chim. et Phys pp. 6–12

† Phil Mag for 1848, p 169

‡ The Nautical Magazine for 1862, p 9 Admiral SPRATT has also obligingly communicated to me the twenty-two unpublished observations to which is attached "*u*" in the Tables.

§ Narrative of the Voyage of H M S. 'Samarang' during the years 1843–46 London, 1848, vol. i p 9.

|| Antarctic Voyage, vol ii p 53

were out in 1852–54, exposed to all the Arctic variations of temperature, and are still perfect. They were made to go inside the water-bottle, and not subjected to jerks of the line, which we found often moved the indices"*.

In 1845–51 Captain KELLETT, in his voyage† to the Pacific and Behring Strait, made 38 observations to depths of 3000 feet, several of them serial, some in mid-Atlantic, others in the Pacific, ranging from near the Equator to Behring Straits, and seven in the Arctic Ocean beyond. SIX's thermometers without protection were used.

Lieutenant (afterwards Commander) DAYMAN, who served on the surveying-ship 'Rattlesnake,' made a series of one hundred and ten observations in the Atlantic, Indian, and Southern Oceans, at depths generally of from 1000 to 2000 feet‡.

While the readings given by other observers who used unprotected self-registering thermometers agree fairly well among themselves, those recorded by DAYMAN are much higher in proportion. But as he gives no particulars of his instruments, or of the mode in which they were used, it is not possible to say how the difference arises or what the error may be, it seems uniformly too high by 1° or 2°. There are also anomalies in the lists, which leads me to suppose that the readings of the lesser and greater depths have sometimes been transposed. The readings, however, have a certain independent value *inter se* as furnishing comparative temperatures at corresponding depths.

Sir A. ARMSTRONG§, who was with Captain M^cCLURE on his memorable voyage along the coast of Arctic America, records three observations made on the voyage out round Cape Horn, and three in the Arctic Ocean after passing Behring Strait. No mention is made of the thermometers, except that they were SIX's "self-registering."

In the series|| of "Reports" to the Government of the United States much valuable information is given with respect to the temperature of the seas off the North-American coast, and especially of the Gulf-stream at various depths. As the original observations are, however, not recorded, but only the diagrams founded on them, I am unable, with two or three exceptions, to give any tabulated details, and must refer to the "Reports" themselves for fuller information Owing to the depth of the Gulf-stream off the American coast, the lines of bathymetrical isotherms lie at very variable depths. The

* With respect to the mode of conducting the observations, Sir EDWARD BELCHER says, "The deep-sea temperatures were observed only in calms The thermometers were all handled by *myself*, and eased overboard with the greatest care The hauling-in was not subject to jerks, as it was done by the aid of a boat astern, the ship drifting by currents, sometimes one to two hundred fathoms from the boat, and great caution observed in getting them detached (by myself) and read off instantly"

† Voyage of the 'Herald,' Captain KELLETT By BERTHOLD SEEMAN. London, 1853, vol i pp. 7, 92, 94, vol ii. p. 107.

‡ Narrative of the Voyage of H M S. 'Rattlesnake,' Captain STANLEY, 1846–50. By JOHN MACGILLIVRAY. London, 1852. Appendix I. vol i, and Edinb. New Phil. Journ. for 1852, vol lii p. 267.

§ A Personal Narrative of the Discovery of the North-West Passage, by ALEXANDER ARMSTRONG, M D, R.N, H.M S. 'Investigator,' Captain M^cCLURE, 1850–54. London, pp 19, 43, 65, 150, 216

|| See Report of the Superintendent of the United States Coast Survey for 1854, by Professor BACHE. Also those for succeeding years

stream forms, as is well known, a trough of warm water, from below which the cold water rises up as a wall in approaching the coast.

Captain MAURY has given* incidentally a few deep-sea temperatures made by the U. S. Coast Survey (DUNSTERVILLE, BROOKE, and RODGERS) during the few years previous to the publication of his work, but it is a subject which he does not treat so fully as other points of ocean physics. It is not stated what instruments were employed.

On the voyage of H.M.S. 'Cyclops' in 1857, forty-one important observations were made by Captain PULLEN in the North and South Atlantic, Indian Ocean, and Red Sea, at depths of from 2400 to about 16,000 feet†. It was on this voyage that the first regular precautions against pressure were taken in this country. Captain PULLEN was furnished, by order of the late Admiral FITZROY, with some instruments constructed purposely for deep-sea observations, the object of which was explained in the following memorandum, communicated to me by Captain PULLEN:—

"In SIX's self-registering thermometer, the long bulb, filled with spirits of wine, is so delicate, that under a great pressure of ocean it is more or less compressed, and drives the spirit against the mercury, which is thus acted on not only by temperature, but by the mechanical pressure of sea-water.

"With a view to obviate this failing, Messrs. NEGRETTI and ZAMBRA undertook to make a case for a weak bulb, which should transmit temperature, but resist pressure

"Accordingly a tube of thick glass is sealed outside of the delicate bulb, between which and the casing is a space all round, which is *nearly* filled with mercury.

"The *small* space not so filled is a vacuum, into which the mercury can be expanded, or forced, by heat or mechanical compression, without doing injury to, or even compressing, the inner and much more delicate bulb

"This provision is meant to guard against possible compression of even the *outer* glass, strong as it is

"One may ask, Why not strengthen the inner tube, the bulb, at once, so as to be equal in power of resistance to the outer casing? Mr. GLAISHER and the makers say no; the bulb will yield a little, on account of its length, be it even as strong as the outer case. (Signed) "ROBERT FITZROY, Admiral.

"May 19th, 1857."

With these instruments Captain PULLEN made a series of observations, and was the first in this country to confirm the observations of the continental observers that so low a temperature as 35° existed in the depths of intertropical seas. In reply to my inquiries, Captain PULLEN informs me that, after comparing the deep-sea thermometers with standards kept on deck and setting the indices, "they were placed in copper cylinders,

* The Physical Geography of the Sea. By M T MAURY, LL.D., U.S.N., 11th edit. London, 1857, pp. 53, 261, 263, and Appendix, p 351 The last edition of 1874 gives no new facts.

† Twelve of these are given in Proc Roy Soc vol IX. p. 189, and the others are abstracts from Captain PULLEN's MS Journal, of which he has kindly given me the particulars, to these latter "*u*" is attached in the Tables

with a valve at each end both opening upwards, so that on going down a column of water passed through. On arriving at the depth, and you commence hauling in, these valves close, thus cutting off a portion of water at that depth, which was brought up and tested for density and its then temperature. Indices read off both from maximum and minimum scale and noted. But I have often found that the maximum index shifted, showing a different reading from what it stood at when started. Now whether this would affect the minimum side is a question"

Captain PULLEN thinks not But there are inequalities in some cases so apparent that they can hardly be accounted for, except by a shifting of the index. In one instance, in fact, while the thermometer at 7890 feet indicated 41°*, the index, owing to rough weather, had shifted to 67° at a depth of 11760 feet. Captain PULLEN speaks also of some of the instruments being more regular in their indications than others. After, however, eliminating those readings, which are evidently too high (marked with a ? in Tables), the value of the other observations remains unaffected

The Austrian Expedition of the 'Novara' in 1857–60 †, under the command of Admiral Von WULLERSTORF, made an extraordinary number of daily meteorological observations, from which it is difficult to extract the few scattered notices respecting the temperature at depths. Although they amount to 33 in number, they are mostly at depths under 1000 feet, and none exceed 1500 feet They embrace eleven observations in the Mediterranean to depths of not more than 760 feet

It was apparently on this occasion that the water-bucket was last used. All that is said on the subject is that "for these observations a wooden cylinder furnished with valves was generally employed, but an English apparatus has also frequently been made use of, which consists of a similarly constructed copper cylinder, with an easily affected maximum and minimum thermometer, so that by it water was not only brought up from a depth, but also the highest and lowest temperatures of the layers of water through which the sounding was made were ascertained" No other particulars are given, and no mention is made in the several observations of which instrument was used WULLERSTORF's observations, as I read them, differ so greatly from those of other observers, that I can only attribute it to some undetected source of error. The readings seem much too high and out of proportion with the others, but still they have a certain value in their comparative temperatures.

In 1859 Captain KÜNDSON‡ made four temperature-soundings between Iceland and Greenland, at depths of 1200 to 1800 feet; and in 1861 Dr. ED LENZ§ records a

* In two other cases also the bottom-temperature is recorded as higher than those at lesser depths

† Reise der osterreichischer Fregatten 'Novara' um die Erde in 1857–59 Wien, 1862. Naturw.-physikalischer Theil, 139–449.

‡ "Voyage of the War Brig 'Queen' from Iceland to Greenland," in the Papers translated for the Hydrographic Office, Washington, 1871

§ Meteorologische Beobachtungen auf den Atlantischen und Grossen Ocean an den Jahren 1847–49 Angestellt von dem Dr ED. LENZ, berechnet von E LENZ Nov 1861 Bulletin de l'Académie Imp des Sciences de St Pétersbourg, tom v p 129 (1863)

series of observations made in the North and South Atlantic, at a uniform depth of 360 feet, the importance of which consists in showing, as HORNER and KOTZEBUE had previously done, that near the equator the water at and beneath the surface is colder than a few degrees further north and south. SIX's self-registering thermometers were used. No protection mentioned.

Dr. WALLICH* gives, in 1862, one temperature-observation at a depth of 600 feet, on the well-known occasion of the deep-sea soundings between England and America.

Between 1860 and 1868 the several other expeditions undertaken to obtain deep-sea soundings in different parts of the world for telegraphic purposes afforded favourable opportunities for temperature-observations. Such were those obtained in 1868 by Capt. SHORTLAND† between Bombay and Aden, which are recorded in a series of means. They extend in one case to the depth of 13,020 feet, and give a reading of 33°·5, and in another, more westward, to 7800 feet, with a reading of 36°. These readings have, I presume, been corrected from the original observations.

Again, in 1868 Commander CHIMMO‡ made a series of observations on the American side of the North Atlantic, at depths extending to 12,000 feet, and recording temperatures of 42°. It is merely stated that the experiments were made with "new and delicate thermometers," which were without protection, and the readings are uncorrected.

In August 1868 the 'Lightning' sailed on the first of that series of deep-sea researches which, conducted under the combined superintendence of Dr. CARPENTER and Professor WYVILLE THOMSON, with the addition afterwards of Mr. GWYN JEFFREYS, and followed up systematically in subsequent voyages, have already yielded such valuable and important information on the natural history and physics of the depths of the sea

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Regarding the relative merits of the several methods employed by the early observers, a few words may be said The water-bucket, when properly constructed, of sufficient size, and when well handled, was not badly contrived to determine the temperature at moderate depths It was free from the errors of pressure and index to which thermometers are liable. The errors depend upon the size of the apparatus, the proper closing of the valves, the rapidity of hauling in, and the difference of temperature between the bottom- and surface-waters When the latter is not great the error can be but small, and such is the case in those Arctic seas where it has been chiefly used. As so considerable a number of observations were made with this apparatus by SCORESBY and FRANKLIN, it might be desirable to determine by experiment the amount of correction required to adjust the error of this particular apparatus.

In the case of LENZ's bathometer, he made a series of experiments to determine

* The North-Atlantic Sea-Bed, 1862, p 145

† Admiral SHERARD OSBORN, "On the Geography of the Bed of the Atlantic and Indian Oceans and Mediterranean Sea," Journ Roy Geogr Soc 1871, vol. xli. p. 58.

‡ Proc Roy Geogr. Soc. 1869, vol xiii. p. 92.

the corrections necessary for his several observations. He showed that a variety of considerations have to be taken into account with HALES's water-bucket or any similar apparatus, and that the scale of corrections must vary with the latitude and the depth. Thus in lat. 21° 14′ N., with a surface-temperature of 79°·5 F. and at a depth of 2635 feet, his corrections amounted to 4° F., while in lat. 45° 53′, with a surface-temperature of 58°·3 and at a depth of 2524 feet, they amounted only to 0°·6 F., and, again, for the lesser depths of 898 and 1252 feet in the same latitude respectively to 0° 4 F. and to 0°·6 F. The same corrections cannot, however, be applied to the observations of ELLIS, COOK, FORSTER, IRVING, SCORESBY, FRANKLIN, and WAUCHOPE, for in the case of the first three and of FRANKLIN the apparatus was not protected by any other non-conducting substances; in the case of WAUCHOPE's and SCORESBY's later experiments the correction must be applied to the enclosed SIX thermometer; and in IRVING's the small size of the apparatus, although protected, necessitates a larger correction. It is, nevertheless, satisfactory to note, from the regular decrease in the value of the corrections from the equator to the pole, that in the higher latitudes, where HALES's apparatus has been most used, the special corrections needed for that apparatus diminish to their minimum, and are so small that probably 0° 5 to 1° would cover all the errors of observation made by the foregoing explorers The main error for correction is that due to pressure in those instances where a SIX's thermometer has been used in conjunction with HALES's apparatus.

The second plan, that of sinking an ordinary thermometer, protected and surrounded by some substances which are bad conductors, has been but little used, as it requires so much time. Independently of this, and for moderate depths, it is trustworthy and useful, and some of the results, as those of SAUSSURE, may be accepted as closely accurate.

The third plan, that of taking the temperature of mud or silt brought up from the bottom, has the advantage that it secures the possession of a body having the exact bottom-temperature; but it has the disadvantage of small bulk, and therefore of being more influenced by the temperature of the water through which it has to pass. For moderate depths, however, the error can only be small.

The first and last of these methods, whatever their inconveniences, had but one main source of error—causing a gain where the surface-temperature is higher, and a loss where lower, than that at depths. Only in one instance, however, was the necessary correction accurately estimated. But with the introduction of the self-registering thermometer two sources of error (the one occasional and uncertain in amount, arising from shifting of the indices; and the other fixed and definite, resulting from pressure) were introduced. Owing also to the want of standard instruments, the observations made on the several voyages have had in themselves different degrees of value, dependent on the care with which the instruments were made, and on the precautions with which they were used. As such precautions were, it is evident, usually enforced, and Admiralty instruments were generally used, a considerable uniformity of result has been nevertheless maintained; and the readings on the different voyages agree sufficiently

well amongst themselves to allow, with reasonable success, of the application of the same correction to all, excepting those of DAYMAN, and perhaps one or two others, which require larger corrections, and WÜLLERSTORF's, which are uncertain.

With these few exceptions, and admitting slight qualifications for each particular case, the larger number of the early observations may, subject to a correction for pressure, be accepted as approximately accurate. The need of this correction for pressure was, as I have before observed, noted so early as 1823; but it was not until the voyage of 'La Vénus' that the necessary precautions were professedly taken against it, and that experiments were made to estimate its amount. Such estimates were then made by DU PETIT-THOUARS in tropical seas, subsequently by MARTINS and BRAVAIS in arctic seas, and afterwards by AIMÉ in an inland sea. The results of the several calculations are as follows:—

DU PETIT-THOUARS made experiments with a protected and an unprotected thermometer at a depth of 1000 brasses or 1620 mètres, which is equal to a pressure of 162 atmospheres, and he was led to adopt a coefficient of 0°·01 Cent. per atmosphere as the measure of correction needed for unprotected thermometers. This gives 1° C. per 100 atmospheres, or of 1°·8 F. per 3200 feet, or 1° FAHR.=1780 feet.

CH. MARTINS concluded from his experiments, which were on a more limited depth, that a coefficient of 0°·13 Cent. per 100 mètres, or of 1°·30 C. per 1000 mètres* (equal to 2°·3 FAHR. per 3280 feet, or 1° for every 1426 feet), was required.

AIMÉ, again, from experiments in the Mediterranean with his special thermometrographs, came to the same conclusion as DU PETIT-THOUARS, viz. that for the pressure of every 100 atmospheres the instrument required a correction of about 1° Cent.

These conclusions agree very closely with the more recent researches of Dr. CARPENTER and the late Dr. MILLER. The latter showed† that under a pressure of $2\frac{1}{2}$ tons (or 374 atmospheres) per square inch, SIX's unprotected self-registering thermometers of three different constructions gave readings from 7°·5 to 10° FAHR. too high. Excluding the effects of the small amount of heat evolved from the water by compression (or some undetermined cause), which was found equal to 0°·9, the mean error of the three was 8°·6 F.−0°·9=7°·7, and, taking the pressure of one ton as equivalent to a depth of 800 fathoms, this would be equal to a rise of 1° F. for every 1560 feet. But in those experiments one instrument (SIX's, with a spherical bulb) gave a variation of 2° in excess of the one with cylindrical bulb and of the Admiralty instrument. Now, as the two latter are of the forms almost always used, and BUNTEN's instruments had also a cylindrical bulb, it is a question whether the one with spherical bulb should not be excluded. In that case the reading of the other two gives a mean of 8° F.−0°·9=7°·1 as the error for pressure of $2\frac{1}{2}$ tons, or equal to 1° FAHR. for every 1690 feet.

It is true that considerable variation was found to exist in the effects of pressure on

* M. MARTINS took the differences between each of the protected and unprotected "thermométrographes," and these he diminished in each case by 0° 1,—"quantité égale à la poussée de l'index."

† Proc. Roy. Soc for 1869, vol. xvii. p. 485. see also Proc. Roy. Soc. for 1870, vol. xviii. p. 409, and Commander DAVIS, R.N., ibid. p. 347.

some other instruments; but with the care taken in the construction of our best thermometers, and of those of BUNTEN, which were generally employed, the chances of greater variation than that here indicated are reduced to a minimum*.

The foregoing estimates show that with good instruments the effect of pressure equals an increase of about 1° F. for every 1400 to 1800 feet of depth, and in adopting a coefficient of 1° F. for every 1700 feet as the necessary correction of all the observations in the Tables, excepting those made with protected instruments or corrected by the original observer, and excepting also those before named as requiring larger corrections in consequence of using unfit or unsuitable instruments or instruments of a different class, I feel that I am below rather than above the true measure of allowance.

§ III. *Summary of the preceding Observations.*

Although the early observers noted the decrease of temperature with the increase of depth, it was not until 1823–26 that LENZ proved that this decrease held good to the greater depths of temperate and tropical seas, and that the water at depths in the open oceans was but little above the zero of Centigrade

The substitution of the self-registering thermometer for the older methods led for a time, owing to the neglected error of pressure, to a retrograde course; for the voyages of BEECHEY, KELLETT, and others which followed between 1826 and 1836, while they added largely to the number of observations at greater depths, gave, in so doing, increased importance to the error, from the circumstance that the pressure on the instrument not only counterbalanced the effect of the greater cold at increased depths, but often gave readings (uncorrected) somewhat higher at those depths than at lesser ones. From this cause, and from inattention to the different properties of sea- and fresh water, an erroneous conclusion was drawn from observations otherwise valuable, which for a time greatly retarded the progress of ocean physics.

The first to fall into this error was D'URVILLE, who, misled by the coincidence of temperature obtained by him in some of his deepest soundings, and of the nearly like minimum temperature (4° to 5° C) so frequently recorded (with his unprotected thermometers) by BEECHEY and others at greater depths, concluded, in ignorance apparently of LENZ's observations, that this uniformity of temperature was the result of a general

* With respect to these variations, Dr. CARPENTER, after speaking of the results obtained on the 'Porcupine' expedition with the MILLER-CASELLA instrument, observes.—"With these results, obtained with thermometers upon which complete reliance can be placed, those obtained last year with the best ordinary thermometers are found to be in close accordance, when the proper correction for pressure is applied to them" He then instances two cases in which experiments were made on both expeditions at nearly the same places and in nearly similar depths. In one case, at a depth of 550 fathoms, the difference exceeded the estimate by about 1°, in the other, at a depth of 550 fathoms, it amounted to 2°·2 F., or was "exactly equivalent to the correction for pressure at that depth in the unprotected thermometers." Dr. CARPENTER concludes —"This very near accordance gave us, of course, a feeling of great satisfaction in our last year's work; and it fully justified our conclusion that, whatever might be the pressure-correction required by the instruments then employed, it would not affect the *differences* obtained at nearly approximating depths." (Proc. Roy. Soc. vol. xviii. p. 455.)

law dependent on the maximum density of water, which he supposed to be alike in fresh and salt water, and he consequently assumed that a temperature of about 4°·4 C. (40° F.) prevailed below a certain depth in open seas*, and that in both hemispheres there was in certain latitudes a zone from the surface downwards of like uniform temperature†.

On the other hand, we have seen that in 1836–39 Du Petit-Thouars fully confirmed the observations of Lenz, that a temperature of from 35° to 37° existed at depths in both the great oceans. Arago, in commenting on these results, testifies to their accuracy and importance, and remarks that "the observations collected by the 'Vénus' will occupy a distinguished place, on account of their number and exactness, and of the great depths at which they were taken." He also observes that, low as some of the readings are, yet all errors must be positive, and that they place on reliable grounds the great fact of the prevalence of the same low temperatures at depths in the Pacific as well as in the Atlantic, and in the equatorial regions of both oceans; and he especially dwells on the circumstance that they tend effectually to disprove the hypothesis which had been advanced, that at great depths there existed a uniform and common temperature of 40° F ‡

It appears, nevertheless, that so little was known of what had already been done and written, that Sir James Ross fell into the very same errors as D'Urville had made thirteen years before. Unfortunately in this case his conclusions were accepted without examination by distinguished writers in two popular works on Physical Geography, and obtained a currency for which it is difficult to account §. Although Sir James Ross's experiments were in themselves valuable, they required both detail and corroboration, and his conclusions were evidently based on an assumption for which there was no warrant. And yet, while his important and positive facts as to the persistence of life to great depths failed to receive the attention they deserved, his physical fallacies were received almost without a question. As with his predecessor, D'Urville, Sir James found in his more numerous and deeper observations that the unprotected thermometer commonly marked a temperature of and about 39° to 40°; and taking the maximum density of fresh water to be 39° 5, he applied the same reasoning to the open seas as had already been applied to freshwater lakes, and assumed, exactly as D'Urville had done, that a uniform temperature of about 39°·5 prevailed at depths varying with the latitude, and that a belt of water of that temperature, extending from the surface downwards, encircles the globe between the 50th and 60th degrees of south latitude, or, as he more definitely fixes it, in a mean latitude of about 56° 26′ S.‖

* Voyage, p 62. † *Ibid* p 59.

‡ "Il faut donc espérer que le fameux nombre +4° 4 si étourdiment emprunté aux observations à la surface et au fond des lacs *d'eau douce* de Suisse cessera de paraître dans les dissertations *ex professo*, comme la limite au-dessous de laquelle la température du fond des mers ne saurait jamais descendre." (Voyage de 'La Vénus,' Physique, vol v p 22, and 'Œuvres Complètes,' vol. ix. p. 254.)

§ I may, however, remark that their mention of the subject is incidental, and confined merely to giving the facts on Ross's authority

‖ "It is therefore evident that about this parallel of latitude there is a belt or circle round the earth,

WILKES, who also explored the Antarctic seas in 1838–42, took the same view, and for the same reasons as D'URVILLE, WAUCHOPE, and ROSS, of the existence of a deep-sea and of a belt of water of the uniform temperature of 39° 5 F.

Commenting on the general results of this great American expedition, BIOT discusses* the question of deep-sea temperatures. He remarks that serial observations should in all cases be made, "that the instruments ought to be protected against pressure by surroundings of great strength and resistance," and that they should be left a considerable time at the bottom. Comparing the observations of ROSS with those of DU PETIT-THOUARS, SCORESBY, PARRY, and MARTINS, he shows their want of agreement He says that the experiments of ROSS depend entirely on his instruments, "of which he had no means of knowing and judging (*aucun moyen d'apprecier*)," while he knew those of DU PETIT-THOUARS and MARTINS to have been prepared with every care. For ROSS, he remarks, "the uniformity of temperature at the bottom of the sea is a necessity;" and he trusts that some steps may be taken to verify his observations, for between them and those of other observers there is, he remarks, "a complete incompatibility"

With respect to the freezing-point and point of greatest density of sea-water, these properties were first more particularly investigated by Dr. MARCET in his well-known paper on the subject published in 1819†. Dr. MARCET ascertained that he could lower the temperature of sea-water (at 1·027 sp. gr.) to 27°, and even, when in large vessels and kept perfectly still, to 18° or 19° F., before freezing, but that when it froze it always rose to 28°; and he states that his experiments "uniformly led him to the conclusion that the law of greatest specific density at 40° did not apply to sea-water, but that, on the

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where the mean temperature of the sea obtains throughout its entire depth, forming a boundary, or kind of neutral ground, between the two great thermic basins of the oceans To the north of this circle the sea has become warmer than its mean temperature, by reason of the sun's heat which it has absorbed, elevating its temperature at various depths in different latitudes So that the line of mean temperature of 39° 5 in latitude 45° S has descended to the depth of 600 fathoms; and at the equatorial and tropical regions this mark of the limit of the sun's influence is found at the depth of about 1200 fathoms, beneath which the ocean maintains its unvarying mean temperature of 39° 5, whilst that of the surface is about 78°.

"So likewise, to the south of the circle of mean temperature, we find that, in the absence of an equal solar supply, the radiation of the heat of the ocean into space occasions the sea to be of a colder temperature as we advance to the south, and near the 70th degree of latitude we find the line of mean temperature has descended to the depth of 750 fathoms, beneath which again, to the greatest depths, the temperature of 39° 5 obtains, whilst that of the surface is 30°

"The experiments which our limited time and means admitted of our making serve to show that the mean temperature of the ocean at present is about 39°·5, or 7½ degrees above the freezing-point of pure water, and as nearly as possible the point of its greatest density. But it would be indispensable that this temperature should be ascertained to the tenth part of a degree, and as we now know where we may send any number of thermometers down to the greatest fathomable depths without an alteration of temperature, even to that small amount, this desideratum might be very easily obtained." (ROSS's 'Voyage to the Antarctic Regions,' vol. ii p. 375.)

* Journal des Savans, 1849, p. 60.

† "On the Specific Gravity and Temperature of Sea-water," Phil. Trans for 1819, p. 161.

contrary, sea-water gradually increased in weight down to the freezing-point, until it actually congealed." Other experiments led him to fix this point of greatest density at 22° F.

ERMAN* in 1828 fixed the maximum density of sea-water of 1·027 specific gravity at 25° F., and found likewise that it did not reach its maximum before congelation. Still more conclusive were the more elaborate experiments of DESPRETZ† in 1837. Taking distilled water at a temperature of 20° C and sea-water of the specific gravity of 1·027 at 20° C., he successfully determined the following important points:—

	Cent.	Fahr.
Maximum density of fresh water	+4°	=39°·2
" sea-water	−3·67	=25·4
Point of congelation of sea-water	−2·55	=27·4
Temperature of sea-water during congelation . . .	−1·88	=28·6

He also showed that the freezing-point and the point of maximum density were proportionate to the quantity of saline matter in the water, and that both therefore varied with the degree of salinity of the sea.

The effects of pressure and the properties of fresh and salt water were therefore perfectly well understood previous to the date of Ross's voyage. How, then, the unsupported opinion of one who, though a most able and enterprising navigator, had not any pretensions to an exact knowledge of physical science could have been accepted by scientific writers of so much eminence is a singular fact. I can only account for it by the circumstance that the subject had not been made in this country one of special investigation, and therefore the results of Ross's work had not been questioned by any competent special authority. In fact they had never been discussed.

The observations of LENZ, DU PETIT-THOUARS, and others, combined with the researches of physicists, had sufficiently established the law of the decrease of temperature with the depth to 2° to 3° above the zero of Centigrade in the temperate and tropical zones of both the great oceans; and their conclusions could hardly be considered as seriously affected by the unsupported though ingenious hypothesis of D'URVILLE and ROSS LENZ had obtained, by means of his bathometer, with corrections for change of medium, the low readings given at p. 599, and subsequently DU PETIT-THOUARS by means of protected thermometers had obtained directly, without correction‡, amongst a number of others at lesser depths, the following deep-sea temperatures:—

* "Nouvelles Recherches sur le maximum de densité de l'eau salée," Annales de Chimie, xxxviii. p. 287.

† "Recherches sur le maximum de densité de l'eau pure et des dissolutions aqueuses," ibid. lxx. p. 5.

‡ Others of his observations were corrected On his return his thermometers were found to give too high a reading by $\frac{2}{10}$ to $\frac{3}{10}$ of a degree Centigrade, so that his observations may require a further slight deduction to this extent.

	Lat.	Long of Paris.	Depth. Mètres.	Temperature at depth.	Temperature at surface
North Atlantic . .	4° 23′ N.	28° 26′ W.	1950	3°·2 C.	27° C.
South Atlantic . .	25 10 S.	5 39 E.	1620	3	19 6
	39 51 S.	41 57 E.	1620	3·2	25 6
North Pacific. . .	4 32 N.	136 54 W.	3740	1 7	27·2
	51 34 N.	159 21 E.	1790	2·5	11 7
South Pacific and Indian Ocean . .	0 55 S.	99 27 W.	1790	3	26 5
	27 47 S.	98 0 E.	1620	2·8	23·8
	37 42 S.	112 38 E.	1620	3	16·7
	43 47 S.	81 26 W.	810	4 1	13 2
	" "	" "	1790	2·3	"

The rate of decrease recorded by the observations of Du Petit-Thouars was confirmed within certain limits for lesser depths by those of Kotzebue, Beechey, D'Urville, Vaillant, and others, and for greater depths by some of the later observations of Captain Pullen, who obtained in the

	Lat.	Long.	Depth fathoms	Temperature at depth	Temperature at surface.
Indian Ocean . . .	5° 31′ S.	61° 31′ E.	2330	35° F.	84° F.
South Atlantic . .	26 46 S.	23 52 W.	2700	35	75
	30 6	20 14	400	43·5	74·5
	" "	" "	1200	38 2	"

These various submarine temperature observations in the several great Oceans, taken in conjunction with the corrected readings for others adopted by Du Petit-Thouars and De Tessan, showed that, whether in temperate or tropical regions, approximately —

The temperature at surface being according to latitude .	60° to 80° Fahr.	
At from 1000 to 2000 feet it was from	40 to 60	"
" 2000 to 5000 " "	37 to 40	"
" 5000 to 12000 " "	35 to 37	" (or less)

Other corrected readings give equally low or still lower temperatures.

On the other hand, in the Arctic seas, the observations of Scoresby and of Martins and Bravais showed that the temperature of the upper strata, down to a depth of 200 to 300 feet, varies greatly with the season, ranging from 8 to 10 degrees above to 3 or 4 degrees under 32° F., and that with increasing depth a more uniform higher temperature prevailed. Scoresby, whose experiments were conducted further northward and westward, found this latter temperature to be generally 3° or 4° above the freezing-point of fresh water, or 7° to 8° above that of sea-water. His two deepest experiments (to the N.W. of Spitzbergen) give the following results:—

Lat.	Long.	Depth.	Temperature. Uncorrected.	Corrected.
79° 4′ N.	5° 38′ E.	4380 feet.	37° Fahr.	34°·5 ?
78 2 N.	0 10 W.	4566 feet.	38 „	35·4 ?

M. Martins's chief experiments were, on the other hand, between the North Cape and Spitzbergen, from 71° to 76° N. lat. The deepest temperature sounding was in 73° 36′ N. and 20° 53′ E., in which instance Walferdin's thermometer registered at 2854 feet 32°·2 F., and Six's thermometer, corrected for pressure, gave 31°·6. This latter is the only recorded instance in the open sea where his reading was below zero of Centigrade. His most northern observations, viz. in 76° 13′ N. and 12° 48′ E., at 1296 feet, and another in the same place in 2103 feet, gave respectively 33° 4 and 32°·3, while one of Scoresby's, in 79° N and 5° 40′ E., at 2400 feet gives, corrected, 34°·6 F., and another in 76° 16′ N., 9° E., at 1380 feet, not far from Martins's position, gives, without allowance for pressure (for in this case Six's thermometer does not appear to have been used), a temperature of 33° 3.

Martins, however, states that on approaching the land in Magdalena Bay, instead of a submarine temperature above zero, he found that in depths of from 110 to 130 mètres the temperature of the water was always below zero; that these bottom-waters there had, in fact, a temperature of −1°·75 C. to −1°·91 C. (28°·6 F.), that of the surface being 0°·1 to 1°·2 Cent.*

The results obtained in another section of the North Atlantic are very different and of much interest The observations in Davis Strait and Baffin Bay by John Ross and Sabine indicate that, after passing the point where the diurnal and annual variation cease, there is a gradual decrease of the temperature with the depth to a point approaching in places to that of the maximum density of sea-water Even taking the readings without correction†, they show:—

From 1000 to 2000 feet, a temperature of	32° to 29°·5 Fahr.
„ 2000 to 3000 „ „	30 to 29 „
„ 3000 to 4000 „ „	29 Fahr.
„ 5000 to 6000 „ „	28¾ „

Besides these, Parry noted, in 68°·29 N. lat. and 63°·43 W. long., at a depth of 4854 feet, a temperature of 27°, and, as before mentioned, Ross and Sabine have recorded‡, in 66° 50′ N., 61° W., at a depth of 4080 feet, a temperature of 25¾° §.

In the Antarctic seas the observations of Cook, James Ross, and Wilkes show that the temperature from the surface down to 600 or 1000 feet varies from 28° to 32°. At greater depths there are, with few exceptions, only the experiments of Ross; and these cannot,

* *Op cit* p. 332

† Probably but very little is needed, *antè*, pp 597 and 598.

‡ Marcet, Phil. Trans 1819, pp 169 & 205

§ This may be rather doubtful (see, however, note, *antè*, p. 596).

for reasons before given, be accepted without reserve. Still they are available after correction for pressure; and the readings then indicate thermal conditions very similar to those which obtain in Arctic seas. To take one of the most southern series of observations, at a spot in the Antarctic Ocean where no soundings were obtained, at a reputed depth of 24,000 feet.—

			Temperature.	
Lat.	Long.	Depth.	Uncorrected.	Corrected for pressure
68° 32′ S.	12° 49′ W.	Surface (March)	30°·8 Fahr.	30°·8 Fahr
		900 feet.	33	32 4
		1800 „	35·5	34·4
		3600 „	38·7	36·5
		4500 „	39·4	36·6
		5400 „	39	35 8
		6300 „	39 5	35·8

Again, another nearer the South Polar land, and in soundings —

63 49 S.	51 7 W	Surface (Feb.).	32 F.	32 F.
		600 feet.	32 2	32
		900 „	33 2	32·6
		1800 „	35·5	35·6
		2700 „	36 4	35
		3600 „	37·3	35 2
		7200 „	39·5	35·2

Still further and closer to another part of the Antarctic continent we have.—

77 49 S.	162 36 W.	Surface (Feb.).	28·5 F.	28 5 F.
		1740 feet.	30·8	29·8

There is only one observation of Du Petit-Thouars in the Southern Ocean for comparison with those of Ross. As the cylinder came up full, I give the reading with the correction:—

59 48 S.	79 56 W.	Surface (March).	42·9 F.	42·9 F.
		2657 feet.	39	37·5

The conditions, therefore, prevailing in the open Arctic and Antarctic seas are apparently closely analogous,—the temperature at a distance from land increasing with the depth until it rises to 35° to 36° F. at 2000 to 3000 feet, below which it seems to remain nearly stationary at about the same temperature; while closer to the land and at less depths it falls nearer to the freezing-point of sea-water (see note, *postea*, p. 635).

The temperatures at depths in inland seas were found at an early period to be very different to those of open seas; and it is singular that the very first observations made

in the Mediterranean by SAUSSURE in 1780, of 55°·8 F. at 944 feet, and 55°·5 at 1918 feet, remain substantially correct to the present day. It was, however, D'URVILLE's more extended observations in 1826 that made better known the fact that the temperature decreased from the surface down to 200 brasses (1066 feet), below which it remained constant at about 13° C., or between 54° and 55° F. Still his greatest depth did not exceed 3189 feet, but BÉRARD in 1831 extended the observations to a depth of 6377 feet, and still found the same degree of temperature.

AIMÉ further showed, from a series of soundings made during 1840–44 in the western basin of the Mediterranean, between Marseilles and Algiers, that the diurnal variation of temperature ceases to be sensible at 16 to 18 mètres, and the annual variation at 300 to 400 mètres. The mean of his series of observations gave the following results:—

Mean Annual Temperature of the Mediterranean at different depths.

Depth.	Temperature.	Extreme monthly variations.
Surface.	18°·2 Cent	10°·2 Cent.
25 mètres.	16·3	6·3
50 „	14·4	2·8
100 „	13·7	2·0
200 „	13 0	1 0
350 „	12·6	0·0

This temperature of 12°·6 (54°·7 F) he showed to be that of the mean of the winter months (or rather that of the months of January, March, and April) of the area, and he was of opinion that the same temperature obtained at greater depths, referring in support of that opinion to other and deeper soundings by BÉRARD The following observations by the latter, made between the Balearic Islands and Algeria, are extracted from D'URVILLE's tables —

Depths of variable temperature.	Surface in August	27°1 Cent.
	„ November	14 6
	At depth of 40 brasses in October . .	16 5
	„ 70 „ „ . . .	14 9
Depths of uniform temperature.	„ 600 „ November . . .	13
	„ 600 „ June	13
	„ 750 „ November . . .	13
	„ 1200 „ June	13

This gives the rather higher reading of 55°·4 at depths; but whether arising from BÉRARD using less perfect instruments or from an actual difference of temperature on this southern side of the Mediterranean, is uncertain. The marked agreement between the observations of BÉRARD in 1831–32 and those of the 'Porcupine' expedition in 1870, leads me to suppose that the latter may be the influencing cause.

D'URVILLE's observations, which were made further north in the western Mediterranean than those of BÉRARD, agree more closely with those of AIMÉ. Thus he found—

At a depth of 300 brasses in March a temperature of	. . .	12°·7 Cent.
„ 600 „ „ „	. . .	12·6

The only temperature-observations made in the eastern basin of the Mediterranean previously to 1869, with the exception of two of WULLERSTORF, are those of Admiral SPRATT. They extend from Malta to Alexandria, and from the Grecian archipelago to the Gulf of Syrtis, forming for this section of the Mediterranean a series complemental to those made in the western section by D'URVILLE, BÉRARD, and AIMÉ. The results he obtained are also, when corrected, in close agreement with those of these several observers. His first experiments were made in Ægina Bay in 1845, in connexion with the natural-history researches of EDWARD FORBES, and extended only to a distance of three miles from shore. Allowing for a gain of 0°·5 or 1° in hauling up the silt, the corrected readings will then give as the general results.—

For FORBES's Zone I (1 to 12 feet) a temperature of	.	55° to 82° FAHR.
„ „ II. (at and near 120 feet)	. . .	69 to 70
„ „ III. („ 330 feet)	. .	56 to 57
„ „ VII („ 1260 feet)	. .	54·5 to 55

Three other experiments in the seas of Greece gave him the following readings.—

1080 feet (four miles off Nio) . .	55°·5	or, allowing for gain in hauling up through warmer waters, of from 54°·5 to 55°
1200 „ (seven miles off Andros)	55·5	
1260 „ (three miles off Ægina)	55·5	

In the shallower waters of the archipelago he found "the temperature of the intermediate depths between 100 fathoms and the surface range from 55° to 76°, and, indeed, in the summer season sometimes up to 80° and 86° in the littoral waters of enclosed gulfs and shallow bays."

A set of serial observations off Crete, made later with unprotected self-registering thermometers, gave readings as under (these, when corrected for pressure, agree, with the exception of the fourth, which seems a doubtful reading, very closely with those of BÉRARD in the southern portion of the western Mediterranean basin*).—

Temperature at depths in the Mediterranean off the N.W. Coast of Crete.

	Uncorrected	Corrected for pressure.
Surface in the month of June . .	73° FAHR.	73° FAHR.
At a depth of 120 feet	68	67·9
„ 300 „	63	62·7
„ 600 „	59¾	59·4
„ 1200 „	59¾ (?)	59 ?
„ 7440 „	59½	55·2

Admiral SPRATT says that he found this temperature of "about 59° in all depths from 300 down to 2000 fathoms." In the extreme eastern portion of the Mediterranean

* Nautical Magazine for 1862, p. 10; 'Travels and Researches in Crete' (London, 1865), vol. ii. App. p. 332.

there are, however, indications of a higher temperature, as the following observations, taken, the first three in November, and the last in April 1861, show:—

Deep-sea Temperatures off the Coast of Egypt, west of Alexandria.

	Uncorrected	Corrected.
At a depth of 180 feet	71° Fahr.	70°·9 Fahr.
„ 300 „	68	67·8
„ 600 „	62½	62·1
„ 1620 „	59½	58·5

Admiral Spratt concluded that "the minimum temperature of their (Eastern Mediterranean, Grecian archipelago, Sea of Marmora, and Black Sea) deeper parts correspond nearly with the mean annual temperature above them." This apparent discrepancy between Aimé and Admiral Spratt evidently arises from the circumstance that the one bases his conclusion on observations made with protected and the other with unprotected thermometers, which gave too high a reading. Subject to correction the results are closely concordant, and both give approximately the mean sub-winter temperature.

The observations of Aimé and others thus proved that in this great inland sea the influence of the variations of temperature at the surface ceases at a depth of from 1000 to 1200 feet, and that below that line a uniform temperature of from 54° to 55°·5 prevails in the western basin, and one possibly 0°·5 to 1° higher in the eastern basin of the Mediterranean.

Some deep temperature-observations have also been made in two other nearly closed seas—the Red Sea and the Sea of Okhotsk,—the latter by Dr. Horner in 1803, and the former by Captain Pullen, with his protected thermometers, in 1858.

The mean winter temperature of the air in the Red Sea may be a little under 70° Fahr. The following observations are not serial, but were taken at intervals in various parts of that sea (see Table III. p. 667).

Temperatures at depths in the Red Sea.

Surface in the months of March and April	78° to 86° Fahr.
At 300 feet	77
„ 2552 „	71
„ 4068 „	70·5

In the Sea of Okhotsk, where the mean winter temperature is doubtlessly under 20° F., the observations were only carried to a depth of 690 feet, with the following results:—

Temperature at depths in the Sea of Okhotsk.

	Uncorrected.	Corrected.
Surface in the month of August	46°·4	46°·4
At 108 feet	31·6	31·6
„ 360 „	29·0	28·8
„ 690 „	29·0	28·6

Parry's observations in Lyon's Inlet are excluded, for the reasons before given (p. 597).

§ IV. *Hypotheses of* Humboldt, Arago, Lenz, *and others.*

Such is a summary of the results obtained between the years 1749 and 1868. From time to time they had been commented on by some of the most eminent physicists of the time, and the cause of the low temperatures prevailing in the depths of tropical seas discussed.

Humboldt, so far back as 1812, and again in his subsequent works*, contended that "the existence of those cold layers in low latitudes proves the existence of an under-current flowing from the poles to the equator." In support of this hypothesis, he showed how it explained the fact, first noticed by Franklin and Williams †, that the water on shoals in the Atlantic was many degrees lower than that surrounding them, from the circumstance that the deeper cold water, flowing and rising over them, displaced the warmer surface-waters ‡. These observations were afterwards confirmed by Du Petit-Thouars, Vaillant, and others He was further of opinion that "in the narrower seas, as well as in the tropical seas which cover the cold waters from Arctic regions, all the mass of water is in a state of movement."

Humboldt also contested the conclusions of those who considered that the ocean is salter under the equator than at a distance from it, and showed that while in lat. 0° to 14° the specific gravity was 1 0272, it was 1·0282 in lat. 15° to 18°, and 1·0278 in lat. 30° to 40° Nor did he fail to note § that the equatorial zone is not the hottest water zone, but that two hotter zones lie a few degrees N. and S. of it

Humboldt subsequently ‖ thus summarized the question as it then stood —

"As fresh and salt water do not attain the maximum of their density at the same degree of temperature, and as the saltness of the sea lowers the thermometrical degree corresponding to this point, we can understand how the water drawn from great depths of the sea during the voyages of Kotzebue and Du Petit-Thouars could have been found to have only the temperature of 37° and 36° 5 This icy temperature of sea-water, which is likewise manifested at the depths of tropical seas, first led to a study of the lower polar currents, which move from both poles towards the equator. Without these submarine currents the tropical seas at those depths could only have a temperature equal to the local maximum of cold possessed by the falling particles of water at the radiating and cooled surface of the tropical sea. In the Mediterranean the cause of the absence of such a refrigeration of the lower strata is ingeniously explained by Arago,

* 'Voyage' Relation Historique (Paris 1814), vol i. p. 73. Climatologie Asiatique (Paris 1831), p 560 'Kosmos,' Otté's translation, 1849, vol i. p. 307.

† On the Use of the Thermometer in Navigation. Philadelphia, 1792.

‡ He instances, for example, a case noticed by himself on the "Signal Bank" off Ferroll, where he found the water to have a temperature of from 54°·5 to 56° F., while the water immediately around was from 59° to 59°·6 F.

§ Ann. Chim. et Phys. xxxiii 1820, p. 40.

‖ Kosmos, vol. i. pp 308, 309 (Sabine's translation, pp. 295, 296)

on the assumption that the entrance of the deeper polar currents into the Straits of Gibraltar, where the water at the surface flows in from the Atlantic Ocean from west to east, is hindered by the submarine counter-currents, which move from east to west, from the Mediterranean into the Atlantic . . . . The zones at which occur the maxima of the oceanic temperature and of the density (the saline contents) of its waters do not correspond with the equator. The two maxima are separated from one another, and the waters of the highest temperature appear to form two nearly parallel lines north and south of the geographical equator. LENZ, in his voyage of circumnavigation, found in the Pacific the maxima of density in 22° north and 17° south latitude, whilst its minimum was situated a few degrees to the south of the equator In the region of calms the solar heat can exercise but little influence on evaporation, because the stratum of air impregnated with saline aqueous vapour, which rests on the surface of the sea, remains still and unchanged "

Similar views were adopted by D'AUBUISSON in 1819*. The whole subject of Oceanic circulation was again discussed from a fresh point of view by D'URVILLE † in his account of the results of his voyage of 1826. After arguing (p. 62) that in open seas the temperature at and below 600 brasses (3198 feet) is nearly constant between 4° C. and 5° C., and that perhaps it may be 4° 4 C. (40° F.), he significantly remarks that in the zone 10° on each side of the equator some particular cause seems to occasion in the water "up to 100 brasses a more sudden and rapid cooling than would have been expected" He afterwards (p 64) proceeds to say that the mass of the equatorial waters, slowly diminished by evaporation, may give rise to a slow and continuous ascensional movement of the lower colder waters, and these so displaced make room for other waters coming from the polar regions, so that "it is rather a transport, nearly in mass and very slow, of the deep waters of high latitudes towards the equator." The point of departure he considered to be between 40° and 60° lat.; and he inferred that the deep cold waters (at 40°) are there directed periodically in two "insensible currents," the one towards the equator and the other towards the pole.

ARAGO ‡ in 1838, reporting to the French Institute on a scientific expedition then in course of preparation to the coast of Africa, thus expresses his opinion.—

"La température des couches inférieures de l'océan, entre les tropiques, est de 22° à 25° centigrades au-dessous du plus bas point auquel les navigateurs aient observé le thermomètre à la surface. Ainsi, cette couche si froide du fond n'est point alimentée par la précipitation des couches superficielles. Il semble donc impossible de ne pas admettre que des courants sous-marins transportent les eaux des mers glaciales jusque sous l'équateur.

* Traité de Géognosie (Strasbourg, 1819), p. 450.

† 'Voyage de l'Astrolabe,' Sect Météorologie, Physique, et Hydrographie, chap iii. pp. 51*–85*. Paris, 1833

‡ "Instructions concernant la Météorologie et la Physique du Globe, par M. ARAGO, Courants Sous-marins," Comptes Rendus, 1838, part 2, tome vii. pp. 212, 213.

"La conséquence est importante. Les expériences faites au milieu de la Méditerranée, la fortifient. Cette *mer intérieure* ne pourrait recevoir les courants froids, provenant des régions polaires, que par la passe si resserrée de Gibraltar; eh bien! dans la Méditerranée, la température des couches profondes n'est jamais aussi faible, toutes les autres circonstances restant pareilles, qu'en plein océan; on peut même ajouter que nulle part cette température du fond de la Mer Méditerranée ne paraît devoir descendre au dessous de la température moyenne du lieu. Si cette dernière circonstance vient à se confirmer, il en résultera qu'aucune partie du flux glacial venant des pôles ne franchit *le seuil du détroit de Gibraltar.*

"Lorsque M. le Capitaine D'URVILLE partit, il y a quelques années, pour sa première campagne de 'l'Astrolabe,' j'eus la pensée qu'il pourrait être utile de rechercher si les phénomènes de l'océan, quant à la température des couches profondes, se présenteraient dans toute leur pureté *dès qu'on se trouverait à l'ouest du détroit.* L'Académie voulut bien accueillir mon vœu Sur sa recommandation expresse, quelques observations de la nature de celles que je désirais, furent faites à peu de distance de *Cadiz.* Eh bien! elles donnèrent précisément ce qu'on aurait trouvé dans la Méditerranée

"Ce fait curieux semble se prêter à deux explications différentes On peut supposer que le courant polaire se trouve complétement refoulé par un courant sous-marin dirigé de la Méditerranée vers l'océan, et dont l'existence est appuyée sur divers événements de mer. On peut supposer aussi que la saillie si forte de la côte méridionale du *Portugal*, ne permet pas au flux d'eau froide venant du nord, de s'infléchir jusqu'à angle droit pour aller atteindre les régions voisines de l'embouchure du *Guadalquivir*"

Again*, in reporting on the observations of the 'Vénus,' ARAGO saw no other explanation of the low deep-sea temperatures recorded in tropical seas, but "the existence of submarine currents carrying to the equator the bottom-water of the icy seas"

It is, however, to LENZ (who had, in his previous papers of 1831 and others, concluded that the temperature of the ocean decreases with the depth, rapidly at first and then gradually, until a point of about 36° F. was reached, when it became insensible) that we are indebted for a more special review and discussion of all the facts known up to 1845†. Speaking of the earlier observations made on the temperature of the sea at great depths, he observes:—"The greater number of these observations, with the exception of those made by myself, were taken with thermometrographs, and especially with SIX's thermometers It is, however, to be observed that all instruments of this kind are liable to a source of error which hitherto investigators have not borne in mind, viz. the compression of the vessel or the bulb which contains the thermometrical substance (spirits of wine), particularly by the enormous pressure in depths of several thousand feet. I was witness of a series of experiments on the action of strong pressure on a thermometer-bulb, which PARROT undertook in order to ascertain the influence of

* Comptes Rendus, 1840, vol xi p 311

† "Bemerkungen uber die Temperatur des Weltmeeres in verschiedenen Tiefen, von EMIL LENZ," Bulletin Acad. Sci. St. Petersburg, v. (1847), cols. 65–74.

a strong pressure on different substances, and which he has made known in the 'Memoirs of the Academy of St. Petersburg' (vi. série, Sc. Math. Ph. et Natur., t. ii. p. 595, 1832) It is there mentioned that a pressure of 100 atmospheres caused the thermometer to rise about 20°·5, without the temperature having altered in the least, as was shown by a second thermometer which was protected from pressure by a brass cylinder."

LENZ proceeds to remark that it necessarily follows that thermometrographs (although in such instruments the effect would be much less owing to their form of construction) exposed in the sea to pressures of 100 to 200 atmospheres must give too high a reading, and that the circumstance of the indications in so many deep soundings remaining uniform, or sometimes increasing with the depth, proves the influence of compression

Reviewing the data furnished by different observers and by himself, and assigning to them, if not an actual, at all events a relative and comparative value for corresponding depths, LENZ notices the circumstance that they all point to the existence of a belt of water at and near the equator cooler than at a short distance to the north and south of it, and in illustration of this he takes the consecutive series of observations at nearly the same depths made by KOTZEBUE in 1815–1818, at short distances apart over a great length of the Atlantic; and he gives a Table, of which the following is an abstract.—

Zones of latitude	North Atlantic.		South Atlantic	
	Mean depth	Mean temperature	Mean depth	Mean temperature
	feet	°F	feet.	°F
0 to 3	435	58·2	480	57
3 „ 6	460	57·8	405	56·4
6 „ 9	400	58	351	61·5
9 „ 12	390	59·4	426	62·7
12 „ 15	390	58·2	351	60·8
15 „ 18	408	66·7	305	60·3
18 „ 21	468	68·2	378	61·7
21 „ 24	414	69·2	420	63·2
24 „ 27	432	69		.
27 „ 30	403	65·7		.
30 „ 33	390	60		.
33 „ 36	447	62·2	.	
36 „ 39	418	61·2		
39 „ 42	438	58·5	.	..
45 „ 48	458	53·6	.	

This, he observes, shows a rapid rise of the isothermal planes in approaching the equator, and taking a definite isotherm of 14°·5 C., he gives the following diagram, in which he shows that this plane, which in lat. 45° to 48° N lies at a depth of 350 feet, sinks gradually to 640 feet in lat. 23° to 26°, and then, rising more abruptly as it

approaches the equatorial regions, reaches to within 390 feet of the surface in lat. 12° to 15° N.*

Depth, ft. N. S

N latitude ...

Temperature..

[redacted]

* Although Kotzebue's observations in the Pacific did not furnish him with the same number of data, he thought there was yet evidence of the same condition prevailing there also, but the observations were much scattered over many parallels of longitude and were made in various currents The results were —

Latitude	Mean depth feet	Mean temperature. ° F
6° to 9° N	600	56
9 „ 12	499	62
12 „ 15	558	61 3
15 „ 18	498	69 5
18 „ 21	402	69 3
27 „ 30	450	64
30 „ 33	600	62
33 „ 36	600	51 8
36 „ 39	600	52 7

Dr Horner had previously noticed, in the Atlantic, this anomaly of a proportionally lower temperature at depths near the equator than 5° S. and 10° N of it, but without offering any explanation, and gave a series of means of some of Krusenstern's observations, of which the following is an extract —

	No of Obs.	Lat.	Long	Depth feet	Temp ° F
April 20 to 26	5	17° 15′ S.	3° 20′ W	342	55 4
27 „ 30	4	10 24	12 2	396	56 2
30 „ 4 M	5	5 12	17 5	402	53 3
May 3 „ 10.	8	0 43 N.	20 28	444	52·5
10 „ 16.	7	4 51	24 38	450	52 5
15 „ 19	5	9 34	29 38	402	52·7
20 „ 24 .	.. 5	19 30	35 7	426	61·0
25 „ 30	6	31 0	36 30	426	58 7
31 „ 6 J.	5	40 30	29 40	408	54·2

Edinb Phil. Journ. 1822, vol. vi p 161

LENZ then proceeds to observe:—" The form of the submarine isothermal line which I have drawn leads us of itself, on the first glance, to an explanation of this striking phenomenon

"The mass of water in the tropics, warmer down to a certain depth from the sun's heat, cannot maintain its equilibrium with the colder waters of the middle and higher latitudes. a flow of the warmer water from the equator to the poles must necessarily take place on the surface; and this surface-flow must be supplied at the equator by a flow of colder water from high latitudes, which at first would flow in an almost horizontal direction, but which under the equator must rise from below to the surface. In this manner, in the northern hemisphere, a great vertical circulation takes place in the ocean, which has its direction above from the equator to the pole, and below from the pole to the equator. Since these flows or currents moving in opposite directions are distinguished by their different temperatures, we obtain in the submarine isotherm an indication of the direction of the lower portion of this flow. A corresponding flow, but moving in the opposite direction, takes place in the southern hemisphere; so that in a zone surrounding the equator where both are united, the water flows almost in the direction from below up to the surface; and thus one meets with cold water in much shallower depths than in those two zones north and south which lie immediately adjoining, and which, in fact, is shown by the observations

"It is not my intention to enter here upon the question, how the original direction of this current to the surface becomes greatly altered by the diminution of the speed of rotation and by the influence of the wind, so that the water first arrived at the polar regions by considerably circuitous ways, or how the lower portion of the current was drawn westward by the entrance of bodies of water into latitudes of greater speed of rotation; in any case the last influence will be very much diminished by the opposition of the west bank of the ocean, in comparison with the corresponding diversion or drawing away which the air-currents undergo. It is sufficient for me to have furnished in the figure of the submarine isothermal line proof of the current from the pole to the equator in the depth of the ocean. It would be highly desirable that future navigators should enlarge our knowledge on this point, by a larger number of observations with one and the same instrument, or with corrected instruments, which could be accomplished with very little trouble and in a very short time. If they would be satisfied with letting down the thermometrographs at always one and the same depth of some 100 fathoms, this observation would be made in fifteen minutes; and in any case, by a frequent repetition of it, results would be arrived at, especially in latitudes ranging from 40° N. to 40° S., which would be far more instructive for physical geography than the observations hitherto made, where one proceeded or reasoned more on the determination of the diminution of the temperature than upon compared determinations of different places.

"From a current underneath of colder waters from the poles to the equator, some important conclusions arise, viz.:—

"1. The diminution (pointed out) of temperature everywhere up to latitude 60° with the increase of depth, in direct opposition to the conditions observed on dry land.

"2. My numerous determinations of the salinity of the ocean have shown that the maximum of the salinity does not occur at the equator, but invariably some degrees north and south from it (in the Atlantic at 23° N. and 17° S.). I have endeavoured to explain this condition from the greater evaporation in these latitudes, which is comprehensible from the cooperation of the trade-wind, in opposition to the region of calms at the equator (see Mém. de l'Acad. Sc. Math. Ph. et Nat. t i. p. 507). According to the above, I do not, however, doubt that also the slight salinity of the uprising polar water in the region of calms contributes materially to this condition.

"3. It is a point which has been determined by HUMBOLDT, JOHN DAVY, and others, that the water of the ocean is colder at the surface over shallows than at some distance from them over very great depths. This phenomenon, the explanation of which hitherto has not been found to be satisfactory (GEHLER's New Lexicon, t. vi. 3. p. 1687), is a simple consequence of the current of colder water at depths from the pole to the equator, for if this runs against any obstruction, such as a shallow would present, it will rise along it as upon an inclined plane, and approach nearer the surface, and in this manner the surface will be cooled down."

A little later POUILLET*, who does not, however, seem to have been aware of LENZ's researches, remarks.—"It seems certain that there is in general a surface-current carrying towards the polar seas the warm water of the tropics, and a lower current bringing back from the poles the cold water of the polar regions, but these currents are modified in their direction and intensity by a number of causes which depend on the depth of the sea-basins, their configuration, and the influence of winds and tides."

I have already referred to BIOT's criticism of ROSS's work. Reasoning afterwards on the different temperatures shown to exist throughout all seas, and on the impossibility, in consequence, of any portion of it being in a state of rest, he observes† ·—"The existence and the initial direction of these constant currents presupposes three things. first, a permanent cause of movement which forces the polar waters towards the equator, secondly, a constant exterior afflux supplying the great polar streams at the origin and along their course; and lastly, some exhausting cause or outflow, preventing the final accumulation of their products" (p. 79). BIOT, however, in consequence, apparently, of the doubts he felt respecting the accuracy of temperature observations at depths, owing to the anomalous results of ROSS's, hesitates to admit "the inference that the bottom of the sea was occupied by a layer of cold water proceeding from the poles and which is unceasingly renewed" (p. 71), and attaches more weight as a cause of this circulation to the inequality of mean pressure of the atmosphere in different latitudes.

BUFF‡ gave in 1850 a good general summary of the question as it then stood.

* Éléments de Physique, 5 ed. vol. ii. p. 666 (1847).

† 'Journal des Savans' for 1849.

‡ 'Physics of the Earth,' translated by HOFMANN. London, 1851, pp. 172–74.

A few years later EMIL VON LENZ* described the observations made by Dr. EDWARD LENZ during a series of voyages across the Atlantic to the west coast of South America at a small but uniform depth, and with the same instruments throughout. For some reason not explained, the temperatures in the low latitudes of the South Atlantic are not given.

North Atlantic					South Atlantic.				
Lat. N	Long W	Feet deep	Temp at depth.	Surface	Lat S	Long W	Feet deep	Temp at depth	Surface
1° 38'	27°	360	58° 2 F.	81 4	° '	° '		°	
3 14	21	„	61 „	80					
6 9	23	„	60 „	84					
6 52	22	„ †	58 2 „	80	13 28	28	360	72 8	80 2
25 35	37	„	66 3 „	72 5	17 17	19 2 (32 ?)	„	76 6	84
31 48	36	„	64 3 „	73	30 13	46	„	64	77
35 35	17	„ †	62 6 „	63 6	33	72 (52 ?)	„	52	56
35 37	35	„	60 „	68 4	53 12	58	„	43	51
35 39	34	„	61 „	67 8	55 19	62	„	41	48 4
40 40	27	„	56 „	62 6	56	64	„	41	46

On these he remarks, "The number of observations here are so few, that no valid general conclusions can be drawn from them, I only mention that this attempt was substantiated by me in results made public on an earlier occasion (Bull. Phys. Math v. 1847), viz that at the equator, or rather in the region of calms, one finds a notably more rapid diminution of temperature at increased depth than even in the tropic or subtropical zone. We also see here that at 4° N. lat. the temperature at 60 fathoms decreases from 21° R. to 12° R., but at 28° (32 ?) lat. only to 14°·8, and it is first at 36° lat in this depth that one finds nearly the same temperature as at the equator, viz. 12° 6 RÉAUMUR In the Southern Atlantic Ocean, the conditions of temperature at depths appear to approximate more nearly to the equator than in the Northern, possibly in consequence of the northern inclination of the region of calms."

§ V *General Conclusions.*

It is evident that the old observations (all before 1868) have very different degrees of value In laying down the lines of Section of the Bathymetrical Isotherms on the Admiralty "Track Chart" of the world, I have selected those observations which appear the most reliable, and which at the same time offer the most continuous series over the greatest number of parallels of latitude, such as the observations of KOTZEBUE in the North and South Atlantic, and those of DAYMAN‡ in the South Atlantic and

* "Meteorologische Beobachtungen auf dem Atlantischen und Grossen Oceane in den Jahren 1847–49 angestellt von dem Dr. ED. LENZ, verechnet von E LENZ," Bull. Acad. Imp. Sci. St. Pétersbourg, IV. 1863, p 130

† These numbers do not quite agree with the text, where they stand as "420" and "180."

‡ Only the correction for DAYMAN's observations should probably be rather higher than that for the others.

Indian Oceans, subject to, as the correction for pressure, the deduction of 1° FAHR. for every 1700 feet of depth. As the 'Challenger' expedition will supply ample data regarding the deeper temperature-soundings in the intertropical seas, the scarcity of them in the earlier voyages is of less importance. Those, on the contrary, collected on the many Arctic and Antarctic voyages under circumstances of so much difficulty, and which bear in so essential a manner upon the intermediate areas, are fortunately much more complete. The lines of Section have therefore been so selected as to embrace the chief observations of the several explorers in both the Arctic and the Antarctic seas For this purpose two lines traverse respectively the length of the Atlantic and of the Pacific, and two others are run through the Indian and Southern Oceans.

Section No. 1 first traverses the North Atlantic from the top of Baffin Bay to the equator in long. 20° W, and shows the low submarine temperatures prevailing in the higher latitudes on that side of the Atlantic. The bathymetrical isotherm of 35° F seems on this line not to extend beyond lat. 63° N. Soundings have been made in Davis Strait and Baffin Bay between lat. 60° and 77° N. to the depth of 6000 feet, and everywhere the temperature decreases with the depth down to 29° and 28°, or even 27°, and in one instance so low a degree as 25° 75 F. has been recorded. The isotherms of 40°, 50°, and 60° F. in the western area of the Atlantic have likewise a less northward extension than in the eastern area traversed by Section No 2, while that of 70° F, which is affected by the Gulf-stream, extends further north.

Section No. 2, which commences in the seas around Spitzbergen, exhibits, to depths within the annual influence, a temperature as low, if not lower, than in No. 1, while below that the temperature, on the contrary, down to the depths hitherto tried (not quite 5000 feet) increases with the depth. Owing to the great diurnal variations of temperature at the surface or to currents, the fluctuations in the upper strata are frequent and rapid. From 1000 down to 3000 feet the temperature is more uniform at 33° to 34°, and reaches, at 4500 to 4600 feet*, 34° to 35° F. or possibly 36°. Off the coast of Greenland the one experiment of SCORESBY shows a decrease of temperature to the full depth tried, viz. to 28°·5 (corr.) at 708 feet.

From the Spitzbergen seas, the bathymetrical isotherm of 35° F. gradually falls until the latitude of about 50° N. is reached, when its depth is twice what it is in lat 76° to 80°. About lat. 40° N. it appears to have attained its maximum depth of about 11,000 feet, at which it remains to lat. 30°, from about which point it again rises gradually, lying in lat. 12° at a depth of about 8000 feet, and reaching probably still nearer the surface at the equator†. The isotherm of 40° F., which, in this north-eastern part of the Atlantic, extends as far as lat. 72° to 73° north, reaches its maximum depth of about 6000 to 7000 feet between lat. 50° to 30° N., and rises to between 3000

* SCORESBY's deepest sounding was in 76° 30′ N., 4° 48′ W, 7200 feet, no bottom

† The depths of these isotherms in the Atlantic will no doubt require correction, but this will not affect their relative position and general bearing.

and 4000 feet near the equator. Of these two and other lower isotherms in temperate and tropical seas the older observations afford, however, very few data, and we need say little. We wait for those of the 'Challenger.'

Of the bathymetrical isotherms of 50°, 60°, 70°, and 80° F., the data are more ample. They seem respectively to set in about lat. 60°, 50°, 25°, and 12° N., and the first two to attain their greatest depths between lat. 40° and 20°—the isotherm of 50° F. falling to 3000 feet, and that of 60° F. to 1200 feet They then rise, and from lat. 12° N. to the equator, the isotherm of 50° F. comes within 1000 to 1200 feet of the surface, and that of 60° F. from 300 to 400 feet.

In the South Atlantic, on the line of section No 1, which now crosses over to the eastern area of the South Atlantic, the bathymetrical isotherms seem to be prolonged southward more nearly on the same level that they have near the equator—the isotherm of 50° lying at from 1000 to 1400 feet, between lat. 7° and 40° S, and that of 60° F. at 500 or 600 feet In the western area (sect. No. 2) the isotherms of 50°, 60°, and 70° F are much more irregular, sinking in lat. 10° to 20° to about 3000, 1800, and 500 feet, and then rising and ending, as in the other line of section, in about lat. 40° and 45° S But while, on the whole, the higher isotherms range rather further south in the western than in the eastern area, the isotherm of 35° F is in both prolonged further south, on a nearly uniform level of from 7000 to 8000 feet, between lat. 20° and 65°.

The Pacific Sections (Nos. 3 & 4) exhibit a much lesser number of observations, but still sufficient to draw some general conclusions. Starting in one case in the Arctic Sea north of Behring Strait, and in the other in the sea south of Behring Strait, one line of section (No. 3) passes through the Eastern Pacific to the equator in long. 120° W., and the other (No. 4) through the Western Pacific to the equator in long. 180° W. North of Behring Strait the sea is so shallow that the observations barely pass beyond the limits of diurnal variations. The width and depth (180 feet) of that strait itself are also so small that the intercommunication through it between the polar seas and the North Pacific can have little or no effect on the thermal condition of the latter, nevertheless it may be a question whether the submarine isotherm of 60° F. in that ocean extends beyond the lat. of 40° to 45° N., and the isotherm of 50° F. beyond about lat. 55° N., being about 5° less in either case of their northern range in the eastern area of the North Atlantic, while the isotherm of 35° F. disappears, as in the western division of the Atlantic, between lat. 60° to 70° N., instead of having the more indefinite northward range it has in the open North Atlantic.

These isotherms also, instead of the remarkable rise which they present near the equator in the North Atlantic, exhibit in the North Pacific a gradual decline to the equator, where, judging from the few data we have at our disposal, they seem to lie—that of 60° F. at 800 to 1000 feet, of 50° F. at 2000 to 2500 feet, of 40° F. at 4000 to 5000 feet, of 35° F. at 7000 to 8000 feet respectively, and pass the equatorial zone without rise or apparent change of level.

On the other hand, in the South Pacific the conditions are much more like those of

the South Atlantic. In the Eastern division (section No. 3) the isotherms of 60° F. and 50° F. are on a nearly uniform level from the equator to about 35° to 45° S. lat., and extending apparently not quite so far southward as in the Atlantic. In the Western division of the Pacific (section No 4) the several isotherms seem to lie rather deeper, and the isotherms of 60° and 50° F to extend some degrees further south. But we again have, as in the South Atlantic, the same expansion of the isotherms of 40° and 35° F. as they range southward, the latter having in lat. 65° S. a depth of 6000 to 7000 feet, from this point it rises rapidly, or is displaced by colder waters, as it approaches the Antarctic continent.

Section No. 5, which crosses the Indian and Southern Oceans from 20° N to 40° S, exhibits conditions analogous to those which obtain in the Pacific, though the isotherms of 40° and 35° appear to lie deeper, viz at depths of about 9000 to 12,000 feet at the equator. They are then prolonged nearly on the same level to about 12° north, and thence to rise as they approach the head of the Arabian Gulf, where they are lost in the heated surface-waters. In the other direction the three higher isotherms on this line of section maintain a more nearly uniform relative depth of about 200, 500, and 1500 feet,—that of 80° F. terminating in about lat. 20° S, that of 70° F. in lat. 30° S., and that of 60° F. in lat 39° S. At this point the isotherm of 50° F lies at a depth of about 1500 feet, that of 40° F. at 4000 to 5000 feet, and that of 35° F. may be at about 7000 to 8000 feet In this section we have no data south of 40° S lat.

Section No 6 traverses the Southern Ocean more to the eastward We there still find the higher isotherms terminating in nearly the same parallels of latitude, but we can follow the lines of 40° F and 35° F. further south—the former at a depth of about 4000 feet in lat. 53° S and becoming lost in about lat. 65° S., and the latter rising and disappearing in about lat. 70° S South of this is a zone in which the temperature of the sea to the depths (1800 feet) yet tried is 30° and 33° F (corr.).

In the preceding observations the position of the bathymetrical isotherms can only be taken as an approximation to the truth, though they are, there is reason to hope, relatively correct. The deeper isotherms have possibly too high a degree, and the upper ones, it must be borne in mind, are, in different meridians, subject to the action of many causes that may produce aberration, such as displacement by the action of surface-currents, which will vary according to their depth ; while another manifest cause, affecting more especially the lower isotherms, arises from the inequalities of the sea-bed, whereby the lower cold strata are deflected and driven nearer to the surface—an effect not only due to submarine banks and some islands, but caused also by continental shores, as on parts of the southern coasts of Africa and of South America*.

Independently, however, of these local variations, certain general conditions have been clearly established by the researches we have had occasion to review,—such as the presence of a stratum of water at and below 35° extending from the Arctic and

* When this takes place the temperature of the sea at or near the surface will be found to become lower on approaching the shore, against which the colder undercurrent rises. Their existence may thus be proved.

Antarctic seas to the equator, and which no doubt has justly been attributed to deep undercurrents carrying the waters of the poles to tropical regions, and the probable rise of these polar waters to the surface in the equatorial zone of the Atlantic. The source of those glacial waters in the North Atlantic lies, probably, in the Arctic Ocean; and the question arises as to the channels by which they travel southward. The comparatively high temperature of 34° to 36° at depths in the seas around Spitzbergen shows that, although a deep body of cold water may move down the east coast of Greenland, the channels of the comparatively shallow sea between Norway and Spitzbergen are entirely, and of the deeper sea between Spitzbergen and Greenland in great part, occupied by a body of warmer water from the south (for without renewal the degree of heat could not be maintained). On the other hand, the constant low temperature at depths in Baffin Bay, and the southward drifting of the large low-sunk icebergs, show that that sea and Davis Strait afford a passage to a deep glacial current derived from the Arctic seas of North America Issuing from these comparatively narrow channels this body of cold water unites with that passing down the east coast of Greenland, and flows southwards, over the great depths of the Atlantic, apparently to the equator

In the South Atlantic, on the contrary, the channel of the deep-seated glacial water is coextensive with the wide expanse open to the Antarctic seas, so that an unbroken undercurrent of such waters may occupy the one broad bed of that ocean

These two great undercurrents of the Atlantic, flowing respectively from the north and the south poles towards the equator, must eventually meet, and, judging from the rise of the bathymetrical isotherms and the low temperature of the sea immediately beneath the heated surface-waters in the equatorial regions, it is probable, as suggested by LENZ, that the meeting is there, and that it is that which in part determines, in conjunction with the excessive evaporation, the surging-up of the polar waters, though other causes presently to be referred to may assist. In whatever way effected, the waters which thus rise to the surface in the equatorial zone necessarily tend to disperse and escape into other areas, whether by a slow movement in mass, or by more rapid currents in shallower and more definite channels, or by both causes combined.

The course of these deep Arctic and Antarctic undercurrents or streams in the Atlantic may be influenced by another cause, viz. by the west to east trend of the South-American continent from the Caribbean Sea to Cape St. Roque, and by that from east to west of the African continent along the coast of Guinea—projections which both contract the width of the Atlantic, and present barriers which may help to deflect sideways and upwards, on the one (American) side the southward flow of the Arctic waters, and on the other (African) side the northward flow of the Antarctic waters, in a manner analogous to that which takes place on shoals and islands.

It is not my intention to enter upon the discussion of the course and magnitude of the Gulf-stream; but I would suggest whether or not the initial start of that great current, together with the others which originate or acquire new power at the equator, such as the Guinea, the South Equatorial, and the Brazilian currents, may not be cradled

by this surging-up of Arctic and Antarctic waters at or near the equator, while other portions of those great bodies of water are deflected back and imperceptibly return the one to the north polar and the other to the south polar seas—in masses unaffected by the more active shallow drifts and currents sweeping over their surface, and whose course is influenced by trade-winds and the earth's rotation, for while the cold waters are found so comparatively near the surface in the equatorial regions, the presence, at depths, in both the polar seas, of bodies of water having a temperature far above not only that of the winter but the annual temperature of those latitudes, is equally well proved. Thus although the mean annual temperature of Spitzbergen does not exceed 18° F (and DOVE estimates* the normal mean temperature of latitudes 80° to 90° at 4°·5 F), we find that in the seas surrounding that island there is a submarine temperature of 34° to 35°, if not rather higher. In the same way in the Antarctic regions and in latitude 60° to 70° we there also find a submarine temperature nearly as high† Thus there is a rise of from 6° to 8° FAHR. in descending from the surface to depths of 3000 feet to 4000 feet in the open polar seas, whereas in like depths in the equatorial regions of the Atlantic there is a fall of not less than 40° F , extending at greater depths to about 50° F

There is every reason to believe that the open seas of the north polar regions are due, as suggested by MAURY and others, to the influence of warm southern waters, though this is not, as supposed by those authors, owing to the action of the Gulf-stream‡, but to the surging-up of these deeper warm strata, and in the same way the open sea found by COOK, WEDDELL, ROSS, and others, after passing the first barrier of ice in the south polar seas, may be due to a similar cause. The great body of water at 32° to 35° or 36° F. extending to the depth of 4000 to 5000 feet or more, and passing by Spitzbergen, must ultimately be displaced and deflected by the colder and denser waters between 32° and 25° 4 of the polar regions, and rise to the surface, and as the influx is constant, an equilibrium can only be maintained by an efflux as great to other areas By Behring Strait, owing to its narrowness and shallowness, comparatively none passes, but the surface-currents through Smith Sound, and the more intricate channels amongst the islands of the North-American coast and so down Baffin Bay, and that down the east coast of Greenland, originate doubtlessly with these effluent waters. The temperature-soundings to depths of 1000 feet in Baffin Bay are in accordance with this view, for after passing the stratum affected by the diurnal variations, the water to about that depth, although there is no surface-current from the south, has generally a temperature of from 30° to 34°, while that at greater depths sinks at places to a point very closely

* The mean summer temperature of Spitzbergen, according to DOVE, is 34° 5 F

† If, as we have reason to think, the observations of Sir JAMES ROSS should require a larger correction than others, then the isotherms in the Antarctic and Southern Oceans will have to be raised, and the isotherm of 35° will be replaced by one of 33° or 32° F

‡ At the same time there cannot, I think, be any doubt of the influence of the Gulf-stream, as a shallow current, on the seas and northern shores of the British Islands and Norway

approaching to the freezing-point or to that of the maximum density of ordinary sea-water. Moving in the same direction as the great body of colder water which it overlies, the warmer surface-water has a greater velocity than it, and moves over it as a surface-current—the causes which effect its impulsion being of a more energetic character than those which originate during the colder months of the year with the descent of the dense waters and their slow outward propulsion in a deep undercurrent.

In the Pacific Ocean the great breadth of open sea, and the almost entire exclusion of the waters of the north polar seas, have produced conditions very different from those which obtain in the Atlantic. The temperature-soundings are too few to lead to any certain conclusion; but, so far as they go, they seem to show that there is no uprising of cold undercurrents at the equator. The observations referred to by LENZ are so scattered and at such small depths, that they may have been affected by the action of the great cold current which passes northward up the west coast of South America, and is deflected westward at the equator, and by various other surface-currents.

In any case, the remarkable rise of the bathymetrical isotherms in the North Pacific, which cannot be accounted for by any current passing through Behring Strait, leads me to infer that the Antarctic waters pass under the whole length of the Pacific, and are thrown up by the barrier presented at its northern extremity by the American and Asiatic coasts. Some of the great currents of the North Pacific may owe their origin to, while others seem to be strengthened by, these distantly derived waters.

Nor is it easy to account in any other way for the rise of the isotherms of 35° and 40° F at the head of the Arabian Sea after traversing the deep bed of the Indian Ocean. The high temperature of the surface-waters, however, prevents the effects being so apparent in the upper strata of that sea. Again, the causes which influence the great currents of the North-Indian Ocean appear to correspond with the area of surging-up, as they approach the Asiatic continent, of the south-polar undercurrents.

The cause of these phenomena in both hemispheres is, in all probability, connected with the intense cold of the polar regions,—the mean annual difference of from 70° to 80° F between the polar and the intertropical regions forming a permanent disturbing cause, owing to the alteration of density to which the affluent waters are unceasingly subjected*. It is a cause, also, which, from the variation in the density of the surface-water in winter and summer, must materially influence the operation of the currents generally, both at the Arctic and Antarctic regions, during the different seasons of the year, increasing the outflow from the polar seas in the cold months, and the influx in the warmer, whence the outflowing current through Behring Strait in the winter or spring, and the inflowing current in the summer. For the same reason we should expect to find the general circulation more active in the one season than in the other. But the discussion of these interesting questions is not our object.

In no way are the effects of the remarkable interchange between the polar and equatorial waters in the great oceans more conspicuous than in the comparison of the

* According to DOVE the mean temperature of the equator is 79°·8 and of the pole 2°·2.

thermal conditions of those oceans with those of inland seas—the one so dependent on local climatal influences, and the other subject to influences so distant, for whereas it is the winter cold of the latitude which regulates the one, it is the cold of the polar winters which affects the other. Thus the temperature of from 54° to 55° F at depths in the Mediterranean below the influence of the annual variations is that of the sub-winter months of that area, as that of 70° is for the Red Sea. But the most striking case is the sea of Okhotsk, where, in the parallel of Great Britain, but with a winter-cold under 20°, or possibly under 15° F., we have a nearly enclosed sea, of which the submarine temperature at 200 to 700 feet in the month of August is under 29° F., or nearly 2° below zero of Centigrade, the surface-temperature being 47° F.

These questions have necessarily a very important bearing on many geological problems, especially those connected with climates and the distribution of species. For example, it is probable that the increased severity of the climate noticed within the historical period on the east coast of Greenland may arise from that elevation of the land which is shown, by the presence of raised beaches and marine remains at heights of from 50 to 300 feet or more on the north-western coast of Greenland* and amongst the islands of the Northern-American archipelago beyond Baffin Bay, to have taken place at a comparatively recent period, for this, by lessening the width and depth of the many small straits opening into Baffin Bay, has thrown a larger volume of the polar waters into the other channels, as that between Greenland and Spitzbergen, and has thus had the effect of increasing and strengthening the ice-bearing current from the north which passes down the east coast of Greenland. The amelioration of climate towards the close of the Quaternary period may also have been locally greatly influenced by the elevation of the land and shallowing of the seas around Britain and Norway, by which any flow over this area of the deep polar currents has been diverted.

The cognate questions also connected with the southward range of an Arctic fauna or the northward range of a tropical fauna, and, to compare the water with the land, the insular-like character of the fauna of inland seas (all so liable to changes with any alteration in the direction and volume of those deep and obscure† undercurrents to which we have been referring, or by their ingress into seas before closed), are of the highest importance in the consideration whether of the later or of the older geological phenomena of the globe. They are, however, beyond the immediate range of this paper, which I submit as a starting-point for further research.

To conclude, the observations recorded in these pages, after subjecting the readings to the necessary corrections, show:—

1.—*a.* That a stratum of water at and under 35° F. extends beneath the Atlantic from the Arctic to the Antarctic seas‡; and, as it traverses all the parallels of latitude

* There is the same evidence of recent elevation on the coasts of Behring Strait

† Using the word in contradistinction to "conspicuous" surface-currents, such as the Gulf-stream, the effects of which are well known, and have so often been reasoned upon in connexion with geological phenomena

‡ This has now been more fully established by the recent expeditions of the 'Porcupine' and 'Challenger.

irrespective of the surface isothermals, it must have an origin dependent not on local influences, but on others at a distance—such, in fact, as accord only with polar influences.

b. That in the North Atlantic the two channels through which the deep-seated cold polar waters pass southward are Baffin Bay and the sea near the east coast of Greenland; while the shallower seas immediately west of Spitzbergen, and between that island and Norway, are occupied to their entire depth by warmer waters flowing northward, from equatorial regions, towards the pole.

2 That in the North Atlantic the isotherm of 35° extends further in the polar seas than in the South Atlantic, but in both its rise is masked by the extreme climatal variations and by surface-currents.

3.—*a.* That in the equatorial regions of the Atlantic the deep-seated north and south polar waters, either owing to their meeting, or from impinging against projecting continental coasts, or from irregularities in the sea-bed, or from the several causes combined, are deflected and surge up at the surface, as shown by the rise of the bathymetrical isotherms

b. That the main portions of the upper strata of these surging waters flow slowly *en masse* from this equatorial zone towards the poles—such bodies of water moving independently of the drifts and surface-currents by which they are traversed and channelled

4.—*a.* That in the Pacific there is a similar deep stratum of cold water at and under 35°, extending from the Antarctic Ocean to Behring Sea without rising, as in the Atlantic, at the equator.

b. That in the North Pacific the submarine temperature is as low as or lower than in the open North Atlantic in the same latitudes.

c Consequently, as the body of cold water in the North Pacific cannot be of north polar origin (comparatively none passing through Behring Strait), there is reason to believe that the south polar waters traverse the whole length of the Pacific, and rise against the coasts bounding that ocean on the north

5 That in the same way the Southern and Indian Oceans are underlaid by the cold waters proceeding from the Antarctic seas, which surge upwards as they approach the Asiatic coast.

6. That there the surging-up of polar waters in the great oceans, and of tropical waters in Arctic and Antarctic seas, is intimately connected with some of the great surface-currents which originate, or acquire additional force, in equatorial and polar seas, although the ultimate course of these currents may be influenced and determined by the action of the prevailing winds and by the movement of rotation of the earth.

7. That the temperature at depths in inland seas is governed by local causes, and tends in each case to assimilate to (or as near as the physical properties of water will allow) that of the mean winter or sub-winter temperature of the place.

Erratum in Mr. PRESTWICH's *Paper on Submarine Temperatures. Paragraph* 6, *to replace lines* 32 *to* 36, *p.* 638.

6. That some of the great surface-currents, which originate or acquire additional force in the equatorial and polar seas, are intimately connected with the surging-up of polar waters in the great oceans and of tropical waters in Arctic and Antarctic seas, although the ultimate course of these currents may be influenced and determined by the action of the prevailing winds and by the movement of rotation of the earth.

Tables of Submarine Temperatures of the Great Oceans and Inland Seas, taken between 1749 and 1868, arranged according to the Latitudes in each Hemisphere, and reduced to English measures and Greenwich Longitude.

References to the original observations will be found in § II., in notes to the several voyages (given in order of date) Those alone to which (*u*) is attached are from unpublished documents Those also where (M.) is added to the name will be found in MARCET's paper (*anté*, p 595), and not in the works of the original observers. The temperature-readings are given as recorded by each observer To obtain an approximately true reading, it is necessary to apply the correction named at p 612, excepting the observations of LENZ (and KOTZEBUE, 2nd voyage), DU PETIT-THOUARS (when stated "cylinder sound"), MARTINS, AIMÉ, SHORTLAND, VAILLANT (in part), D'URVILLE (in part), and some of PULLEN's, and probably ROSS's and PARRY's of 1818–19 The correction consists in a deduction of 1° FAHR for every 1700 feet of depth The figures in parentheses attached to DU PETIT-THOUARS's observations give his original corrections of temperature and depth A separate list of the voyages on which the observations were taken will be found, in connexion with the names in column VIII, in "Explanation of Map," p 671

TABLE I.—Northern Hemisphere.

I	II.	III	IV	V.	VI.	VII.			VIII	IX
	Date.	North Latitude.	Longitude of Greenwich	Sea.	Depth in feet	Temperature in degrees of Fahr. At depth	Surface	Air	Name of observer	Remarks
1	Mar, 1828	0° 0′	99° 40′ W	N Pacific	480	71°	83°	83°	Beechey	Under the Equator 8° west of the Galapagos Islands
2	" "	" "	" "	" "	960	63·5	"	"	"	
3	22 Apr, 1825	0 ?	179 43 W	N Pacific	4800	45·5	83·7		Kotzebue, 2d voy	Just N of the Equator, between the Sandwich Islands and Australia
4	May, 1824	Near the	above place	N Pacific	6000	36·5	86		"	
5	6 May, 1818	0 7	20 26 W	N Atlantic	339	59·1	83·3	84·1	Kotzebue	Between Brazil and Sierra Leone
5a	21 Apr, 1848	0 30		N Pacific	6000	43·5	80·5		Belcher	1° W of Albemarle Island
6	Oct, 1836	0 33	8 16 E	N Atlantic	3918	43	78·7		Wauchope	Gulf of Guinea. Rope vertical
7	6 May, 1818	0 36	20 29 W	N Atlantic	416	58	83·3	84·5?	Kotzebue	Between Brazil and Sierra Leone
8	5 Sept, 1772	0 52	8 W ?	N Atlantic	510	66	74	75·5	Forster	Between the coast of Guinea and Ascension
9	8 Aug, 1828	1	126 40 E.	N Pacific	1541	54·8	82·4	81·2	D'Urville	In the Straits of Molucca.
10	12 Jan, 1847	1 5	22 32 W	N Atlantic	2010	52	83	77	Dayman	In mid-ocean W of No 7
11	12 May, 1816	1 17	177 5 W	N Pacific	1800	55	82·5	83	Kotzebue	N of Island of New Nantucket
12	1847–49	1 38	27 W	N Atlantic	360	58·2	81·4		E Lenz	Near the Island of St Paul
13	8 May, 1818	1 58	21 6 W	N Atlantic	467	57·5	82·6	74	Kotzebue	Not far from No 10
14	2 Dec, 1857	2 20	28 44 W	N Atlantic	4080	46·2	80		Pullen	In soundings 90 miles off the Island of St Paul.
15	" "	" "	" "	" "	6480	38·5	"		"	
16	4 Feb, 1829	2 30	19 10 W	N Atlantic	5101	43·6	80·5	79·2	D'Urville	In mid-ocean, between the north-west of Brazil and the coast of Guinea.
17	9 May, 1818	2 32	21 13 W	N Atlantic	480	58·5	84·3	81·8	Kotzebue	
18	5 Feb, 1829	3	19 10 W	N Atlantic	2657	45·6	78·8	80·5	D'Urville	
19	" "	"	" "	" "	1594	59	83·2	82·8	"	
20	10 May, 1818	3 5	21 24 W	N Atlantic	480	58·9	84·5	84·4	Kotzebue	
21	1847–49	3 14	21 W	N Atlantic	360	61	80		E Lenz	
22	11 May, 1818	3 30	21 53 W	N Atlantic	463	59	83	79·3	Kotzebue	
23	6 Feb, 1829	3 30	19 20 W	N Atlantic	53	80·2	81	80·5	D'Urville	
24	" "	" "	" "	" "	133	79·5	"	"	"	
25	" "	" "	" "	" "	266	70·8	"	"	"	
26	" "	" "	" "	" "	531	67·8	81·8	"	"	
27	" "	" "	" "	" "	797	65·3	82	81·5	"	
28	" "	" "	19 10 W	" "	1062	60·6	81	80·5	"	
28a	22 May, 1803	3 27	145 W	N Pacific	600	60·2	82	82	Krusenstern	Among the Society Islands
29	23 Sept., 1858	3 37	160 52 E	N Pacific	1200	71·2	85·8	82·8	Wullerstorf	Between Marshall and Salomon Islands
30	22 Feb, 1804	4	16 ? W	N Atlantic	2274	45·5	88	88·3	Péron	Therm remained down 1h 15m
31	22 Sept., 1858	4 2	160 41 E	N Pacific	600	81·6	84·8	81·6	Wullerstorf	Near the Caroline Islands
32	30 Dec, 1838	4 14	91 8 E.	Indian Ocean	600	70	82·5	82	Pratt	Between Sumatra and Ceylon

TABLE I.—Northern Hemisphere (continued).

I	II	III	IV.	V	VI	VII.			VIII	IX.
	Date.	North Latitude	Longitude of Greenwich	Sea	Depth in feet	Temperature in degrees of Fahr			Name of observer.	Remarks.
						At depth	Surface	Air		
		° ′	° ′			°	°	°		
33	12 May, 1818	4 16	22 42 W	N Atlantic	480	59·2	82·5	77·1	Kotzebue	In mid-ocean, between the coasts of Guinea and of Guayana.
34	1 Dec, 1857	4 16	28 42 W	N Atlantic	6000	42·5	80		Pullen	
35	„ „	„ „	„ „	„ „	9000	39·4	„		„	
36	24 May, 1839	4 23	26 06 W	N Atlantic	6398 (5151?)	42·8 (39·7)	80·6	77·5	DuPetit Thouars	Cylinder full of water
37	„ „	„ „	„ „	„ „	6398 (6037?)	37·8	„	„	„ „	Cylinder sound
38	27 June, 1837	4 32	134 34 W	N Pacific	12273	35	81	78·8	DuPetit Thouars	Instrument crushed Index fixed
39	13 May, 1818	4 33	24 11 W	N Atlantic	471	57·9	82·6	81·9	Kotzebue	Between Island of St Paul and Sierra Leone
40	Oct 1823	5	22 ? W	N Atlantic	3000	43·2	83·8		Kotzebue, 2d voy	In mid-ocean near No 33
41	19 Feb, 1804	5	18 ? W	N Atlantic	1280	48·6	87	90	Péron	Therm remained down 1h 50m
42	7 Jan 1847	5 8	22 19 W	N Atlantic	2040	49	83	82	Dayman	Mid-ocean, between Brazil and Sierra Leone
43	14 May, 1818	5 29	26 9 W	N Atlantic	479	56·5	82·9	83·9	Kotzebue	Between Guayana and Liberia
44	6 Jan, 1847	5 54	22 34 W	N Atlantic	2166	50	82	79	Dayman	In mid-ocean near No 42
45	15 May, 1818	6	27 34 W	N Atlantic	414	55·9	81·9	81·8	Kotzebue	Between Guayana and Liberia
46	1847–49	6 9	23 W	N Atlantic	360	60	84		E Lenz	
47	5 Jan, 1847	6 28	22 39 W	N Atlantic	1110	51	84	82	Dayman	Between Sierra Leone and Guayana
48	1847–49	6 52	22 W	N Atlantic	420	58·2	80		E Lenz	Between Cape-Verd Is & St Paul
49	24 Nov, 1800	7	20 W?	N Atlantic	320	61·2	86		Péron	Therm broke and replaced
50	16 May, 1818	7 13	28 32 W	N Atlantic	368	58	81·5	82·3	Kotzebue	In mid-ocean, between Brazil and Sierra Leone
51	10 Oct, 1823	7 20	21 59 W	N Atlantic	3435	35·9	78·5		Lenz	
52	16 Mar, 1858	7 47	93 18 E	Indian Ocean	510	68·8	81·8	81·8	Wullerstorf	Near the Nicobar Islands
53	22 Nov, 1800	8	Par of Cape Verd Isl	N Atlantic	532	77	86·5	86	Péron	Therm down only 5m
54	14 Sept, 1858	8 2	158 10 E	N Pacific	1200	66·5	84·5	82	Wullerstorf	Amongst the Caroline Islands
55	5 Mar, „	8 29	93 33 E	Indian Ocean	480	78·2	83	80·3	„	East of Nicobar Islands
56	3 Jan, 1847	8 55	22 38 W	N Atlantic	1116	59	82	78	Dayman	In the parallel of Sierra Leone
57	13 Nov, 1817	8 59	155 36 E	N Pacific	600	56·2	87	85	Kotzebue	Between the Radack and the Mariana Islands
58	6 July, 1826	9	20 40 W	N Atlantic	2125	41·2	80·3	76·8	D'Urville	7° W of Sierra Leone
59	12 May, 1846	9	97 W	N Pacific	60	85	87	84	Kellett	Between the Galapagos Islands and Acapulca, Mexico
60	„ „	„	„	„ „	120	83	„	„	„	
61	„ „	„	„	„ „	180	81	„	„	„	
62	„ „	„	„	„ „	240	77	„	„	„	
63	„ „	„	„	„ „	300	66	„	„	„	
64	„ „	„	„	„ „	600	56	„	„	„	
65	„ „	„	„	„ „	1200	53	„	„	„	
66	„ „	„	„	„ „	1800	48	„	„	„	
67	„ „	„	„	„ „	2400	46	„	„	„	
68	„ „	„	„	„ „	3000	44	„	„	„	
69	14 Nov 1817	9 20	155 16 E	N Pacific	150	77	83	84	Kotzebue	Amongst the Caroline Islands
70	15 Nov, 1817	9 26	154 59 E	N Pacific	90	79	87·4	85·7	Kotzebue	Between the Radack and the Mariana Islands.
71	„ „	„	„ „	„ „	300	59·1	„	„	„	
72	„ „	„	„ „	„ „	114	51·4	„	„	„	
73	„ „	„	„ „	„ „	606	49·5	„	„	„	
74	17 May, 1818	9 27	29 7 W	N Atlantic	132	58·4	79·2	80	Kotzebue	Between the Cape-Verd and St Paul Islands.

Table I.—Northern Hemisphere (continued).

I	II.	III	IV.	V	VI	VII			VIII.	IX
	Date	North Latitude	Longitude of Greenwich	Sea	Depth in feet	Temperature in degrees of Fahr			Name of observer	Remarks
						At depth	Surface	Air		
75	12 Apr., 1828	10° 3′	80° 31′ E	Bay of Bengal	531	63°	86°	86°	Blosseville	Off the north coast of Ceylon
76	" "	" "	" "	" "	3082	47	"	"	"	
77	17 Nov., 1817	10 3	153 17 E	N. Pacific	438	57.4	84.2	83.5	Kotzebue	N. of the Caroline Islands
78	Aug., 1816	10 14	29 9 W	N. Atlantic	5796	51	80		Wauchope	Corrected depth 2880 feet
79	18 Nov., 1817	10 41	152 07 E	N. Pacific	366	59.9	83.9	83.2	Kotzebue	N. of the Caroline Islands.
80	19 " "	11 4	150 56 E	" "	192	58.6	83.7	82.7	"	
81	18 May, 1818	11 35	30 56 W	N. Atlantic	393	59.4	78.8	79.8	"	S.W. of Cape Verd Islands
82	20 Nov., 1817	11 42	150 09 E	N. Pacific	516	63	84	84½	Kotzebue	Between the Mariana and the Caroline Islands
83	30 Aug., 1858	11 55	149 53 E	" "	270	84.8	84.5	83.8	Wullerstorf	
83a	3 Sept., 1836	11 59	111 55 W	" "	4266	42.8	78.8	81	Vaillant	Between Mexico and the Marquesas Islands
83b	4 " "	12 6	112 40 W	" "	2133	49	80	82.4	"	
84	21 Nov., 1817	12 28	149 06 E	" "	468	66.9	83.3	81.1	Kotzebue	Between the Mariana and the Caroline Islands
85	22 July, 1843	12 36	25 35 W	N. Atlantic	900	52	79.5		James Ross	140 miles W. of Cape Verd Islands. No soundings in 11 100 feet
86	" "	" "	" "	" "	1800	47.6	"		"	
87	" "	" "	" "	" "	8100	39.5	"		"	
88	" "	" "	" "	" "	11100	39.6	"		"	
89	Before 1857	13 9	78 W	Caribbean Sea	1440	48	83		Dunsterville	Quoted by Maury without date or exact position.
90	" "	" "	" "	" "	2316	43	"		"	
91	" "	" "	" "	" "	2700	42	"		"	
92	" "	" "	" "	" "	3000	43	"		"	
93	Apr., 1859	13	48 10 E	Gulf of Aden	7200	45	81.5		Pullen (n)	In soundings
94	19 May, 1818	13 24	32 2 W	N. Atlantic	393	58.1	76.5	77.7	Kotzebue	Between Cape Verd Islands and Guayana.
94a	27 Mar., 1837	13 27	83 20 E	Bay of Bengal	3200	46	78.8	78	Vaillant	Uncertain
95	22 Nov., 1817	13 28	147 18 E	N. Pacific	396	69.9	83	83.1	Kotzebue	E. of the Mariana Islands.
96	13 Dec., "	13 51	119 36 E	China Sea	561	61.5	82.2	84.5	"	West of Luzon
97	23 Nov., "	13 52	145 11 E	N. Pacific	270	71.1	82.9	83.7	"	Between the Mariana and the Philippine Islands
98	Mar., 1828	14 22	99 35 W	N. Pacific	600	57	88	91	Beechey	Off the south-west coast of Mexico
99	" "	" "	" "	" "	1200	55	"	"	"	
100	" "	" "	" "	" "	1800	48.5	"	"	"	
101	" "	" "	" "	" "	2400	49.5	"	"	"	
102	Apr., 1859	14 26	54 5 E	Arabian Sea	9000	43.5	82.5		Pullen (n)	In soundings. Entrance of the Gulf of Aden
103	28 June, 1826	15	22 40 W	N. Atlantic	425	64.8	73.4	73.4	D'Urville	E. of the Cape-Verd Islands.
104	28 June, 1858	15 5	118 3 E	China Sea	510	83	84.7	84.5	Wullerstorf	1¼° W. of Luzon
105	2 Jan., 1847	15 28	23 22 W	N. Atlantic	1080	53	73	72	Dayman	E. of the Cape Verd Islands
106	20 May, 1818	15 51	32 56 W	N. Atlantic	384	65.5	76.5	77.8	Kotzebue	W. of Cape Verd Islands
107	Jan., 1827	16 5	133 35 W	N. Pacific	1992	40	75	76	Beechey	Between Mexico and the Sandwich Islands
108	" "	" "	" "	" "	2592	45	"	"	"	
109	1 Dec., 1817	16 32	140 56 E	N. Pacific	534	68.7	82.5	82.5	Kotzebue	Between the Mariana and the Philippine Islands.
110	12 " "	16 42	119 26 E	China Sea	483	60.1	80.5	81.7	"	West of Luzon.
110a	11 Sept., 1836	16 47	115 40 W	N. Pacific	6930	42.4	82	84.7	Vaillant	5° N.W. of No. 83b
111	Apr., 1859	16 57	64 21 E	Arabian Sea	11280	44.4	82		Pullen (n)	In soundings
112	10 Oct., 1827	17 5	83 12 E	Bay of Bengal	1647	50.4	85.7	90.5	Blosseville	Off the Circars coast
113	8 July, 1857	17 19	29 50 W	N. Atlantic	1200	60.3	80.8	78.9	Wullerstorf	W. of the Cape-Verd Islands

Table I.—Northern Hemisphere (continued).

I.	II. Date.	III North Latitude	IV Longitude of Greenwich	V. Sea	VI Depth in feet	VII Temperature in degrees of Fahr. At depth	Surface	Air	VIII. Name of observer.	IX. Remarks.
		° ′	° ′			°	°	°		
114	2 Dec, 1817	17 23	139 14 E	N Pacific	456	70·2	81·4	81·2	Kotzebue	Between the Mariana and the Philippine Islands.
115	June, 1825	17 30	27 1 W	N Atlantic	480	60·5	76·5	77	Beechey	N W of Cape-Verd Islands.
115a	20 Jan, 1837	17 54	119 47 E	China Sea	3733	42·4	81·2	78	Vaillant	In sight of Luzon
116	11 Dec., 1817	18	119 53 E	China Sea	570	60	82	82	Kotzebue	W of Luzon
117	21 May, 1818	18 1	34 24 W	N Atlantic	432	68·6	78·8	77·7	”	Between Senegal and Martinique
118	10 Nov, 1836	18 21	134 20 E	N Pacific	3182	42	80	81·5	Vaillant	Between the Philippine and the Mariana Islands
119	27 ” ”	” 27	134 18 E	” ”	4201	40·6	”	79	”	
120	3 Dec, 1817	18 25	137 56 E	N Pacific	366	71·8	81·5	79·6	Kotzebue	
121	1 Jan, 1847	18 40	23 18 W	N Atlantic	468	70	73	68	Dayman	N of the Cape-Verd Islands
122	” ”	” ”	” ”	” ”	1068	57	”	”	”	
123	Mar, 1827	18 51	163 58 E	N Pacific	600	67	79·5	75	Beechey	Between Lamira and the Marshall Islands, Polynesian Archipelago
124	” ”	” ”	” ”	” ”	1200	54	”	”	”	
125	” ”	” ”	” ”	” ”	1800	48	”	”	”	
126	Mar, 1827	18 51	161 30 E	N Pacific	2520	44	79	76	Beechey	
127	” ”	18 53	148 54 E	” ”	1200	57	79·5	82	”	E of the Mariana Islands
127a	Feb 1804	19	114 E	Indian Ocean	420	58	72·2		Horner	Off the N W coast of Australia.
128	4 Dec, 1817	19 20	134 32 E	N Pacific	270	70·0	80·8	79·8	Kotzebue	Between the Mariana and Philippine Islands
129	5 ” ”	19 44	132 15 E	” ”	438	67·1	79	79·8	”	
130	6 ” ”	19 44	130 35 E	” ”	498	67·6	79	77·3	”	
131	22 May, 1818	19 59	35 10 W	N Atlantic	471	68·5	76·2	76·9	Kotzebue	In mid-ocean
132	13 Nov, 1822	20 30	83 30 W	Caribbean Sea	7476	45·5	83		Sabine	Corrected depth 6000 feet.
133	9 July, 1837	21 06	155 59 W	N Pacific	531	65·4	77	76	DuPetit Thouars	Cylinder sound
134	31 Dec, 1846	21 13	22 1 W	N Atlantic	1158	61	71	66	Dayman	Between the Canaries and Cape Verd Islands
135	18 May, 1824	21 14	164 E	N Pacific	898	61·5	79·5		Lenz	Between the Sandwich Islands and the coast of China.
136	” ”	” ”	” ”	” ”	2035	37·6	”		”	
137	” ”	” ”	” ”	” ”	4236	37·3	”		”	
138	” ”	” ”	” ”	” ”	5835	36·4	”		”	
139	23 May, 1818	21 40	36 14 W	N Atlantic	368	68·8	75·8	76·7	Kotzebue	Between Canaries and West-Indian Islands
140	13 Jan, 1837	21 50	19 33 W	N Atlantic	2657 (1607?)	50	70·2	71·6	DuPetit Thouars	Cylinder full of water
141	June, 1825	22 2	21 14 W	N Atlantic	240	63	72	74	Beechey	4° W of Cape Blanco
141a	Nov 1804	23	132 E	N Pacific	300	72	74·2		Horner	Between the Loo choo and the Mariana Islands
141b	” ”	”	”	” ”	780	60·2	”		”	

The later observations in the Indian Ocean by Capt Shortland are given as a whole without separate particulars:—

1868 Between Jan 28 and Feb 13	Between Koorla-Moorla and Bombay (17° to 20° lat N and 45° to 70° long E)	600	60	75	74·5	Shortland	Mean of all the observations between these dates.
		3000	50·9	”	”	”	
		6000	42·8	”	”	”	
		9000	35·3	”	”	”	
		12240	33·7	”	”	”	
		13020	33·5	”	”	”	
Feb 22 to March 6	Between Koorla-Moorla and Aden (13° to 17° lat N and 45° to 55° long E)	600	67·7	76·5	78·8	”	Mean of the observations between these dates.
		3000	54·2	”	”	”	
		6000	45·4	”	”	”	
		7800	36	”	”	”	

TABLE I.—Northern Hemisphere (continued).

I.	II. Date.	III. North Latitude	IV. Longitude of Greenwich	V. Sea.	VI. Depth in feet	VII. Temperature in degrees of Fahr. At depth	Surface	Air	VIII. Name of observer	IX. Remarks.
141c	June, 1804.	23° '	178° ' E.	N. Pacific	150	76·4	78°	°	Horner	Between the Sandwich Islands and Japan
141d	„	„	„	„ „	300	70·8	„		„	
141e	„	„	„	„ „	750	62	„		„	
142	24 May, 1818	23 6	36 51 W	N. Atlantic	471	69·6	76·9	77	Kotzebue	In mid-ocean
143	May, 1827	23 6	124 52 E	China Sea	1260	55·5	80·5	82	Beechey	Off the east coast of Formosa
144	„ „	„ „	„ „	„ „	1800	47	„	„	„	
145	„ „	„ „	„ „	„ „	2100	45	„	„	„	
146	30 Dec., 1846	23 22	20 58 W	N. Atlantic	396	66	69	68	Dayman	Between the Cape-Verd and Canary Islands
147	„ „	„ „	„ „	„ „	1140	61	„	„	„	
147a	22 June, 1804	On the N. tropic	178 4 E	N. Pacific	150	76	78		Krusenstern	Between the Sandwich Islands and Japan
147b	„ „	„ „	„ „	„ „	300	71	„		„	
147c	„ „	„ „	„ „	„ „	750	62·6	„		„	
148	June, 1826	24 57	163 21 W	N. Pacific	1200	67	77	76	Beechey	Between Sandwich and Gardner Island
149	8 Feb., 1825	25 6	156 58 W	N. Pacific	1070	57·5	71		Lenz	N. of the Sandwich Islands
150	1749	25 13	25 12 W	N. Atlantic	3900	53	84	84	Ellis	43m to haul up.
151	„	„ „	„ „	„ „	5346	53	„	„	„	
152	25 May, 1818	25 23	37 W	N. Atlantic	435	68·9	76	76	Kotzebue	Between the Canaries and the W.-India Islands
153	1847–49	25 35	37 W	N. Atlantic	360	66·3	72·5		E. Lenz	
154	Dec., 1827	25 38	117 48 W	N. Pacific	300	62	63	62·5	Beechey	3° distant from the coast of Lower California
155	„ „	„ „	„ „	„ „	900	50	„	„	„	
156	„ „	„ „	„ „	„ „	1200	47·5	„	„	„	
157	„ „	„ „	„ „	„ „	1800	47·5	„	„	„	
158	June, 1853	Off Cape	Florida	N. Atlantic	3300	49			Bache	12 miles E. of the lighthouse
158a	June, 1803	26	37 W	N. Atlantic	420	65·7	74·2		Horner	Between Africa and the West India Islands.
158b	„ „	„	„	„ „	1200	63	„		„	
159	6 June, 1846	26 38	133 26 W	N. Pacific	60	69	71	70	Kellett	Between Lower California and the Sandwich Islands
160	„ „	„ „	„ „	„ „	120	68	„	„	„	
161	„ „	„ „	„ „	„ „	180	68	„	„	„	
162	„ „	„ „	„ „	„ „	240	68	„	„	„	
163	„ „	„ „	„ „	„ „	300	68	„	„	„	
164	„ „	„ „	„ „	„ „	600	64·5	„	„	„	
165	„ „	„ „	„ „	„ „	1200	50	„	„	„	
166	„ „	„ „	„ „	„ „	1800	46	„	„	„	
167	„ „	„ „	„ „	„ „	2400	44·5	„	„	„	
168	„ „	„ „	„ „	„ „	3000	43	„	„	„	
168a	Nov., 1804	27	147 W	N. Pacific	180	70·8	78		Horner	10° N.E. of the Sandwich Islands
168b	„ „	„	„	„ „	540	64·7	„		„	
168c	„ „	„	„	„ „	600	64·4	„		„	
168d	„ „	„	„	„ „	720	64·4	„		„	
169	4 Mar., 1829	27	31 40 W	N. Atlantic	2657	51·2	69·3	68	D'Urville	Between Teneriffe and Bermuda
170	27 June, 1857	27 2	24 7 W	N. Atlantic	600	72·2	74·4	73·8	Wullerstorf	No soundings in 24,300 feet
171	„ „	„ „	„ „	„ „	1440	63·7	„	„	„	

Table I.—Northern Hemisphere (continued).

I.	II. Date.	III. North Latitude	IV. Longitude of Greenwich	V. Sea	VI. Depth in feet	VII. Temperature in degrees of Fahr. At depth	Surface	Air	VIII. Name of observer.	IX. Remarks
172	9 Nov., 1857	27° 31′	21° 39′ W	N Atlantic	3000	50°	72°	°	Pullen (u)	Between the Canaries and Cape-Verd Islands.
173	" "	" "	" "	" "	4800	44·5	"		" (u)	
174	26 May, 1818	27 38	37 10 W	N Atlantic	448	65·7	74·5	75	Kotzebue	In mid-ocean.
175	22 Sept., 1817	27 50	152 21 W	N Pacific	30	75	77	76·1	Kotzebue	6° N E from the Sandwich Islands
176	" "	" "	" "	" "	60	74·5	"	"	"	
177	" "	" "	" "	" "	150	73·7	"	"	"	
178	" "	" "	" "	" "	300	67·2	"	"	"	
179	" "	" "	" "	" "	600	61	"	"	"	
180	" "	" "	" "	" "	1200	51·5	"	"	"	
181	June, 1826.	28 22	172 17 W	N. Pacific	900	57	76·5	77	Beechey	Off Bunker Island
182	1854	95 miles off Cape Canaveral.		N Atlantic	2100	50	82?		Craven	Exact position not given
183	30 Dec., 1846	28 34	18 38 W	N Atlantic	780	63	67	66	Dayman	W of the Canaries
184	June, 1826	28 52	173 9 W	N Pacific	2400	47	78	81	Beechey	N of Bunker Island, Polynesian Archipelago
185	" "	" "	" "	" "	3600	41	"	"	"	
186	" "	" "	" "	" "	4704	42·8	"	"	"	
187	10 Oct., 1837	29 32	34 40 W	N Atlantic	8838	44	75	79	Vaillant	Between the Canaries and Florida
188	1 June, 1816	29 24	160 34 E.	N Pacific	600	62	74	75	Kotzebue	7° N N W of the Sandwich Islands
189	" "	" "	" "	" "	1800	52·5	"	"	"	
190	17 Nov., 1837	29 25	118 51 W	N Pacific	2657	43·3	65·3	69·5	DuPetitThouars	Cylinder full
190a	June, 1803	30	40 W	N Atlantic	90	70·2	72·5		Horner	Between the Canaries and Bermuda.
190b	" "	"	"	" "	180	68·5	"		"	
190c	" "	"	"	" "	378	65·7	"		"	
190d	" "	"	"	" "	840	62	"		"	
190e	" "	"	"	" "	1020	62	"		"	
190f	" "	"	"	" "	1200	62	"		"	
191	27 May 1818	30 3	37 24 W	N Atlantic	368	66·5	73	75·5	Kotzebue	
192	25 Oct., 1815	30 12	15 14 W	N Atlantic	1176	56·3	74·3	74·3	Kotzebue	Between the Canaries and Madeira
193	22 June, 1857	30 50	23 6 W	N Atlantic	576	67	71	71	Wullerstorf	Between the Canaries and the Azores
194	23 July, 1817	31 1	123 46 E	Yellow Sea	240	65	74	76	Abel	E of Chusan
195	1847–49	31 48	36 W	N Atlantic	360	64·3	73	"	E Lenz	Between the Azores and West-India Islands
196	31 Aug., 1825	32 6	136 48 W	N Pacific	578	56	70·6		Lenz	About 3° distant from the south coast of Japan
197	" "	" "	" "	" "	1364	43·6	"		"	
198	" "	" "	" "	" "	2870	38·8	"		"	
199	" "	" "	" "	" "	3773	35·9	"		"	
200	6 Nov., 1857	32 13	19 5 W	N Atlantic	2400	51·5	70·5	"	Pullen	Between Madeira and the Canaries.
201	6 May, 1826	32 20	42 30 W	N Atlantic	6470	36	69·7		Lenz	Between Madeira and Bermuda.
202	28 May, 1818	32 36	36 35 W	N Atlantic	393	67·1	72	72·7	Kotzebue	
203	1844?	32 46	165 53 W	N Pacific	600	55·7			Belcher	Between the Sandwich and the Aleutian Islands Quoted by Jas. Ross, vol. ii p. 58.
204		" "	" "	" "	900	52·7			"	
205		" "	" "	" "	1800	48·1			"	
206		" "	" "	" "	2700	43·2			"	
207		" "	" "	" "	3600	43·2			"	

TABLE I.—Northern Hemisphere (continued).

I.	II. Date.	III North Latitude.	IV. Longitude of Greenwich	V Sea.	VI Depth in feet.	VII. Temperature in degrees of Fahr At depth	Surface	Air	VIII. Name of observer.	IX Remarks
207a	July, 1804	33 ° ′	170 ° ′ E	N Pacific	330	60°5	70°2	°	Horner	Between the Sandwich Islands and Japan.
207b	" "	"	"	" "	1200	53 6	"		"	
208	7 June, 1857	33 38	14 4 W	N Atlantic	720	59 6	69	68 6	Wüllerstorf	Between Madeira and Morocco
209	29 May, 1818	34 34	35 55 W	N Atlantic	452	62	69 2	69 3	Kotzebue .	Between Bermuda and Spain
210	June, 1826	34 51	165 39 E.	N Pacific	1920	54·7	78	69	Beechey	Between Japan and the Sandwich Islands
211	" "	" "	" "	" "	3450	43	"	"	"	
212	" "	" "	" "	" "	4560	43 5	"	"	"	
213	17 Dec., 1846	34 52	16 24 W	N Atlantic	792	61	61	59	Dayman	N of Madeira
213a		about 35°?	65 W ?	N Atlantic		35	80		Maury	Bottom of Gulf-stream
214	June, 1826	35 11	165 21 E.	N Pacific	900	62	72	78	Beechey	In mid-ocean
215	" "	" "	" "	" "	1500	57·2	"	"		
216	4 June, 1857	35 20	8 55 W	N Atlantic	420	59 6	65 5	66 8	Wullerstorf	Near the Strait of Gibraltar
217	1847–49	35 35	17 W	N Atlantic	180	62·6	63 6		E Lenz	N of Madeira
218	"	35 37	35 W	" ,	360	60	68 4		"	S W of the Azores
219	"	35 39	34 W	" "	360	61	67 8		"	S W of the Azores.
220	30 May, 1818	35 41	35 12 W	N Atlantic	445	62 3	74 5	69 9	Kotzebue	7° S W of the Azores.
221	14 Sept, 1817	35 51	147 38 W	N Pacific	24	72	72 2	75	Kotzebue	Between the Sandwich Islands and the coast of California
222	" "	" "	" "	" "	48	70·9	"	"	"	
223	" "	" "	" "	" "	90	68 1	"	"	"	
224	" "	" "	" "	" "	150	57 6	"	"	"	
225	" "	, "	" "	" "	300	54	"	"	"	
226	" "	" "	" "	" "	600	51	"	"	"	
227	" "	" "	, "	" "	2448	42 8	"	"	"	
228	18 Sept, 1817	36 9	148 9 W	N Pacific	150	57 1	71 9	73	Kotzebue	
229	" "	" "	" "	" "	600	52 8	"	"	"	
230	" "	" "	" "	" "	1800	44	"	"	"	
231	24 July, 1817	36 24	122 59 E.	Yellow Sea	90	67	71	75	Abel	S of Staunton Island
232	6 June, 1816	37 3	160 43 E.	N Pacific	60	59 5	61	63	Kotzebue	Between the Polynesian Archipelago and Kamtschatka
233	" "	" "	" "	" "	150	56 8	"	"	"	
234	" "	" "	" "	, "	600	52 7	"	"	"	
235	" "	" "	" "	" "	1800	43	"	"	"	
236	31 May, 1818	37 9	34 31 W	N Atlantic	378	62 2	68 7	73	Kotzebue	6° W of the Azores
237	25 July, 1817	37 30	122 40 E.	Yellow Sea	90	66	69	74	Abel	Upper part near the coast
238	" "	" "	" "	" "	120	62	67	72	"	
239	26 " "	37 38	121 34 E.	, "	90	66	67	74	"	
240	1 June, 1818	38 9	33 8 W	N Atlantic	445	61 5	68 9	68 7	Kotzebue	W of Fayal
241	27 July, 1817	38 12	120 20 E	Yellow Sea	90	72	74	75	Abel	Gulf of Petchili
242	June, 1826	38 55	165 48 E	N Pacific	1080	44	61	64	Beechey	Between the Polynesian Archipelago and Aleutian Islands
243	" "	" "	" "	"	2280	41 5	"	"	"	
244	16 Oct., 1815	39 4	13 8 W	N Atlantic	828	55	69 1	72 5	Kotzebue	4° W. of Lisbon.
245	" "	" "	" "	" "	576	56 ·	"	"	"	
246	2 June, 1818 .	39 15	31 3 W	N Atlantic	432	60 1	67 5	65	,	Mid-ocean, W of the Azores.
247	15 Oct., 1815	39 27	12 57 W	N Atlantic	600	55·7	68 5	71 1	"	Off the coast of Portugal

TABLE I.—Northern Hemisphere (continued).

I	II Date	III North Latitude	IV Longitude of Greenwich	V Sea	VI Depth in feet.	VII Temperature in degrees of Fahr At depth	Surface	Air	VIII Name of observer.	IX Remarks.
248	1847-49	40° 40'	27° ' W	N Atlantic	360	58°	62° 6	°	E Lenz	N of the Azores.
249	24 Aug, 1825	41 12	141 58 W	N Pacific	1308	41 2	66 5		Lenz	Between the Sandwich Islands and British Columbia
250	" "	" "	" "	" "	3263	35·8	"		"	
250a	1854	160 miles off Nantucket		N Atlantic	120	67	82?		Craven	Exact position not given
251	"			" "	5400	35	"		"	
252	May, 1825	41 20	14 40 W	N Atlantic	840	58	64	62	Beechey	6° W of the coast of Portugal
253	19 Aug, 1837	41 42	162 42 E	N Pacific	1066 (905½)	41 2	58	53 6	DuPetitThouars	Cylinder sound.
254	4 June, 1818	41 43	27 23 W	N Atlantic	442	58 6	64 1	64	Kotzebue	4° N of the Azores
255	26 Oct, 1837	42 32	34 58 W	N Atlantic	4688	46	61	64	Vaillant	Between Portugal and New York.
256	18 Aug, 1837	42 1	163 38 E	N Pacific	1066 (640?)	41 5	58 6	60 2	DuPetitThouars	South of Kamtschatka Cylinders sound
256a	15 July, 1868	43 30	38 50 W	N Atlantic	600	62	73	77	Chimmo	Soundings in 13,680 feet
256b	" "	" "	" "	" "	1800	52	"	"	"	
256c	" "	" "	" "	" "	6000	42	"	"	"	
256d	Sept, 1868	43 40?	38 0 W?	N Atlantic	600	59	69	68	Chimmo	Near the Grand Bank of Newfoundland
256e	" "	" "	" "	" "	2400	49	"	"	"	
256f	" "	" "	" "	" "	6000	43	"	"	"	
256g	" "	" "	" "	" "	12000	42	"	"	"	
256h	29 Aug, 1868	44 3	48 7 W	N Atlantic	300	43	61	61	Chimmo	Soundings in 9900 feet
256i	" "	" "	" "	" "	6000	39 5	"	"	"	
256j	July, 1868	Western edge of Bank of Newfoundland			3000	39 5	60		"	Soundings in 9000 feet
256k	" "				6000	40 3	"		"	
256l	" "	Between Flemish Gap and Grand Bank			1500	38	50	50	"	
257	21 Aug, 1837	45 5	161 48 E	N Pacific	958 (479?)	39 2	54 7	55 4	DuPetitThouars	South of Kamtschatka Cylinders sound
258	30 June, 1846	45 30	133 W	N Pacific	60	48	52	51	Kellett	10° W of the mouth of the Columbia River, Oregon.
259	" "	" "	"	" "	120	48	"	"	"	
260	" "	" "	"	" "	180	48	"	"	"	
261	" "	" "	"	" "	240	47	"	"	"	
262	" "	" "	"	" "	300	47	"	"	"	
263	" "	" "	"	" "	600	45	"	"	"	
264	" "	" "	"	" "	1200	42	"	"	"	
265	" "	" "	"	" "	1800	42	"	"	"	
266	" "	" "	"	" "	2400	42	"	"	"	
267	" "	" "	"	" "	3000	42	"	"	"	
268	26 May, 1826	45 53	15 17 W	N Atlantic	1252	50 7	58 3		Lenz	Near the Bay of Biscay.
269	" "	" "	" "	" "	2524	49 9	"		"	
270	6 June, 1818	45 57	21 23 W	N Atlantic	357	54·7	60 6	65	Kotzebue	Between Portugal and Azores.
270a	Sept 1804	47	158 E	N Pacific	480	33	60		Horner	Outside the Kurile Islands
270b	12 Sept, 1868	47 11	23 14 W	N Atlantic	12000	42			Chimmo	Between Ireland & Newfoundland.
271	7 June, 1818	47 18	20 30 W	N Atlantic	402	54 5	60	60 7	Kotzebue	Between Ireland and the Azores
272	" "	47 32	20 24 W	" "	462	54 7	60 3	61 3	"	

TABLE I.—Northern Hemisphere (continued).

I.	II. Date.	III North Latitude	IV. Longitude of Greenwich	V. Sea	VI Depth in feet	VII Temperature in degrees of Fahr At depth	Surface	Air	VIII. Name of observer.	IX Remarks.
273	9 June, 1818	48° 2'	19° 42' W	N Atlantic	451	54°.2	62°.5	60°.4	Kotzebue	Between Ireland and the Azores
274	10 " "	48 2	17 56 W	" "	472	54.2	62	62.3	"	
275	11 " "	48 8	15 33 W	" "	480	52.3	62.5	63.5	"	
276	12 " "	48 22	13 45 W	" "	429	51.1	63.8	69.9	"	
277	13 " "	48 42	10 50 W	" "	492	52.1	59.5	68	"	5° W of the Scilly Isles
278	May, 1845	48 45	6 19 W	N Atlantic	408	52	57	59	Beechey	Entrance of British Channel
279	19 July, 1850	50	170 W ?	N Pacific	1080	40	51	50	Armstrong	No soundings S of Aleutian I^s
280	18 Sept, 1837	51 34	161 41 E	N. Pacific	5872 (5741½)	36.4	53	51.8	DuPetit Thouars	Cylinder sound
280a	July, 1805	52	160 E	N Pacific	600	31.2	43.1		Horner	Off Petropaulovski
281	Oct, 1826	53 12	163 39 W	N Pacific	600	39	47.5	46	Beechey	3° to the S E of the Aleutian Islands
282	" "	" "	" "	" "	1200	39.7	"	"	"	
283	" "	" "	" "	" "	2136	40.7	"	"	"	
284	" "	" "	" "	" "	2736	40	"	"	"	
285	11 June, 1773	55?	0 37 W ?	N Atlantic	192	49	51	55	Phipps	North Sea, off Whitby
286	4 June, 1819	55 1	35 36 W	N Atlantic	1500	44.5	44.2	43	Parry	No soundings
287	27 May, 1819	56 59	24 33 W	" "	6120	45.5	48.5	49	"	Between Ireland and Greenland
288	25 " "	57 4	17 52 W	" "	600	49	50	50.5	"	Near Rockall Marcet's bottle
289	24 " "	57 42	14 16 W	" "	840	47.7	49.5	50	"	Do do In soundings
290	28 " "	57 26	25 16 W	" "	780	48	49	49	"	Marcet's bottle used
291	5 May, 1828	57 35	36 36 W	N Atlantic	660	44.4	46.4	51.3	Graah	Betw Ireland and Newfoundland
292	17 June, 1819	57 51	41 5 W	N Atlantic	1410	39	40.5	41.5	Parry	Off the south of Greenland
293	July, 1827	58 48	175 2 E	N Pacific	600	45	54	57	Beechey	Off the Siberian coast, Behring Sea
294	" "	" "	"	" "	1200	41.5	"	"	"	
295	" "	" "	" "	" "	1962	40.5	"	"	"	
296	" "	" "	" "	" "	2052	40.5	"	"	"	
297	17 June, 1819	58 52	48 12 W	N Atlantic	1740	38.7	38.5	38.5	Parry	Entrance to Davis Strait
298	23 May, 1818	59	44 W	N Atlantic	480	37	39	40	Sabine (M)	No soundings Off C Farewell
299	1860	59 27	26 41 W	N Atlantic	600	48.5	48	44	Wallich	Soundings in 7560 feet
300	30 June, 1859	59 35	38 9 W	N Atlantic	1800	44.4	44.6		Kundson	Off S of Greenland
301	18 June, 1819	59 40	47 46 W	N Atlantic	1560	39	37	35	Parry	Entrance to Davis Strait No soundings
302	12 June, 1773	60	0 10 E ?	N Atlantic	390	44	50	50	Phipps	Off Shetland
304	4 Oct, 1818	60	58 W	Davis Strait	5400	35.7	40	37	Sabine (M)	No soundings
305	8 Aug, 1859	60 10	36 21 W	N Atlantic	1800	45	48.6		Kundson	Parallel of Cape Farewell
306	7 Sept, 1773	60 14	2 30 E ?	N Atlantic	336	50	57	60	Irving	Between Shetland and Norway.
307	29 June, 1859	60 27	35 34 W	N Atlantic	1800	44.1	48		Kundson	Parallel of Cape Farewell
308	27 Oct., 1818	61	7 W	N Atlantic	2820	47	49.5	50.5	Sabine (M)	No soundings
308a	14 Aug, 1858	62 ?	55 W ?	Baffin's Bay	150	31.5	38		Walker	Doubtful about position
308b	" "	"	" "	" "	300	29.5	"		"	
308c	" "	"	" "	" "	684	30	"		"	
309	July, 1827	61 10	176 32 E.	N Pacific	30	41.5	43.5	45	Beechey	Off the coast of Siberia northern part of Behring Sea.
310	" "	" "	" "	" "	60	38	"	"	"	
311	" "	" "	" "	" "	120	29.5	"	"	"	

TABLE I.—Northern Hemisphere (continued).

I.	II. Date.	III. North Latitude	IV. Longitude of Greenwich	V. Sea	VI. Depth in feet	VII. Temperature in degrees of Fahr. At depth	Surface	Air	VIII. Name of observer	IX. Remarks.
312.	July, 1827	61° 10′	178° 32′ E	N Pacific	120	30° 5	43° 5	45°	Beechey	Off the coast of Siberia, northern part of Behring Sea
313.	,, ,,	,, ,,	,, ,,	,, ,,	180	30 5	,,	,,	,,	
314.	,, ,,	,, ,,	,, ,,	,, ,,	180	30 5	,,	,,	,,	
315	,, ,,	,, ,,	,, ,,	,, ,,	312	32·5 ?	,,	,,	,,	
316	,, ,,	,, ,,	,, ,,	,, ,,	600	32 5	,,	,,	,,	
317	,, ,,	,, ,,	,, ,,	,, ,,	1200	32 5	,,	,,	,,	
318.	11 Oct., 1820	61 11	31 12 W	N Atlantic	1920	44 2	47 5	48	Parry	15°W of Cape Farewell Marcet's bottle used
319	28 June, 1859	61 12	33 W	N Atlantic	1200	43 7	46 4		Kundson	Between Cape Farewell and Iceland
320	27 Oct, 1818	61 48	1 52 W	N Atlantic	2838	47	49	50	Parry (M)	N of the Shetland Islands
321	1 June, 1818	63 50	55 30 W	Davis Straits	870	32	36	35 5	Parry (M)	
322	3 Sept, 1821	64 to 64 30?	84 to 85 W ?	Arctic America	900	30	30	38	Parry	Beset in ice, in and near Lyon Inlet, Fox Channel Hudson Bay Soundings were obtained in each case at a further depth of from 30 to 100 feet Marcet's water-bottle supposed to have been used
323	,, ,,	,, ,,	,,	,, ,,	1080	30	30 5	40	,,	
324.	4 Sept ,,	,, ,,	,,	,, ,,	600	30 5	30	37	,,	
325	,, ,,	,, ,,	,,	,, ,,	840	31	31	42	,,	
326	,, ,,	,, ,,	,,	,, ,,	1020	30·5	30 5	39	,,	
327	,, ,,	,, ,,	,,	,, ,,	1200	30 5	30 5	37	,,	
328	5 Sept, ,,	,, ,,	,,	,, ,,	960	31 4	31 7	37	,,	
329	6 Sept, ,,	,, ,,	,,	,, ,,	690	29 5	30	30	,,	
330	,, ,,	,, ,,	,,	,, ,,	750	30 7	30 7	36	,,	
331	,, ,,	,, ,,	,,	,, ,,	780	30	30 5	34	,,	
332	,, ,,	,, ,,	,,	,, ,,	810	30	30 5	33	,,	
333	7 Sept, ,,	,, ,,	,,	,, ,,	600	30 5	31	36	,,	
334	,, ,,	,, ,,	,,	,, ,,	630	29 5	29 5	32	,,	
335	,, ,,	,, ,,	,,	,, ,,	690	29 5	30 2	33	,,	
336	,, ,,	,, ,,	,,	,, ,,	744	30·2	31	36	,,	
337	8 Sept, ,,	,, ,,	,,	,, ,,	630	29	30	34	,,	
338	,, ,,	,, ,,	,,	,, ,,	648	29 6	30 5	35	,,	
339	,, ,,	,, ,,	,,	,, ,,	660	29 7	30	36	,,	
340.	,, ,,	,, ,,	,,	,, ,,	720	29 5	29 7	33	,,	
341	9 Sept, ,,	,, ,,	,,	,, ,,	600	30	30	35	,,	
342	,, ,,	,, ,,	,,	,, ,,	720	30	30 5	38	,,	
343	10 ,, ,,	,, ,,	,,	,, ,,	840	30	30	37	,,	
344	11 ,, ,,	,, ,,	,,	,, ,,	720	30	30	35	,,	
345	4 Sept, 1773	65 ?	2 21 E	N Atlantic	4098	40	55	66 5	Phipps	Between Iceland and Norway
346	26 Sept, 1818	65 50	59 30 W	Davis Strait	1860	29	34	36	Sabine (M)	Soundings in 2220 feet
347	24 Sept, 1818	66 35	5 33 E.	N Atlantic	1560	41 5	43	44 5	Franklin	Between Iceland and Norway
348	24 Sept, 1818	66 38	5 44 E	N Atlantic	1560	41 5	43 5		Beechey (M)	
349	19 Sept, 1818	66 50	61 W	Davis Strait	600	30	33	35	Sabine (M.)	Soundings in 4500 feet
350	,, ,,	,, ,,	,,	,, ,,	1200	29	,,	,,	,,	
351	,, ,,	,, ,,	,,	,, ,,	2400	29	,,	,,	,,	
352	,, ,,	,, ,,	,,	,, ,,	4080	25 75		,,	Ross & Sabine (M)	
353	21 Sept., 1820	67 38	59 1 W.	Davis Strait	1200	33 2	34 5	30	Parry	In soundings. Marcet's bottle used.
354.	20 ,, ,,	68 12	60 50 W	,, ,,	1908	33	32	31 5	,,	Parallel of Disco Island.

TABLE I.—Northern Hemisphere (continued).

I.	II. Date.	III North Latitude	IV Longitude of Greenwich	V Sea.	VI. Depth in feet	VII Temperature in degrees of Fahr. At depth	Surface	Air	VIII. Name of observer	IX Remarks
355	11 Sept., 1820	68° 19′	66° 05′ W	Davis Strait	876	34°	32°	34°	Parry	Near the American coast
356	" "	" "	" "	" "	990	34	"	"	"	
357	18 " "	68 24	63 08 W	Davis Strait	1908	30	30	29	"	Near mid-channel
358	15 " "	68 24	63 32 W	" "	1020	30·5	30·5	31	"	No soundings in 3660 feet
359	16 " "	68 29	63 48 W	" "	4854	27	31	34	Parry & Fisher	Six's therm. used. In soundings
360	17 Aug., 1855	68 42	174 27 W	Arctic Ocean	120	38	45	48·6	Rodgers (Maury)	Off the Asiatic coast. Soundings very near the bottom
361	" "	" "	" "	" "	108	40·2	"	"	" "	
362	9 Sept., 1820	69 24	67 05 W	Davis Strait	210	31	32·5	34	Parry	Between Disco Island and Cape Kater
363	Aug., 1827	70 2	164 40 W	Arctic Ocean	126	37	49	57	Beechey	Off Icy Cape
364	10 Aug., 1850	70 30?	148 W?	Arctic Ocean	540	29·5	29?	33?	Armstrong	No soundings
365	4 July, 1839	70 40	23 36 E	Arctic Ocean	567	30	40		Bravais	Bay of Hammerfest.
366	15 July, 1839	70 40	23 35 E	Arctic Ocean	390	39·5	41	48	Martins	Bay of Hammerfest. Temperature at bottom
367	" "	" "	" "	" "	640	39	"		"	
368	6 Sept., 1820	70 47	67 56 W	Baffin Bay	456	31·3	33	32	Parry	Near the American coast
369	" "	" "	" "	" "	1170	31·5	"	"	"	
370	22 Aug., 1839	71 1	23 23 E	Arctic Ocean	266	39·1	45	45	Martins	Off N. coast of Norway, bottom temperature
371	" "	" "	" "	" "	788	38·9	"	"	"	
372	16 Aug., 1855	71 16	176 5 W	Arctic Ocean	90	31·6	38·2	37·5	Rodgers (Maury)	Between Kellet Land and Siberia, being near the bottom
373	" "	" "	" "	" "	186	34	"	"	" "	
374	14 Aug., 1855.	71 21	175 22 W	Arctic Ocean	60	33·4	44	45	Rodgers (Maury)	The next day's reading gave 3° higher
375	" "	" "	" "	" "	150	37·3	"	"	" "	
376	3 Sept., 1820	71 24	70 58 W	Baffin Bay	528	33	35·5	38	Parry	In soundings
377	9 Sept., 1850	71 30?	120 W?	Arctic Seas	210	29?		35?	Armstrong	Amongst ice, Prince of Wales Strait
378	" "	" "	"	" "	450	31?		"	"	
379	13 Aug., 1855	72 2	174 37 W	Arctic Ocean	120	34	43·7	45·2	Rodgers (Maury)	Within 2 feet of bottom
380	" "	" "	" "	" "	240	41	"	"	" "	
381	6 Aug., 1822	72 7	19 11 W	Arctic Seas	708	29	34	42	Scoresby	Off the east coast of Greenland
382	2 Sept., 1820	72 9	73 58 W	Baffin Bay	450	32·2	32	33	Parry	Marcet's bottle used
383	7 Sept., 1818	72 16	71 18 W	Baffin Bay	6000	28·7	35	33	Sabine (M.)	Soundings in 6000 feet
384	6 Sept., 1818	72 22	73 06 W	Baffin Bay	1476	30	36	41	Parry (M.)	Near Pond's Bay
385	7 Sept., 1818	72 22	73 58 W	Baffin Bay	6000	28·7	35		John Ross	
386	6 Sept., 1818	72 23	72 55 W	Baffin Bay	1476	30	36	37	Sabine (M.)	No soundings
387	21 Aug., 1839	72 29	19 54 E	Arctic Ocean	531	40·1	43·4	43·8	Martins	Between Norway and Bear Island
388	" "	" "	" "	" "	1279	38·5	"	"	"	
389	5 Sept., 1818	72 37	74 6 W	Baffin Bay	1140	30·2	35	35·5	Sabine (M.)	Soundings in 1140 feet
390	5 Sept., 1818	72 39	74 30 W	Baffin Bay	1140	30·2	35	39	Parry (M.)	West side of the Bay
391	28 July, 1849	72 51	163 W	Arctic Ocean	30	33	36		Kellett (Seeman)	Off the American coast, near the ice-pack
392	" "	" "	"	" "	60	32	"		" "	
393	" "	" "	"	" "	90	29	"		" "	
394	" "	" "	"	" "	120	29	"		" "	
395	" "	" "	"	" "	150	29	"		" "	
396	" "	" "	"	" "	180	29	"		" "	
397	" "	" "	"	" "	210	29·5	"		" "	

TABLE I.—Northern Hemisphere (continued).

I.	II. Date.	III. North Latitude	IV. Longitude of Greenwich	V. Sea	VI. Depth in feet	VII. Temperature in degrees of Fahr. At depth	Surface	Air	VIII. Name of observer.	IX. Remarks.
398	1 Sept., 1820	72° 55′?	75° 19′ W ?	Baffin Bay	660	30·2°	30·5°	31°	Parry	Pond Bay Marcet's bottle used.
399	" "	73	90 W	Arctic Sea						Prince Regent Inlet.
400	14 Aug., 1819	73 35	80 1 W	Baffin Bay	1110	34	34	39	Parry	Lancaster Sound soundings in 1200 feet Marcet's bottle used
401	20 July, 1839	73 36	20 52 E	Arctic Ocean	2854	32·2	42·2	42·2	Martins	Mean of four experiments
402	1 Sept., 1818	73 38	77 19 W	Baffin Bay	750	30·5	35	36	Parry (M.)	Near Pond Bay
403	8 Aug., 1838	73 52	16 23 E	Arctic Ocean	1010	38·3	41·8	34·3	Bravais et Martins	Mean of two experiments Between Norway and Bear Island.
404	30 Aug., 1818	74 4	79 W	Lancaster Sd	1410	29·2	36·5	37	Sabine (M.)	No soundings
405	Aug., 1818	74 20 ?	80 W ?	Baffin Bay	3900	29			John Ross	Entrance of Lancaster Sound
406	31 " "	74 8	"	" "	4044	29·5			"	
407	Sept., 1818	74 21	112 48 W	Arct America	630	32	31·2	34	Parry (M.)	Entrance to Banks Strait In soundings
408	18 July, 1838	74 45	15 E	Arctic Ocean	1493	34·5	39·2	37·3	Bravais et Martins	Mean of two experiments Between Norway and Bear Isl
409	6 Nov., 1819	74 47?	110 48 W ?	Arct America	30	30	28	−16	Parry	South of Melville Island
410	9 " "	74 47?	110 48 W ?	" "	30	31	28		"	
411	6 July, 1818	74 48	10 15 E	Arctic Ocean	204	34·5	34	36	Franklin	At bottom, near land. Query, lat 79° 48′?
412	18 July, 1818	74 50	59 30 W	Baffin Bay	1182	29·5	32	37	Parry (M.)	Off the Greenland coast
413	19 Aug., 1839	74 52	12 57 E	Arctic Ocean	397	37·8	41·2	39·8	Martins	Bottom temperature Between Norway and Spitzbergen
414	" "	" "	" "	" "	1598	33·4	"	"	"	
415	20 Aug., 1818	74 58	77 42 W	Baffin Bay	1020	31	36	34	Parry (M.)	Near Lancaster Sound
416	29 Aug., 1818	74 59	76 37 W	Baffin Bay	1020	31	36	34	Sabine (M.)	Soundings in 1020 feet.
417	27 Aug., 1820	75 2	105 14 W	Arct America	564	31·7	30	31	Parry	In soundings Marcet's bottle used
418	10 Sept., 1818	75 14	3 53 E ?	Arctic Ocean	4536	36	35	37	Franklin (M.)	Between Spitzbergen and Iceland.
419	14 Aug., 1818	75 50	66 W	Baffin Bay	1200	30·1	32	38	Sabine (M.)	Melville Bay, soundings in 2700 feet.
420	" "	" "	"	" "	2532	29·7	"	"	"	
421	3 Aug., 1818	75 52	63 W	Baffin Bay	2490	29	34	38	Sabine (M.)	Melville Bay Soundings.
422	26 July, 1839	75 55	9 16 E	Arctic Ocean	2395	32·7	38·2	38·2	Martins	Mean of four experiments, bottom temperature
423	14 Aug., 1818	75 56	66 31 W	Baffin Bay	1200	30·2	32	36	Parry (M.)	Melville Bay
424	" "	" "	" "	" "	2532	29·2	"	"	" "	
425	25 July, 1839	75 59	9 51 E	Arctic Ocean	2142	32	38·2	38	Martins	Mean of two experiments Bottom
426	1 Aug., 1818	76?	62 W ?	Baffin Bay	1260	29·5			John Ross	Top of Melville Bay
427	2 " "	75 51	62 59	" "	2520	29·5			"	
428	" "	76?	65 W ?	" "	2730	29·5			"	Near Melville Bay
429	25 Aug., 1818	76 8	[illegible] 21 W	Baffin Bay	324	29·5	32·5	31·5	Sabine (M.)...	Soundings in 336 feet.
430	25 Aug., 1818	76 8	78 31 W	Baffin Bay	324	20·5	32	31·5	Parry (M.)	In soundings Entrance to Jones Sound
431	18 Aug., 1839	76 13	12 48 E	Arct Ocean A	308	37·2	40·4	40·2	Martins	Between Bear Island and Spitzbergen Mean of two experiments
432	" "	" "	" "	" "	1296	33·4	"	"	"	
433	" "	" "	" "	" "	2103	32·3	"	"	"	Mean of four experiments, at bottom.
434	9 April, 1810	76 16	9 0 E	Arctic Ocean	300	31·3	28·8	12	Scoresby	In ice, 1° S W of Spitzbergen.
435	" "	" "	" "	" "	738	33·8	"	"	"	
436	" "	" "	" "	" "	1380	33·8	"	"	"	
437	23 Apr., 1810	76 16	10 50 E	Arctic Ocean	120	28	28·3	16	Scoresby	Frozen up.
438	" "	" "	" "	" "	300	28·3	"	"	" ..	
439	" "	" "	" "	" "	738	30	"	"	"	
440	24 Aug., 1818	76 22	77 38 W	Baffin Bay	600	30·2	31·5	33	Parry (M.)	Near Cobourg Island.

TABLE I.—Northern Hemisphere (continued).

I.	II. Date.	III. North Latitude	IV. Longitude of Greenwich	V. Sea	VI. Depth in feet	VII. Temperature in degrees of Fahr. At depth	Surface	Air	VIII. Name of observer	IX. Remarks
441	24 Aug., 1818	76 22	77 38 W	Baffin Bay	1440	29·5	31·5	33	Parry (M.)	Near Cobourg Island
442	22 " "	76 33	77 10 W	" "	612	29·5	32	36	"	Entrance of Smith Sound
443	23 Apr., 1811	76 34	10 E.	Arctic Ocean	120	31	30	25	Scoresby	Frozen up off the S.W. of Spitzbergen
444	" "	" "	"	" "	240	35	"	"	"	
445	" "	" "	"	" "	360	34	"	"	"	
446	" "	" "	"	" "	600	34·7	"	"	"	
447	24 Aug., 1818	76 35	78 W	Baffin Bay	600	30·2	31·5	33	Sabine (M.)	Entrance of Smith Sound
448	" "	" "	"	" "	1440	29·5	"	"	"	
449	26 May, 1818	76 48	12 26 E	Arctic Ocean	4200	43	33	29	Franklin & Buchan	Off S. of Spitzbergen. Franklin ascribes the high temperature to the water bucket being examined in the cabin
450	28 July, 1839	76 57	13 29 E	Arctic Ocean	515	37·4	37·8	37·3	Martins	Mean of 2 expts. each, off Spitzbergen, bottom temperature
451	" "	" "	" "	" "	1040	34·7	"	"	"	
452	1 May, 1811	77 15	8 10 E	Arctic Ocean	120	29·3	29·3	16	Scoresby	In ice off the W. coast of Spitzbergen
453	" "	" "	" "	" "	240	29·3	"	"	"	
454	" "	" "	" "	" "	360	30	"	"	"	
455	" "	" "	" "	" "	600	30	"	"	"	
456	20 May, 1813	77 40	2 30 E	Arctic Ocean	300	29·3	29	30	Scoresby	Amongst floes, between Spitzbergen and Greenland
457	" "	" "	" "	" "	660	31	"	"	"	
458	15 Aug., 1839	77 43	12 11 E	Arctic Ocean	397	34·3	36·4	35·9	Martins	Mean of 4 expts. at bottom
459	7 June, 1817	78 2	0 10 W	Arctic Ocean	4566	38	32	36	Scoresby	Ice near, between Spitzbergen and Greenland
460	14 Aug., 1839	78 41	9 39 E	Arctic Ocean	321	33·5	34·7	36·3	Martins	Mean of 4 expts. at bottom
461	20 May, 1816	79	5 40 E.	Arctic Ocean	78	31	29	34	Scoresby	Moored to a floe, N.W. of Spitzbergen
462	" "	"	" "	" "	222	33·8	"	"	"	
463	" "	"	" "	" "	342	34·5	"	"	"	
464	" "	"	" "	" "	600	36	"	"	"	
465	" "	"	" "	" "	2400	36	"	"	"	
466	21 May, 1816	79 4	5 38 E	Arctic Ocean	4380	37	29	38	Scoresby	Amongst floes
467	13 Aug., 1839	79 33	10 54 E	Arctic Ocean	213	34·2	35·7	38	Martins	4 expts. Off Magdalena Bay, Spitzbergen
468	" "	" "	" "	" "	404	34·1	"	"	"	4 expts.
469	28 June, 1818	79 44	9 33 E	Arctic Ocean	90	34	34	35	Frankl. & Buch.	At bottom
470	3 Aug., 1839	In	Magdalena	Bay	79	32·4	33·2	38·6	Martins	West coast of Spitzbergen
470a	" "	"	"	"	361	28·6	"	"	"	
471	4 June, 1827	79 49	15 11 E	Arctic Ocean	441	29·2	30	38	Parry	Beset in the ice off the north coast of Spitzbergen.
472	5 June, 1827	79 49	15 17 E	Arctic Ocean	459	29·7	30·5	43	Parry	
473	" "	" "	" "	" "	471	29·8	31	43	"	
474	" "	" "	" "	" "	480	29·8	31	44	"	
475	" "	" "	" "	" "	492	28·7	30	41	"	
476	" "	" "	" "	" "	492	30	31	43	"	
477	" "	" "	" "	" "	507	29·5	30	43	"	
478	6 June, 1827	79 49	15 22 E	Arctic Ocean	408	30	31	39	Parry	
479	" "	" "	" "	" "	408	29	30·2	39	"	
480	" "	" "	" "	" "	408	30	30·5	39	"	
481	" "	" "	" "	" "	408	29·2	30·5	38	"	

Table I.—Northern Hemisphere (continued).

I	II. Date.	III North Latitude	IV. Longitude of Greenwich.	V. Sea.	VI. Depth in feet	VII. Temperature in degrees of Fahr. At depth.	Surface.	Air	VIII. Name of observer.	IX. Remarks.
482	6 June, 1827..	79° 49′	15° 22′ E	Arctic Ocean	408	29°·2	30°·5	37°·5	Parry	
483	" "	" "	" "	" "	456	29·5	30·5	38	"	
484	" "	" "	" "	" "	504	30	30	37	"	
485	" "	" "	" "	" "	408	29·5	29·7	41	"	
486	" "	" "	" "	" "	420	20·7	32	41	"	
487	" "	" "	" "	" "	438	30	30	38	"	
488	" "	" "	" "	" "	474	29·2	31	42	"	
489	7 June, 1827 .	79 50	15 30 E.	Arctic Ocean	312	29	31·5	38	Parry	
490	" "	" "	" "	" "	312	29	32	40	"	
491	" "	" "	" "	" "	315	30	33	37	"	
492	" "	" "	" "	" "	318	29·5	31·5	39	"	Beset in the ice off the north coast of Spitzbergen.
493	" "	" "	" "	" "	318	29	31	41	"	
494	" "	" "	" "	" "	324	30	30·5	40·5	"	
495	" "	" "	" "	" "	336	29	31·5	41	"	
496	" "	" "	" "	" "	336	30	32	42	"	
497	" "	" "	" "	" "	348	29·8	31	39	"	
498	" "	" "	" "	" "	384	29	31·5	41	"	
499	" "	" "	" "	" "	408	30	31·5	42	"	
500	" "	" "	" "	" "	468	29·5	31	40	"	
501	8 June, 1827	79 50	15 30 E.	Arctic Ocean	288	28·8	30	42	Parry	
502	" "	" "	" "	" "	312	29	32	40	"	
503	" "	" "	" "	" "	321	29·2	31·5	40	"	
504	25 June, 1818	79 51	10 E	Arctic Ocean	102	34	33	34	Frankl. & Buch	N W of Spitzbergen Soundings
505	" "	" "	"	" "	360	34	33	34	" "	
506	29 June, 1818	79 51	10 E	" "	102	34	34	39	" "	Near the land in a current.
507	" "	" "	"	" "	114	34	34	37	" "	
508	27 June, 1818	79 51½	"	" "	432	34·5	34	36	" "	Near ice
509	19 May, 1827	79 55	13 46 E	Arctic Ocean	372	29	28·5	13	Parry	Beset N of Spitzbergen.
510	" "	" "	" "	" "	426	28	28	14	"	
511	18 May, 1827	79 56	13 39 E	Arctic Ocean	570	30	28	22	Parry ..	Beset N of Spitzbergen.
512	" "	" "	" "	" "	432	28·5	28	15	"	
513	21 June, 1818	79 56	11 30 E.	Arctic Ocean	114	31	30	30	Frankl. & Buch	Ice around, bottom
514	20 " "	79 58	11 25 E	" "	144	31	31·5	30	" "	At bottom; beset.
515	23 June, 1818	79 59	10 12 E	Arctic Ocean	126	32·5	31·5	30	Frankl. & Buch.	Beset.
516	22 " "	80	11 14 E	" "	198	31	30	30	" "	Beset off the land
517	7 June, 1816	80	5 E	Arctic Ocean	720	36·3	29·7	40	Scoresby ..	Beset N.W of Spitzbergen.
518	16 May, 1827	80 1	13 5 E.	Arctic Ocean	564	29·5	28·7	18	Parry ..	
519	" "	" "	" "	" "	576	28·5	28·5	18·5	" ..	Beset N of Spitzbergen.
520	15 " "	" "	" "	" "	606	30	28	17	"	
521	15 May 1827	80 4	12 39 E	Arctic Ocean	690	32	29·5	41	Parry	N of Spitzbergen
522	22 July, 1818	80 13	11 31 E.	Arctic Ocean	498	35·8	31	41	Frankl. & Buch	At bottom N W of Spitzbergen.
523	21 " "	80 14	11 12 E.	" "	570	35·3	32·5	41·5	" "	At bottom. " "
524	23 " "	80 15	11 36 E	" "	438	36·8	32·5	37	" "	At bottom " "

TABLE I.—Northern Hemisphere (continued).

I.	II. Date.	III. North Latitude	IV. Longitude of Greenwich.	V. Sea.	VI. Depth in feet.	VII. Temperature in degrees of Fahr. At depth.	Surface	Air	VIII. Name of observer.	IX. Remarks
525	25 July, 1818...	80° 18′	11° 40′ E	Arctic Ocean	549	38°	32°·5	34°	Frankl. & Buch.	At bottom.
526.	7 „ „	80 18	11 10 E.	„ „	720	36	33	35	„ „	North-west of Spitsbergen At bottom beset The temperature of the air is taken from Marcet
527.	10 „ „	80 19	11 24 E.	„ „	714	36	32		„ „	
528	8 „ „	80 20	11 10 E.	„ „	780	36·5	31·5	35	„ „	
529.	9 „ „	80 20	10 55 E	„ „	660	35·5	30·5	30·5	„ „	
530	12 „ „	80 20	11 7 E	„ „	870	35·8	32	36	„ „	
531	26 „ „	80 20	11 25 E	„ „	330	36	32·5	36	„ „	
532	20 „ „	80 21	10 12 E	„ „	648	35·5	32·5	34·5	„ „	
533	11 „ „	80 22	10 30 E	„ „	720	36	32	40	„ „	
534.	13 „ „	80 22	10 2 E	„ „	1410	35·5	32	40·5	„ „	
535	18 July, 1818 ..	80 22	11 E	Arctic Ocean	1302	37	32·5		Franklin (M.)	N of Spitsbergen rocky bottom
536	„ „	„ „	10 55 E	„ „	1422	35·5	31·5	40	„ „	N of Spitsbergen beset
537	19 July, 1818	80 24	11 14 E	Arctic Ocean	618	36·5	31·5	41	Frankl & Buch	At bottom beset
538	14 „ „	80 26	10 45 E	„ „	1398	35·5	32	39	„ „	
539	16 „ „	80 26	11 25 E	„ „	1038	36·3	36·5	39	„ „	
540	18 „ „	80 26	10 30 E	„ „	1986	36	32·5	36	„ „	
541	9 July, 1818	80 26	11 38 E	Arctic Ocean	720	36	31	35	Franklin (M.)	N W of Spitzbergen beset
542.	15 July, 1818	80 27	10 20 E	Arctic Ocean	1188	36	32	38	Frankl & Buch	At bottom beset
543	17 „ „	80 27	11 E	„ „	1710	35·5	34		„ „	
544	15 July, 1818	80 28	10 20 E	Arctic Ocean	1110	36·2	32·5		Franklin (M.)	Beset
545	4 Aug., 1773	80 30	16 E	Arctic Ocean	360	39	36	32	Phipps	Under the ice
546	14 June, 1827...	80 47	18 22 E	Arctic Ocean	570	29·8	31	26	Parry	N of Spitsbergen
547	15 „ „	80 49	19 7 E	„ „	450	29	30	27	„	
548	„ „	„ „	„ „	„ „	402	28·6	29	26	„	

NOTE.—The observations where it is said that MARCET's water-bottle has been used are not reliable There is an ambiguity in the few remarks in WULLERSTORF's 'Voyage of the Novara' "On the Temperature and Density of Sea-water at Depths," which perhaps should exclude those observations also The irregularity of the readings would seem to indicate that the temperature is rather that at time of taking the specific gravity than that at time of emersion of the apparatus Some may be about right, others much wrong.

Table II.—Southern Hemisphere.

I	II.	III	IV	V.	VI	VII.			VIII.	IX.
	Date.	South Latitude.	Longitude of Greenwich.	Sea.	Depth in feet.	Temperature in degrees of Fahr. At depth	Surface	Air	Name of observer.	Remarks.
1a	May, 1804	° ′ 0 0	° ′ 146 0 W	Pacific	1200	° 57·5	° 84·5	°	Horner	10° N W of the Marquesas Islands.
1	5 Mar, 1858	0 13	58 26 E.	Indian Ocean	7980	41	81		Pullen (u)	North of Seychelles Islands Weather bad, and readings uncertain
2	,, ,,	,, ,,	,, ,,	,, ,,	11760	67?	,,		,,	
3	,, ,,	,, ,,	,, ,,	,, ,,	14280	40	,,		,,	
4.	7 Feb, 1838	0 31	97 19 W	S Pacific	2656 (1706?)	45	80·7	84·2	DuPetitThouars	Cylinder sound
5.	30 Dec, 1857	0 46	82 43 E	Indian Ocean	720	77·4	82	81·6	Wullerstorf.	In mid-ocean
6	5 May, 1818	0 53	20 28 W	S Atlantic	480	57·3	83	83	Kotzebue	Between N Brazil and Guinea.
7	8 Feb, 1838	0 55	97 7 W	S Pacific	5872 (2296?)	37·4	79·7	86	DuPetitThouars	Cylinder sound
7a	May, 1804	1 5	146 W	S Pacific	600	59	82·5		Horner	Near No 1a
8	4 May, 1818	2 17	19 50 W	S Atlantic	480	57·1	83	82·5	Kotzebue	In mid-ocean
9	7 Aug, 1845	2 32	30 53 W	S Atlantic	2400	50·5	78	80	Kellett	Soundings in 17,970 feet.
10	23 Mar, 1843	2 32	8 11 W	S Atlantic	1800	46	79		Belcher	Between Ascension Island and the coast of Guinea
11	,, ,,	,, ,,	,, ,,	,, ,,	2400	38?	,,		,,	
12	,, ,,	,, ,,	,, ,,	,, ,,	3000	46	,,		,,	
13	,, ,,	,, ,,	,, ,,	,, ,,	3600	45·5	,,		,,	
14	,, ,,	,, ,,	,, ,,	,, ,,	4200	46	,,		,,	
15	,, ,,	,, ,,	,, ,,	,, ,,	4800	45	,,		,,	
16	,, ,,	,, ,,	,, ,,	,, ,,	5400	40·2	,,		,,	
17	,, ,,	,, ,,	,, ,,	,, ,,	6000	42·7	,,		,,	
18	14 Jan, 1847	2 37	26 15 W	S Atlantic	1608	53	80	79	Dayman	Between St. Paul and Ascension
19	Sept., 1816	3 26	7 39 E	S Atlantic	8610	42	73		Wauchope	Off the coast of Congo Corrected depth 6060 feet.
20	3 May, 1818	3 42	18 41 W	S Atlantic	426	56	82·6	83·5	Kotzebue	Between Brazil and Guinea.
21	28 Sept, 1827	3 48	128 7 E.	S Pacific	425	74	83	81·6	D Urville	Amongst the Molucca Islands
22	Aug, 1836	3 58	1 37 W	S Atlantic	1800	52	73		Wauchope	32m to haul in
23	18 July, 1827	4 42	152 40 E	S Pacific	212	81	83·2	85·1	D'Urville	Off New Ireland
24	2 May, 1818	5 8	17 14 W.	S Atlantic	378	57·6	81·6	82	Kotzebue	In mid-ocean
25	15 Jan, 1847	5 9	27 51 W	S Atlantic	918	54	80	78	Dayman	East of Juan Fernando Reading probably reversed
26	,, ,,	,, ,,	,, ,,	,, ,,	1758	60	,,	,,	,,	
27	28 Feb, 1858	5 31	61 31 E	Indian Ocean	13980	35	84		Pullen (u)	In soundings
27a	10 Mar, 1836	5 59	24 35	S Atlantic	3733	43·7	79·5	79·5	Vaillant ..	Cylinder full
28	1 May, 1818	6 35	15 34 W	S Atlantic	339	59	81·5	81·7	Kotzebue	N. of Ascension
29	26 Feb, 1858	7 12	60 52 E	Indian Ocean	12000	38·2	81·5	..	Pullen (u)	No bottom at 13,524 feet.
30	21 Dec, 1838	7 29	85 18 E	Indian Ocean	600	78	83	82·5	Pratt	Between Sumatra and the Mauritius
31	20 ,, ,,	7 54	85 20 E	,, ,,	240	81·5	84	82·5	,,	
31a	July, 1839	7 54	112 53 W	S Pacific	2700	44·5	74		Wilkes ..	Between Peru and Marquesas Islands.
32	16 Jan, 1847	7 55	29 11 W	S Atlantic	1098	53	80	79	Dayman	Between Ascension and Brazil.
33	,, ,,	,, ,,	,, ,,	,, ,,	1638	47	,,	,,	,,	
34.	30 Apr, 1818	8 15	14 3 W	S Atlantic	367	64	80·3	80·3	Kotzebue	Near Ascension Island.
35	17 Oct., 1858	8 21	162 56 E	S Pacific	300	83·5	84·2	83·2	Wüllerstorf	East of the Salomon Isles
36	,, ,,	,, ,,	,, ,,	,, ,,	600	81·1	,,	,,	,,	
37	,, ,,	,, ,,	,, ,,	,, ,,	900	77·7	,,	,,	,,	
38	,, ,,	,, ,,	,, ,,	,, ,,	1140	73·8	,,	,,	,,	

Table II.—Southern Hemisphere (continued).

I.	II. Date.	III. South Latitude.	IV. Longitude of Greenwich.	V. Sea.	VI Depth in feet.	VII Temperature in degrees of Fahr. At depth	Surface	Air	VIII. Name of observer.	IX. Remarks.
39	8 Dec, 1857	9° 30′	30° 38′ W	S Atlantic	5280	41°·5	80°	°	Pullen (u)	Soundings in 7680 feet
40	29 Apr., 1818	9 39	12 46 W	S Atlantic	420	60·1	79·4	80·7	Kotzebue	Between Ascension and St Helena
41	19 June, 1839	10?	Off Peru	S Pacific	498	57	63		Wilkes	Latitude estimated
42	23 Feb, 1858	10 54	58 44 E	Indian Ocean	2640	51·5	83		Pullen (u)	No bottom in 7920 feet Probable error in reading of last depth, from shifting of index
43	,, ,,	,, ,,	,, ,,	,, ,,	5280	41·5	,,		,,	
44.	,, ,,	,, ,,	,, ,,	,, ,,	7920	51·5?	,,		,,	
45	28 Apr, 1818	11 11	11 21 W	S Atlantic	432	65·5	78·5	80·3	Kotzebue	Between Ascension and St Helena
46.	April, 1836	12	97 E ?	Indian Ocean	2178	45			FitzRoy	Near Keeling Island
47	15 July, 1839	12?	Off Callao	S Pacific	1800	51	67		Wilkes	Latitude estimated
48	18 ,, ,,	12?	,, ,,	,, ,,	1740	50	70		,,	
49	27 Apr, 1818	12 30	9 58 W	S Atlantic	368	59·8	77·2	78·3	Kotzebue	Between Ascension and St Helena
50	23 May, 1837	12 39	77 7 W	S Pacific	682	55·7	67·8	64·4	DuPetitThouars	Cylinder sound in soundings
51	17 Jan, 1847	12 49	32 23 W	S Atlantic	354	80	81	79	Dayman	Off the coast of Brazil
52	1847–49	13 28	28 W	S Atlantic	360	72·8	80·2		E Lenz	Between Bahia and Ascension
53	22 May, 1837	13 50	76 41 W	S Pacific	688	55·4	65	68	DuPetitThouars	Cylinder sound off Pisco Bay
54	26 Apr, 1818	14 12	7 55 W	S Atlantic	330	62	75·6	74	Kotzebue	North of St. Helena
54a	December	15	31 W	S Atlantic	360	74·6	78		Horner	No soundings in 27,600 feet
55	3 June, 1843	15 3	23 14 W	S Atlantic	5400	40·3	77		James Ross	
56	,, ,,	,, ,,	,, ,,	,, ,,	7200	39·5	,,		,,	
57	10 Jan, 1847	15 5	34 44 W	S Atlantic	1356	59	80	79	Dayman	Off Bahia, Brazil reading probably reversed
58	,, ,,	,, ,,	,, ,,	,, ,,	1902	62	,,	,,	,,	
59	13 Apr, 1816	15 26	133 42 W	S Pacific	60	79	80	79·8	Kotzebue	North of the Low Islands, or 15° E of the Society Islands
60	,, ,,	,, ,,	,, ,,	,, ,,	120	79	,,	,,	,,	
61	,, ,,	,, ,,	,, ,,	,, ,,	300	78·8	,,	,,	,,	
62	,, ,,	,, ,,	,, ,,	,, ,,	600	72	,,	,,	,,	
63	,, ,,	,, ,,	,, ,,	,, ,,	1200	56	,,	,,	,,	
64	27 Oct, 1827	15 40	120 50 E	Indian Ocean	2136	46·2	82·4	80·3	D'Urville	Between Australia and Java
65	8 May, 1839	15 54	10 23 W	S Atlantic	1066	53·6	74·5	74·8	DuPetitThouars	Cylinder sound near St Helena.
66	28 July, 1826	16	26 40 W	S Atlantic	960	51·5	73·6	71·6	D'Urville	Between St Helena and Brazil
67	24 Apr, 1818	16 14	5 7 W	S Atlantic	276	62·8	74·3	72·5	Kotzebue	Near St Helena.
68	28 Oct, 1827	16 40	120 20 E	Indian Ocean	1068	69·3	82·8	81·6	D'Urville	Off N W coast of Australia
69	23 Apr, 1818	17 55	3 8 W	S Atlantic	327	58·1	73·7	75	Kotzebue	S E of St Helena
70	1847–49	17 17	192 (32?)	,, ,,	360	76·6	84		E Lenz	Apparent error of longitude
71	11 Nov, 1827	17 30	135 20 E ?	Indian Ocean	1602	55·8	80·1	78·8	D'Urville	Probable error of longitude, should be 114° 20′ E
72	20 Oct, ,,	17 30	120 20 E	,, ,,	640	73·8	80·6	81	,,	Near the Rowley Shoals
73	20 Jan., 1847	17 48	36 20 W	S Atlantic	792	67	81	80	Dayman	Off the coast of Brazil
74	29 July, 1830	17 54	112 53 W	S Pacific	2700	44·5	74		Wilkes	Amer Journ Sc, January, 1848 In mid-ocean
75	30 Oct., 1827	18	119 50 E	Indian Ocean	480*	75·5	78·8	80·1	D'Urville	Near the north-west coast of Australia
76	,, ,,	,,	,, ,,	,, ,,	506*	76·8	80·3	79		
77	4 Aug., 1839	18?	120 W ?	S Pacific	300	74	75		Wilkes	Between the coast of America and the Society Islands
78.	,, ,,	,,	,,	,, ,,	600	73·5	,,		,,	
79	,, ,,	,,	,,	,, ,,	1200	61	,,		,,	
80	,, ,,	,,	,,	,, ,,	1800	50	,,		,,	

TABLE II.—Southern Hemisphere (continued).

I.	II. Date.	III. South Latitude.	IV. Longitude of Greenwich	V. Sea.	VI. Depth in feet	VII. Temperature in degrees of Fahr. At depth	Surface	Air	VIII. Name of observer.	IX. Remarks.
81.	31 July, 1857	18° 7′	37° 16′ W	S. Atlantic	240	75·8°	76·3°	74·5°	Wüllerstorf	Off the coast of Brazil.
82	7 Aug., 1839	18 14	125 W	S. Pacific	600	75	77		Wilkes	In mid-ocean, between the Marquesas and Easter Island.
83	7 Apr., 1816	18 17	124 56 W	S. Pacific	750	68·5	78·5	79·2	Kotzebue..	
84	„ „	„ „	„ „	„ „	750	68	79·6	80	„	
85	„ „	„ „	„ „	„ „	1050	57·5	78·5	79·2	„	
86	„ „	„ „	„ „	„ „	1200	54	79·6	80	„	
87	Jan., 1826	18 38	136 1 W	S. Pacific	1410	70	76	76·5	Beechey	Near Clermont Tonnerre Isl.
88	22 Apr., 1818	19 18	1 25 W	S. Atlantic	393	62·8	73	72·1	Kotzebue	Between St. Helena and the Cape
89	14 Dec., 1857	19 34	27 19 W	S. Atlantic	2400	53	76		Pullen (*u*)	Between Rio Janeiro and St. Helena.
90	„ „	„ „	„ „	„ „	4800	38·5	„		„	
91	„ „	„ „	„ „	„ „	7200	41·2?	„		„	
92	23 Sept., 1828	20	70 20 E.	Indian Ocean	6194	45·3	73·4	71·2	D'Urville	E. of Rodrigue Isl. Thermometer wrong, too high
93	17 Apr., 1827	20 20	174 10 W	S. Pacific	1602	51·7	77·8	73·8	„	Amongst the Friendly Islands.
94	21 Jan., 1847	20 10	37 58 W	S. Atlantic	876	59	80	78	Dayman	Off the coast of Brazil.
95	„ „	„ „	„ „	„ „	1836	50	„	„	„	
96	16 Feb., 1858	20 14	59 35 E.	Indian Ocean	2880	50·5	80		Pullen (*u*)	Off the east coast of Mauritius.
97	„ „	„ „	„ „	„ „	5610	40	„		„	
98	„ „	„ „	„ „	„ „	8250	40·5?	„		„	
99	31 July, 1820	20 32	29 20 W.	S. Atlantic	426	69·4	72·6	71	D'Urville	Near Martin-Vaz Island
100	20 Apr., 1818	20 33	0 54 E.	S. Atlantic	367	60·8	73·5	71·8	Kotzebue	Between St. Helena and the Cape
101	July, 1825	20 38	38 46 W	S. Atlantic	2760	43·5	73	71	Beechey	Near the coast of Brazil
102	3 May, 1847	20 42	58 47 E.	Indian Ocean	840	74	77	76	Dayman	E. of the Mauritius.
103	„ „	„ „	„ „	„ „	1800	57	„	„	„	
104.	Feb., 1826	21 19	140 23 W	S. Pacific	1200	58·5	81·5	76	Beechey	Between the Society Islands and Pitcairn Island
105	„ „	„ „	„ „	„ „	1800	51	„	„	„	
106	„ „	„ „	„ „	„ „	2400	45	„	„	„	
107	6 Feb., 1859	21 51	149 59 W	S. Pacific	1080	71·3	81·6	82·4	Wüllerstorf	Between the Society and Tubuai Isles
108	18 May, 1847	21 53	56 45 E	Indian Ocean	1002	63	77	77	Dayman	Near the Isle of Bourbon.
109	18 Jan., 1819	22 31	40 31 W	S. Atlantic	600	69·5	74·7		Wauchope	Near the coast of Brazil.
110	17 Feb., 1837	23 30	43 21 W	S. Atlantic	373	60	72·5	74·3	DuPetitThouars	Cylinder full
111	July, 1825	23 32	41 12 W	S. Atlantic	1200	56	75	71	Beechey	Off Rio Janeiro.
112	15 Sept., 1857	33 49	0 39 W	S. Atlantic	840	56·3	57·5	55	Wüllerstorf	(Entered in wrong place.)
113	31 July, 1837	24 7	54 20 E.	Indian Ocean	4740	43	69·8	70·4	Vaillant	Between Bourbon and Madagascar
114.	19 May, 1847	24 16	56 58 E	Indian Ocean	1092	71	75	76	Dayman	S. of Isle of Bourbon.
115	Dec., 1825	24 35	127 W	S. Pacific	1440	60·5	76	76·5	Beechey	Near Elizabeth Island.
*116	27 Sept., 1772 .	24 44	24 54 W	S. Atlantic	480	68	70	72·5	Forster	Between Brazil and the Cape.
117	6 Aug., 1857	24 54	43 10 W	S. Atlantic	84	72·8	72·2	72·3	Wüllerstorf	
118	1 May, 1839 . (morning)	25 10	7 59 E.	S. Atlantic	5316	37·4	67·3	69	DuPetitThouars	1st cylinder sound south of St. Helena. 2nd cylinder full. Corr. 1=1790° feet.
119	1 May, 1839 (noon)	„ „	„ „	„ „	5316	40·4 (37·4)	67·2	67·5	„ „	
120	Apr., 1828	25 30	108 – W	S. Pacific .	600	69	80	80	Beechey .	To the N. E. of Easter Island.
121.	„ „	„ „	„	„ „	1200	58	„	„	„	
122	„ „	„ „	„	„ „	1860	50	„	„	„ .	

Table II.—Southern Hemisphere (continued).

I.	II. Date.	III South Latitude	IV. Longitude of Greenwich.	V Sea	VI Depth in feet	VII Temperature in degrees of Fahr At depth	Surface	Air	VIII Name of observer.	IX Remarks
123	Apr., 1828	25° 30′	108° ′ W	S. Pacific	2460	44°	80°	80°	Beechey	To the N E of Easter Island.
124	31 Mar., 1840	25 40	160 E.	S Pacific.	60	70	75		Wilkes	Between New South Wales and New Caledonia.
125	" "	" "	"	" "	120	72	"		"	
126	" "	" "	"	" "	180	73	"		"	
127	" "	" "	"	" "	240	71·5	"		"	
128	" "	" "	"	" "	300	72	"		"	
129	" "	" "	"	" "	360	71?	"		"	
130	" "	" "	"	" "	420	71·5	"		"	
131	" "	" "	"	" "	480	71·5	"		"	
132	" "	" "	"	" "	540	69·5	"		"	
133	" "	" "	"	" "	600	73?	"		"	
134	" "	" "	"	" "	1200	63·5	"		"	
135	" "	" "	"	" "	1800	56	"		"	
136	" "	" "	"	" "	2400	52	"		"	
137	" "	" "	"	" "	3000	49	"		"	
138	1 May, 1847	25 48	61 6 E.	Indian Ocean	900	62		74	Dayman	Between Mauritius and Isl of Amsterdam.
139	" "	" "	" "		1920	59		"	"	
140	4 Feb., 1847	26 7	40 30 W	S Atlantic	1386	60	77	66	Dayman	Off the coast of Brazil
141	" "	" "	" "	" "	2106	51	"	"	"	
142	20 May, 1847	26 9	58 45 E	Indian Ocean	840	63	71	74	Dayman	S of the Mauritius. Reading probably reversed
143	" "	" "	" "	" "	2160	73	"	"	"	
144	22 May, 1850	26 34	101 28 W	S Pacific	660	65	72	71	Armstrong	Between the Society Islands and Chili.
145	" "	" "	" "	" "	1110	53	"	"	"	
146	Nov., 1825	26 36	112 40 W	S. Pacific	2598	44	74·5	71	Beechey	West of Easter Island.
147	" "	" "	" "	" "	3240	43	"	"	"	
148	" "	" "	" "	" "	3840	44·5?	"	"	"	
149	20 April, 1839	26 36	7 32 E	S Atlantic	5315	41·7 (38·5)	68	69·8	DuPetit Thouars	Cylinder full, near the Cape.
150	19 Dec., 1857	26 46	23 52 W	S Atlantic	16200	35	75		Pullen	In soundings, descent 1 hour, ascent 2 hours
151	14 Feb., 1839	26 47	98 30 E	Indian Ocean	5316 (5200)	42·8 (38·7)	73·8	74·3	DuPetit Thouars	Cylinder full, in mid-ocean.
152	30 Sept., 1838	26 53	174 31 W	S Pacific.	5316 (4987)	45·2 (42·1)	66·7	66·7	DuPetit Thouars	Cylinder full, Kermadec Island.
153	28 April, 1847	26 56	57 31 E.	Indian Ocean	1200	60	74	70	Dayman	South of Mauritius
154	" "	" "	" "	" "	2100	57	"	"	"	
155	27 Nov., 1827	27	98 40 E	Indian Ocean	1602	52·3	70·5	69	D'Urville	Between Mauritius and Australia.
156	Nov., 1825	27 17	103 W	S Pacific	600	64·5	68·5	66	Beechey	Between the Society Islands and Chili
157	" "	" "	"	" "	1260	51·5	"	"	"	
158	" "	" "	"	" "	1800	46	"	"	"	
159	5 Feb., 1847	27 21	38 1 W	S Atlantic	1092	65	76	73	Dayman	Between the River Plata and the Isl of Tristan d'Acunha
160	" "	" "	" "	" "	2052	51	"	"	"	
161	15 May, 1836	27 30	41 E	Indian Ocean	30	74·5	75·6		FitzRoy	Between Natal and Madagascar.
162	" "	" "	"	" "	48	74·2	"		"	
163	" "	" "	"	" "	108	74	"		"	
164	" "	" "	"	" "	120	74	"		"	

TABLE II.—Southern Hemisphere (continued).

I	II Date	III South Latitude.	IV. Longitude of Greenwich	V. Sea.	VI Depth in feet.	VII. Temperature in degrees of Fahr. At depth	 Surface	 Air.	VIII. Name of observer.	IX. Remarks.
165	15 May, 1836	° ′ 27 30	° ′ 41 E	Indian Ocean	168	° 73	° 75·6	°	FitzRoy	Southern entrance of Mozambique Channel, between Madagascar and Natal. The 4th, 7th, 8th, and 10th observations, on being repeated, gave exactly the same results. The 1st, however, gave 74·4.
166	„ „	„ „	„	„ „	240	72·5	„		„	
167	„ „	„ „	„	„ „	288	71	„		„	
168	„ „	„ „	„	„ „	300	70	„		„	
169	„ „	„ „	„	„ „	450	68	„		„	
170	„ „	„ „	„	„ „	600	64·5	„		„	
171	„ „	„ „	„	„ „	1200	58·5	„		„	
172	„ „	„ „	„	„ „	1800	55·5	„		„	
173	„ „	„ „	„	„ „	2400	52·5	„		„	
174	„ „	„ „	„	„ „	2520	52	„		„	
175	21 May, 1847	27 36	61 9 E.	Indian Ocean	1998	54	73	69	Dayman	Between Madagascar and Isl of Amsterdam
176	11 Feb., 1839	27 47	100 20 E	Indian Ocean	5316 (5282)	37	74·8	76·2	DuPetit Thouars	Cylinder sound. W. of Australia.
177	24 May, 1847	28 1	67 28 E	Indian Ocean	1716	54	69	67	Dayman	Between Mauritius and the Island of Amsterdam
178	22 „ „	28 6	63 30 E	„ „	1800	53	69	68	„	
179	27 April, 1847	28 16	57 18 E.	Indian Ocean	1260	60	73	70	Dayman	Between Madagascar and Isl of Amsterdam
180	„ „	„ „	„ „	„ „	2160	57	„	„	„	
181	April, 1828	28 40	96 W	S Pacific	600	71	74	73	Beechey	Between Valparaiso and Easter Island.
182	„ „	„ „	„	„ „	1200	53	„	„	„	
183	„ „	„ „	„	„ „	1800	49	„	„	„	
184	„ „	„ „	„	„ „	2400	45	„	„	„	
185	14 Dec., 1857	28 45	84 48 E.	Indian Ocean	720	63·7	69·9	67·7	Wullerstorf	Between Madagascar and Australia
186	4 Oct., 1838	28 49	177 18 W	S Pacific	5316 (3740?)	44·7 (42·6)	67	66·2	DuPetit Thouars	Cylinder full. Near No. 152
187	27 July, 1827	29 13	12 54 W	S Atlantic	1655	50·4	66·6	69·5	Blosseville	Between the Cape and Paraguay
188	18 Nov., 1827	19 20	107 30 E	Indian Ocean	4378	40	73·4	71·2	D'Urville	(Entered in wrong place.)
189	13 Dec., 1857	29 25	85 2 E	Indian Ocean	720	68	69·1	66·6	Wullerstorf	Between Natal and Australia
190	26 April, 1839	29 33	10 57 E	S Atlantic	6396 (6133)	41·2 (37·6)	66·2	62·7	DuPetit Thouars	Full, between the Cape and Tristan d'Acunha
191	25 May, 1847	29 49	67 14 E	Indian Ocean	2160	54	66	66	Dayman	Between Madagascar and Isl of Amsterdam
192	10 Aug., 1826	30	22 40 W	S Atlantic	1602	50	63·6	59	D'Urville	Between the Cape and Uruguay
193	7 Dec., 1828	30	44 20 E	Indian Ocean	1602	58·8	72·8	73·6	„	Between Madagascar and the Cape
194	17 Aug., 1826	30	13 40 W	S Atlantic	1494	51·8	64	55·4	„	In mid-ocean
195	21 Dec., 1857	30 6	20 14 W	S Atlantic	2400	43·5	74·5		Pullen (*)	Mid-ocean, between the Cape and Brazil
196	„ „	„ „	„ „	„ „	4800	40·2	„		„	
197	„ „	„ „	„ „	„ „	7200	38·2	„		„	
198	1847–49	30 13	46 W	S Atlantic	360	64	77		E. Lenz	Off coast of Uruguay
199	26 April, 1847	30 13	56 50 E	Indian Ocean	972	61	71	65	Dayman.	Between Mauritius and the Cape
200	„ „	„ „	„ „	„ „	1698	60	„	„	„	
201	Nov., 1825	30 21	89 34 W.	S Pacific	600	62·5	63	66·5	Beechey	Between Easter Island and Valparaiso.
202	„ „	„ „	„ „	„ „	1320	50	„	„	„	
203	„ „	„ „	„ „	„ „	1920	45·2	„	„	„	
204	13 April, 1818	30 39	14 27 E.	S Atlantic	150	66·1	67·5	68	Kotzebue	West of the Colony of the Cape, northern part.
205	„ „	„ „	„ „	„ „	300	60·8	„	„	„	
206	„ „	„ „	„ „	„ „	1200	49·5	„	„	„	

TABLE II.—Southern Hemisphere (continued).

I.	II. Date.	III South Latitude	IV. Longitude of Greenwich	V Sea	VI Depth in feet	VII Temperature in degrees of Fahr At depth	Surface	Air	VIII. Name of observer.	IX Remarks.
207	8 Feb., 1847	30° 52′	36° 48′ W	S. Atlantic	1200	61°	73°	71°	Dayman	In mid-ocean, parallel of Uruguay
208	" "	" "	" "	" "	2160	51	"	"	"	
209	8 March, 1840	45 miles	W of Cape	S Atlantic	762	45	56	65	James Ross	Lat and long not given
210	10 " "	60 miles	" "	" "	1200	43·5	61	64	"	
211	7 " "	120 miles	" "	" "	2400	?	70	71	"	
212	Aug, 1825	31 29	45 57 W	S Atlantic	1860	46·5	66	62	Beechey	Off the coast of Uruguay
213	23 March, 1839	31 33	33 30 E	Indian Ocean	5316 (4659)	39·5	75·2	72·2	Du Petit Thouars	Cylinder sound, off Natal
214	9 Dec, 1828	32	35 50 E	Indian Ocean	2136	56·4	70·4	66·5	D'Urville	Between the Cape and Madagascar
215	26 May, 1847	32 4	68 6 E	Indian Ocean	2040	55	65	65	Dayman	Midway between the Cape and Western Australia
216	27 Jan, 1859	32 21	157 18 W	S Pacific	1200	60·3	73·1	71·2	Wullerstorf	Between Australia and Valparaiso
217	7 Oct, 1838	32 51	176 42 E	S Pacific	5316 (4692)	44·4 (41·7)	61·3	66·2	Du Petit Thouars	Cylinder full N of N Zealand
218	12 June, 1827	32 54	11 26 W	S Atlantic	2455	56·4	72	72·5	Blosseville	Between the Cape & Rio Janeiro
219	1847-49	32	72 (52?) W		360	52	56		E Lenz	Apparent error in longitude
220	16 Dec, 1828	33	30 20 E	Indian Ocean	801	64·2	69·7	69	D'Urville	Off the S coast of Natal
221	16 " "	33	29 20 E	" "	1014	69	74·2	72·3	"	
222	24 Mar, 1818	33 14	29 59 E	Indian Ocean	870	62·7	71·9	76·1	Kotzebue	On the Bank off the Cape
223	9 Feb, 1847	33 22	36 54 W	S Atlantic	1104	60	70	68	Dayman	Between Monte Video and the Cape
224	" "	" "	" "	" "	1944	50	"	"	"	
225	1 Mar, 1840	33 23	7 41 E	S Atlantic	600	56	70	71	James Ross	Vol ii p 53 In mid-ocean between the Cape and the Island of Tristan d'Acunha.
226	" "	" "	" "	" "	900	53·2	"	"	"	
227	" "	" "	" "	" "	1800	47·4	"	"	"	
228	" "	" "	" "	" "	2700	43	"	"	"	
229	" "	" "	" "	" "	3600	41·7	"	"	"	
230	24 Apr, 1837	33 26	72 03 W	S Pacific	853	49·1	54·7	51·8	Du Petit Thouars	Cylinder sound, soundings in 960 feet Off Valparaiso
231	27 Mar, 1827	33 30	175 50 E	S Pacific	3204	44·5	69·3	68·2	D'Urville	North of New Zealand
232	11 Aug, 1841	33 32	167 40 E	S Pacific	900	53			James Ross	In soundings on a bank between New Zealand and N S Wales
233	" "	" "	" "	" "	1200	51			"	
234	" "	" "	" "	" "	1800	48·1			"	
235	" "	" "	" "	" "	2400	45·3			"	
236	9 Aug., 1841	33 40	164 18 E	S Pacific	900	55·8	59		James Ross	Between New Zealand and N S Wales
237	" "	" "	" "	" "	1800	49·7	"		"	
238	10 Aug, 1841	33 41	166 23 E	S Pacific	12	58·7	59·7		James Ross	Between the North Island of New Zealand and New South Wales. No bottom in 4920 feet.
239	" "	" "	" "	" "	300	57·6	"		"	
240	" "	" "	" "	" "	600	56·7	"		"	
241	" "	" "	" "	" "	900	53·6	"		"	
242	" "	" "	" "	" "	1800	49·5	"		"	
243	" "	" "	" "	" "	2700	45·6	"		"	
244	" "	" "	" "	" "	3600	42·7	"		"	
245	" "	" "	" "	" "	4500	40·4	"		"	
246	27 May, 1847	33 48	70 11 E	Indian Ocean	2100	54	63	63	Dayman	Between the Cape and Australia.
247	17 Dec., 1828	34	27 20 E	Indian Ocean	334*	66·9	69·5	74·2	D'Urville	Near Algoa Bay
248	20 Mar., 1818	34 2	28 12 E	Indian Ocean	324	64	71·1	72	Kotzebue	On the Bank off the Cape

Table II.—Southern Hemisphere (continued).

I.	II. Date.	III. South Latitude.		IV. Longitude of Greenwich		V. Sea	VI. Depth in feet	VII Temperature in degrees of Fahr. At depth	Surface	Air	VIII. Name of observer.	IX. Remarks.
		°	′	°	′			°	°	°		
249	24 Apr, 1847	34	24	54	14 E	Indian Ocean	942	60	64	60	Dayman	Between Island of Amsterdam and the Cape.
250	„ „	„	„	„	„	„ „	1772	58	„	„	„	
251	19 Nov, 1838	34	34	161	2 E	S Pacific	4264 (3347?)	40·8	65	65 3	DuPetitThouars	Between New Zealand and Sydney Cylinder sound.
252	? „	34	37	171	1 E	„ „	3214 (2920)	42 8	62 7	61 2	„ „	
253	24 Feb, 1847	34	42	4	15 W	S Atlantic	2184	51	70	69	Dayman	Between St. Helena and Tristan d'Acunha.
254	„ „	„	„	„	„	„ „	3900	44	„	„	„	
255	12 Oct., 1772	34	48?	7	E.?	S. Atlantic	600	58	59	60	Forster	In the parallel of the Cape.*
256	12 Oct., 1838	34	54	174	5 E	S Pacific	1607 (951?)	50 5	62	60 8	DuPetitThouars	Soundings N of New Zealand.
256a	20 April, 1836.	34	57	52	30 W	S Atlantic	266	55	62 2	56 7	Vaillant	Entrance of Rio de la Plata
256b	„ „	35	1	„	„	„ „	250	60 6	61	62 6	„	
257	1 June, 1847	35		80	56 E	Indian Ocean	2076*	55	59	61	Dayman	Between the Cape and King George's Sound
258.	19 Dec, 1828	35		23	20 E	Indian Ocean	378	60 4	68 2	70 7	D'Urville	S W of Algoa Bay
259	21 Dec, 1828	35		18	20 E	S Atlantic	694	59 6	67 6	68	„	Off the Cape
260	4 Oct, 1826	35		111	20 E	Indian Ocean	480	56 3	57	59 8	„	Off the S W. of Australia
261	„ „	35	7	118	5 E	„ „	229*	58	62	58	„	Off King George's Sound.
262	27 Mar, 1818	35	17	22	56 E.	Indian Ocean	516	51 7	68 1	77 5	Kotzebue	Off the Cape.
263	10 Feb, 1847	35	21	35	31 W	S Atlantic	1008	62	68	68	Dayman	Between Monte Video and Tristan d'Acunha Island.
264.	„ „	„	„	„	„	„ „	1854	49	„	„	„	
265	25 Feb, 1847	35	28	3	6 W	S Atlantic	1170	54	69	68	Dayman	Between the African coast and Tristan d'Acunha Island.
266	„ „	„	„	„	„	„ „	2010	46	„	„	„	
267	4 Jan, 1827	35	30	137	20 E	S Pacific	1869	46 2	66 3	63 4	D'Urville	Off South Australia.
268	17 Feb, 1847	35	30	19	34 W	S Atlantic	1290	58	69	64	Dayman	Between the Cape and Monte Video ? Reading reversed
269	„ „	„	„	„	„	„ „	2196	61	„	„	„	
270	28 May, 1847	35	33	72	6 E	Indian Ocean	2100	55	60	61	Dayman	Between the Mauritius and the Island of Amsterdam
271	20 Dec, 1858	35	34	175	31 E	S Pacific	1020	60	67	63 7	Wüllerstorf	N of New Zealand.
272	5 Sept, 1826	36		33	20 E	Indian Ocean	1174	55 4	62 1	59	D'Urville	Off Natal
273	27 Oct, „	36		121	20 E	„ „	1708	45 3	56·6	54 9	„	Near King George's Sound.
274.	23 Feb, 1847	36	4	4	53 W	S Atlantic	1230	61	67	62	Dayman	Between the Cape and Monte Video
275	„ „	„	„	„	„	„ „	2070	48	„	„	„	
276	29 May, 1847	36	6	74	15 E	Indian Ocean	2100	52	59	60	Dayman	Near the Island of Amsterdam.
277	16 Feb, 1837	36	7	21	4 W	S Atlantic	1176	55	66	59	Dayman	Between the Cape and Monte Video
278	„ „	„	„	„	„	„ „	2016	47	„	„	„	
279	5 Feb, 1858	36	11	54	12 E	Indian Ocean	3600	46 8	66 5		Pullen (u)	Between the Cape and Island of Amsterdam.
280	„ „	„	„	„	„	„ „	6000	40·8	„		„	
281	13 Apr, 1847	36	17	26	43 E	Indian Ocean	1290	62	68	61	Dayman	South of Algoa Bay
282	„ „	„	„	„	„	„ „	2160	60	„	„	„	
283	20 Sept, 1857	36	22	5	29 E	S Atlantic	1320	53	51 8	56 8	Wüllerstorf.	Between Tristan d'Acunha and the Cape.
284.	20 Oct, „	36	22	17	34 E	„ „	600	63 3	63 5	59 9	„	South of the Cape.
285.	5 Mar, 1847	36	22	13	40 E	S Atlantic	1302	52	68	66	Dayman	Off the Cape.
286	„ „	„	„	„	„	„ „	2202	46	„	„	„	
287	6 Mar, 1847	36	24	14	42 E.	S Atlantic	882	65	70	71	Dayman	
288	„ „	„	„	„	„	„ „	1704	56	„	„	„	

Table II.—Southern Hemisphere (continued).

I.	II. Date.	III. South Latitude.	IV. Longitude of Greenwich.	V. Sea.	VI. Depth in feet.	VII. Temperature in degrees of Fahr. At depth.	Surface	Air	VIII. Name of observer.	IX. Remarks.
289.	17 Feb., 1827	36° 30′	176° 40′ E	S Pacific	907	60°·8	65°·5	64°·2	D'Urville	North of New Zealand.
290	15 Feb., 1847	36 31	24 7 W	S Atlantic	1164	58	64	63	Dayman	In mid-ocean
291	" "	" "	" "	" "	2034	45	"	"	"	
292.	27 Jan., 1839	36 36	118 28 E	Indian Ocean	5316 (5282)	37	64·2	63·5	DuPetitThouars	Cylinder sound. Near King George's Sound.
293	4 Mar., 1847	36 41	12 1 E	S Atlantic	1128	55	64	66	Dayman	S W of the Cape.
294	" "	" "	" "	" "	1968	46	"	"	"	
295	6 June, 1847	36 42	97 54 E	Indian Ocean	1920	51	56	55	Dayman	Between Amsterdam Island and Australia.
296	18 Feb., 1847	36 47	18 47 W	S Atlantic	768	57	68	54	Dayman	Between the Cape and Buenos Ayres
297	" "	" "	" "	" "	1542	50	"	"	"	
298.	3 Mar., 1847	36 47	10 24 E.	S Atlantic	1248	54	66	63	Dayman	S W of the Cape
299	" "	" "	" "	" "	2088	46	"	"	"	
300	30 Oct., 1857	36 48	18 11 E.		900	52	63·5	62·2	Wullerstorf	S of the Cape
301	13 Feb., 1847	36 50	27 50 W	S Atlantic	1290	62	66	66	Dayman	Between Tristan d'Acunha and Monte Video
302	" "	" "	" "	" "	2220	45	"	"	"	
303	14 April, 1847	36 53	27 49 E	Indian Ocean	1290	65	69	66	Dayman	Off Algoa Bay
304.	" "	" "	" "	" "	2160	56	"	"	"	
305	26 Feb., 1847	36 57	1 31 W	S Atlantic	1170	53	67	65	Dayman	Between Tristan d'Acunha Island and the Cape
306	" "	" "	" "	" "	2010	40	"	"	"	
307	18 Feb., 1827	37	176 20 E	S Pacific	801	57·7	67·1	64·2	D'Urville	North of New Zealand
308	12 July, 1841	37 20	151 36 E.	S Pacific	3300	46·2	60		James Ross	Off Port Jackson, no soundings
309	11 " "	17 miles	off C Howe	S Pacific	1752	49·7	59	59	"	In soundings
310	July, 1828	37 20	48 47 W	S Atlantic	600	57	60	57	Beechey	Off Rio de la Plata.
311	" "	" "	" "	" "	1140	56·5	"	"	"	
312	" "	" "	" "	" "	1740	48·5	"	"	"	
313	12 Feb., 1847	37 20	30 58 W.	S Atlantic	1230	57	66	69	Dayman	Between La Plata and the Cape
314	" "	" "	" "	" "	2130	45	"	"	"	
315	13 Jan., 1827.	37 30	157 20 E.	S Pacific	3257	42	67	65·5	D'Urville	Between New Zealand and New South Wales
316	8 Feb., "	37 30	178 55 E	" "	1922	46	67·4	65·3	"	Near the N E of New Zealand
317	1 Feb., 1839	37 42	114 58 E.	Indian Ocean	5315 (5282)	37·4	62·2	61·1	DuPetitThouars	S of Cape Leeuwin, Australia.
318	19 April, 1847	37 49	39 50 E.	Indian Ocean	1596	51	59	64	Dayman	Between the Cape and Crozet Island ? Reading reversed
319	" "	" "	" "	" "	1896	53	"	"	"	
320	21 Feb., 1847	37 54	10 28 W	S Atlantic	1230	53	62	59	Dayman	Between the Cape and La Plata
321	" "	" "	" "	" "	2070	43	"	"	"	
322.	1 Sept., 1826	38	24 20 E	Indian Ocean	587	54·7	63·2	54·7	D'Urville	S W of Algoa Bay
323	" "	"	" "	" "	2776	41·3	"	"	"	
324.	21 Nov., "	38	149 20 E	S Pacific	934	55·4	60	60·8	"	Bass's Strait
325.	19 Feb., 1847	38 7	16 43 W	S. Atlantic	2220	48	63	65	Dayman	Between La Plata and the Cape
326.	16 April, 1847	38 8	32 54 E.	Indian Ocean	768	64	69	69	Dayman	Between Port Natal and Prince Edward Island
327	" "	" "	" "	" "	1668	60	"	"	"	
328	3 Dec., 1857	38 9	77 46 E.	Indian Ocean	720	55·4	56·7	54	Wüllerstorf	Near the Island of Amsterdam
329	15 April, 1847	38 10	29 39 E.	Indian Ocean	1230	67	69	67	Dayman	South of Algoa Bay
330.	" "	" "	" "	" "	2100	58	"	"	"	
331.	26 Feb., 1837	38 12	33 40 W	S Atlantic	2132 (1968?)	37·4	62·3	71·6	DuPetitThouars	Cylinder full. In mid-ocean

TABLE II.—Southern Hemisphere (continued).

I	II. Date	III South Latitude	IV. Longitude of Greenwich	V. Sea	VI Depth in feet	VII. Temperature in degrees of Fahr At depth	Surface	Air	VIII. Name of observer.	IX. Remarks.
332	21 April, 1847	38° 13′	45° 36′ E.	Indian Ocean	948	55°	60°	66°	Dayman	Between the Cape and Island of Amsterdam
333	" "	" "	" "	" "	1758	52	"	"	"	
333a	27 Feb., 1847	38 22	0 28 W	S Atlantic	1152	55	62	64	Dayman	In open ocean
333b	" "	" "	" "	" "	2028	45	"	"	"	
334	1 Mar., 1847	38 25	4 1 E	S Atlantic	1170	48	55	56	Dayman	Between the Cape and Gough Island.
335	" "	" "	" "	" "	2010	44	"	"	"	
336	14 Nov., 1826	38 30	145 16 E	S Pacific	58	60.2	63.6	65.3	D'Urville	In Bass's Strait
337	Oct., 1825	38 30	75 44 W	S Pacific	540	51	55.5	54	Beechey	Off the south coast of Chili
338	" "	" "	" "	" "	1200	44.5	"	"	"	
339	" "	" "	" "	" "	1800	43.5	"	"	"	
340	" "	" "	" "	" "	2400	44	"	"	"	
341	26 Nov., 1857	38 41	77 45 E	Indian Ocean	720	55.4	56	54.7	Wullerstorf	Near the Island of Amsterdam.
342	10 Nov., 1826	39	141 50 E.	S Pacific	1708	47.5	56.2	63.6	D'Urville	Off the S coast of Australia.
343	23 Jan., 1839	39 4	123 22 E.	S Pacific	1870	48.5 (47.5)	60.8	58.7	DuPetit Thouars	Cylinder full S of Australia
344	27 Nov., 1841	39 16	177 25 W	S Pacific	900	53.5	58		James Ross	Off the east coast of the North Island of New Zealand no soundings
345	" "	" "	" "	" "	1800	49.2	"		"	
346	" "	" "	" "	" "	2700	46.8	"		"	
347	" "	" "	" "	" "	3600	44.9	"		"	
348	Aug., 1825	39 31	45 2 W	S Atlantic	1482	55	59	47	Beechey	Open ocean
349	15 Mar., 1839	39 51	44 17 E.	Indian Ocean	5316 (3051?)	37.8	78	80.6	DuPetitThouars	Cylinder sound Open ocean
350	12 June, 1847	39 57	118 E	Indian Ocean	1920	45	54	48	Dayman	S of King George's Sound
351	14 Nov., 1857	40 44	60 8 E	Indian Ocean					Wullerstorf	To the N W of Kerguelen I. No soundings in 37,020 feet
352	14 June, 1847	40 46	123 26 E	S Pacific	2280	50	53	49	Dayman	South of W Australia
353	11 Nov., 1857	40 52	49 57 E	Indian Ocean	600	54.0	54.3	47.7	Wullerstorf	Between the Cape and Kerguelen Island
354	20 Jan., 1827	40 58	173 5 E	S Pacific	32*	63.5	64.4	65	D'Urville	In Cook Strait.
355	5 April, 1850	41	54 35 W	S Atlantic	900	40	59		Armstrong	Off La Plata
356	2 Feb., 1827	40 31	176 48 E	S Pacific	506*	58.7	65.2	65	D'Urville	Cook's Strait, New Zealand.
357	2 Mar., 1837	41 56	55 6 W	S Atlantic	1066 (591?)	38.5	60.8	62.7	DuPetitThouars	Cylinder full.
358	4 Jan., 1827	42	171 E.	S Pacific	534	55.8	63	61	D'Urville	Near the W coast of N Zealand
359	Sept., 1825	42 2	46 8 W	S Atlantic	1200	41	47.5	47	Beechey	In the parallel of Rio Negro
360	27 Dec., 1838	42 34	153 10 E.	S Pacific	5316 (3904?)	43.5 (41.4)	55.7	55.4	DuPetit Thouars	Cylinder full
361	17 Jan., 1839	43 2	131 54 E	S Pacific	5872	44.6 (41.2)	55.4	53.6	DuPetitThouars	S of Australia, cylinder full
362	26 March, 1843	43 10	14 44 E	S Atlantic	1800	44	53	52.7	James Ross	Between the Cape of Good Hope and Bouvet Island, no soundings.
363	" "	" "	" "	" "	2700	41.1	"	"	"	
364	" "	" "	" "	" "	6300	39.8	"	"	"	
365	" "	" "	" "	" "	7200	39.5	"	"	"	
366	17 Dec., 1827	43 25	147 7 E	S Pacific	100*	55.6	59	59	D'Urville	Off east coast of Tasmania.
366a	9 July, 1847	15 miles	E of Cape	Pillar	2250	48	55	53	Dayman	Tasmania
367	16 April, 1838	43 47	79 6 W	S Pacific	2656	39.3	55.7	55.5	DuPetit Thouars	Off the Isle of Chiloe Cylinder sound
368	" "	" "	" "	" "	5872	36.1	55.4	55	"	

TABLE II.—Southern Hemisphere (continued).

I.	II. Date.	III. South Latitude.	IV. Longitude of Greenwich	V. Sea.	VI Depth in feet	VII. Temperature in degrees of Fahr At depth	Surface	Air	VIII. Name of observer.	IX. Remarks.
369	27 Mar, 1843 .	43° 52′	13° 23′ E.	S Atlantic ..	900	44°	47·5°	49·8°	James Ross	Between the Cape and Bouvet Island.
370	,, ,,	,, ,,	,, ,,	,, ,,	1800	40·3	,,	,,	,, ,,	
371	,, ,,	,, ,,	,, ,,	,, ,,	2700	39·8	,,	,,	,, ,, ..	
372	,, ,,	,, ,,	,, ,,	,, ,,	3600	39·5	,,	,,	,, ,,	
373	8 Jan, 1816 .	44 47	57 31 W	S Atlantic	1176	38·8	54·9	57·6	Kotzebue	Between Monte Video and the Falkland Islands
374	5 Mar, 1837 (8 A M)	45 38	61 10 W	S Atlantic	160	48·2	57·5	59	DuPetitThouars	Cylinder sound In soundings N of the Falkland Islands
375	,, (7 A M)	,, ,,	,, ,,	,, ,,	213	42·4	57·2	55·4	,, ,,	
376	,, (6 A M)	,, ,,	,, ,,	,, ,,	374	41·3	,,	55	,, ,,	
377	,, (noon)	,, ,,	,, ,,	,, ,,	374	41·3	58·6	63	,, ,,	
378	Sept, 1825	46 15	51 53 W	S Atlantic	1080	41	51	55	Beechey	N E of the Falkland Islands.
379	14 Nov, 1840	?	?	S Pacific	900	49·8	51	46·8	James Ross	Two days' sail south of Van Diemen Land no soundings
380	,, ,,	,, ,,	,, ,,	,, ,,	1800	48	,,	,,	,, ,,	
381	,, ,,	,, ,,	,, ,,	,, ,,	2700	46·5	,,	,,	,, ,,	
382	,, ,,	,, ,,	,, ,,	,, ,,	3600	45·6	,,	,,	,, ,,	
383	Sept, 1825	47 18	53 30 W	S Atlantic	1620	44·7	49·8	43	Beechey	Open sea to the N E. of the Falkland Islands
384	,, ,,	,, ,,	,, ,,	,, ,,	3618	39·2	,,	,,	,,	
385	,, ,,	,, ,,	,, ,,	,, ,,	4398	40·1	,,	,,	,,	
386	,, ,,	,, ,,	,, ,,	,, ,,	5124	39·4	,,	,,	,,	
387	4 Dec, 1841	49 17	172 28 W	S Pacific	900	48·7	53	49·7	James Ross	Near Antipodes Island No soundings in 6600 feet.
388	,, ,,	,, ,,	,, ,,	,, ,,	2700	44·5	,,	,,	,, ,,	
389	,, ,,	,, ,,	,, ,,	,, ,,	3600	42·2	,,	,,	,, ,,	
390	,, ,,	,, ,,	,, ,,	,, ,,	4500	41	,,	,,	,, ,,	
391	,, ,,	,, ,,	,, ,,	,, ,,	5400	40·2	,,	,,	,, ,,	
392	,, ,,	,, ,,	,, ,,	,, ,,	6300	40	,,	,,	,, ,,	
392a	Feb., 1804	52	68 W	S Atlantic	330	46	53·4		Horner	Off Patagonia.
393	2 April, 1841	52 10	136 56 E	South'n Ocean	900	42	43		James Ross	Between Australia and the Antarctic Land Soundings in 9240 feet
394	,, ,,	,, ,,	,, ,,	,,	1800	41	,,		,, ,,	
395	,, ,,	,, ,,	,, ,,	,,	2700	40	,,		,, ,,	
396	,, ,,	,, ,,	,, ,,	,,	3600	39·8	,,		,, ,,	
397	23 Dec, 1772	52 26	?	S Ocean .	600	34·5	32	33	Forster	20° south of the Cape
398	1847–49	53 12	55 W	S Ocean	360	43	51		E Lenz	S E of the Falkland Islands
398a	9 May, 1836	53 47	62 45 W	S Atlantic	957	39·2	41	38	Vaillant	Off Terra del Fuego
399	16 Sept., 1842	54 41	55 12 W	S Ocean	900	39·8	39·5	33·5	James Ross	In soundings. 10° east of Cape Horn
400	,, ,,	,, ,,	,, ,,	,,	1080	39·8	,,	,,	,, ,,	
401	10 Jan, 1840	55?	157 E ?	S Ocean ..	1800	39	43		Wilkes	Off Macquarie Island Mud at bottom of No 402
402	22 ,, ,,	?	?	,,	1920	27·5	32		,,	
403	16 ,, ,,	?	157 46 E	,,	5100	31·5	31		,,	
404	16 Mar, 1839 .	55	65 W	,,	2400	37	44		,,	Near Le Maire Strait
405	15 Dec, 1772	55 8	22 E ?	S. Ocean	600	34	30	32	Cook	Amongst ice S of the Cape
406	30 Mar, 1841 ..	55 9	132 28 E	S Ocean	900	39	38·5	39 ?	James Ross	Open ocean, on the parallel of Macquarie Island no soundings
407	,, ,,	,, ,,	,, ,,	,,	1800	39·5	,,	,,	,, ,,	
408	,, ,,	,, ,,	,, ,,	,,	2700	39·8	,,	,,	,, ,,	
409	,, ,,	,, ,,	,, ,,	,,	3600	39·8	,,	,,	,, ,,	

TABLE II.—Southern Hemisphere (continued).

I.	II. Date.	III. South Latitude	IV. Longitude of Greenwich.	V. Sea.	VI. Depth in feet.	VII. Temperature in degrees of Fahr. At depth	Surface	Air	VIII. Name of observer	IX. Remarks.
410.	13 Dec., 1841	55° 18′	149° 20′ W	S. Ocean . .	900	39°·6	39°	42°·5	James Ross	Between Cape Horn and New Zealand no soundings.
411	" "	" "	" "	"	1800	39·9	"	"	" "	
412	13 Dec., 1841	" "	" "	"	2700	39·7	39	42·5	" "	
413	" "	" "	" "	"	3600	39·7	"	"	" "	
414.	1847–49	55 19	62 W	S Ocean	300	41	48·4		E Lenz .	Between Cape Horn and the South Shetlands.
415	23 Dec., 1772	55 20	31 30 E	S Ocean	600	34·5	32	33	Cook	Near ice between Cape and Enderby Land
416	18 Sept., 1842	55 40	63 8 W	S Ocean	1800	37·2	40·2	31·9	James Ross	No soundings.
417	20 Dec., 1842	55 48	54 40 W	S Ocean .	900	40	40	45·4	James Ross	Between the Falkland Islands and Elephant Island no soundings.
418	" "	" "	" "	"	1800	39·6	"	"	" "	
419	" "	" "	" "	"	2700	39·6	"	"	" "	
420	" "	" "	" "	"	3600	39·4	"	"	" "	
421	" "	" "	" "	"	4500	39·3	"	"	" "	
422	" "	" "	" "	"	6000	39·5	"	"	" "	
423	Sept., 1825	55 58	72 10 W	S Ocean	600	42·5	43·5	37	Beechey	Off the south coast of Terra del Fuego
424.	" "	" "	" "	"	1380	42·5	"	"	"	
425	" "	" "	" "	"	1980	40·5	"	"	"	
426	" "	" "	" "	"	2580	41·6	"	"	"	
427	1847–49	56	64 W		300	41	46		E. Lenz	East of Cape Horn
428	14 Dec., 1841	56 20	148 8 W	S Ocean	900	38	35·8	41	James Ross	In mid-ocean no soundings.
429	" "	" "	" "	"	1800 to 7200	39·7	"	"	" "	
430	18 Mar., 1843	56 41	6 5 W	S Ocean	900	35·2	33·5	33·2	James Ross	In mid ocean, between Bouvet Island and Sandwich Isl. no soundings.
431	" "	" "	" "	"	1800	36·8	"	"	" "	
432	" "	" "	" "	"	2700	37·8	"	"	" "	
433	" "	" "	" "	"	3600	39	"	"	" "	
434.	5 April, 1837	56 58	82 16 W	S Ocean	13124 (12828)	?	44·6	42·6	DuPetitThouars	Cylinder crushed index fixed
435	21 Dec., 1840	57 52	170 30 E	S Ocean .	1380	39·5	42	39	James Ross.	
436	23 Mar., 1837	58 32	73 29 W	S Ocean	2132 (1608)	39·5	44	45	DuPetitThouars	Near No 448 Cylinder sound
437	23 Mar., 1842	58 36	104 40 W	S Ocean	300	40·8	41	32	James Ross	Between Dougherty Island and Cape Horn no soundings.
438	" "	" "	" "	"	600	40·8	"	"	" "	
439	" "	" "	" "	"	900	40·7	"	"	" "	
440	" "	" "	" "	"	1800	40·8	"	"	" "	
441	" "	" "	" "	"	2700	40·5	"	"	" "	
442	" "	" "	" "	"	3600	40	"	"	" "	
443	1 April, 1837	58 40	79 15 W	S Ocean	2657 (1870)	38·6	42·4	42·4	DuPetitThouars	Cylinder full.
444	28 Mar., 1842	58 55	83 16 W	S Ocean	900	40·8	42	40	James Ross	Open sea to the S W of Cape Horn
445	" "	" "	" "	"	1800	40·8	"	"	"	
446	" "	" "	" "	"	2700	40·5	"	"	"	
447	" "	" "	" "	"	3600	40	"	"	"	
448	22 Dec., 1840	59	171 E.	S Ocean	900	38·5	37	37·4	James Ross	Between New Zealand and South Victoria Land
449	" "	"	" "	"	1800	39·5	"	"	"	
450	"	"	" "	"	2700	39·7	"	"	"	
451	" "	"	" "	"	3600	39·7	"	"	"	

Table II.—Southern Hemisphere (continued).

I.	II. Date.	III. South Latitude	IV. Longitude of Greenwich.	V. Sea.	VI. Depth in feet	VII. Temperature in degrees of Fahr. At depth	Surface	Air	VIII. Name of observer	IX. Remarks.
452	26 Mar., 1837	59° 48′	79° 56′ W	S Ocean	2657 (2395)	39°	42°·9	42°·4	Du Petit Thouars	Near his last. Cylinder full
453	25 Mar., 1841	60 22	131 28 E	S Ocean	900	37	35	34·4	James Ross	Between Australia and Adélie Land no soundings.
454	„ „	„ „	„ „	„	1800	38	„	„	„	
455	„ „	„ „	„ „	„	2700	39·5	„	„	„	
456	„ „	„ „	„ „	„	3600	40·5	„	„	„	
457	Mar., 1839	61 ?	55 W ?	South Seas	1800	33	36		Wilkes	Off Elephant Island
458	22 Feb., 1843	61 30	22 30 W	S Ocean	4500	30·2	32	30	James Ross	In mid-ocean no soundings.
459	8 Mar., 1842	62 15	163 50 W	S Ocean	600	32·2	35	32	James Ross	Between the Society Islands and Antarctic Continent. From the surface to 600 ft. the temperature was 30° 8.
460	„ „	„ „	„ „	„	900	35·5	„	„	„	
461	„ „	„ „	„ „	„	1800	37·2	„	„	„	
462	„ „	„ „	„ „	„	2700	38·5	„	„	„	
463	„ „	„ „	„ „	„	3600	39	„	„	„	
464	1839	63 ?	57 W ?	S Ocean	600	20	29	28	Wilkes	Near the South Shetlands.
465	27 Dec., 1840	63	174 30 E	S Ocean	900	35·5	30	31·9	James Ross	Between New Zealand and S Victoria Land no soundings
466	„ „	„	„ „	„	1800	38·2	„	„	„	
467	„ „	„	„ „	„	3600	39·7	„	„	„	
468	4 Mar., 1839	63 18	55 W	S Ocean	600	30	31		Wilkes	See Amer Journ of Sc Jan 1848
469	20 Dec., 1841	63 47	151 34 W	S Ocean	900	35·6	30	27·7	James Ross	Amongst ice Between Dougherty Island and South Victoria Land Soundings in 10,200 feet
470	„ „	„ „	„ „	„	1800	38·4	„	„	„	
471	„ „	„ „	„ „	„	3600	40	„	„	„	
472	„ „	„ „	„ „	„	4500	39·6	„	„	„	
473	„ „	„ „	„ „	„	5400	39·8	„	„	„	
474	8 Feb., 1843	63 49	51 7 W	S Ocean	600	32·2	32	33	James Ross	Near Louis Philippe Land, Antarctic Continent Off the pack No soundings in 7260 feet
475	„ „	„ „	„ „	„	900	33·2	„	„	„	
476	„ „	„ „	„ „	„	1800	35·5	„	„	„	
477	„ „	„ „	„ „	„	2700	36·4	„	„	„	
478	„ „	„ „	„ „	„	3600	37·3	„	„	„	
479	„ „	„ „	„ „	„	7260	39·5	„	„	„	
480	18 Mar., 1841	63 51	151 47 E	S Ocean	900	35·5	30·4	28·3	James Ross	Between Van Diemen Land and South Victoria Land. Near the pack no soundings
481	„ „	„ „	„ „	„	1800	37·5	„	„	„	
482	„ „	„ „	„ „	„	2700	38·5	„	„	„	
483	„ „	„ „	„ „	„	3600	39·2	„	„	„	
484	18 Jan., 1843	63 59	54 35 W	S Ocean	900	30	32		James Ross	Off the pack soundings in 1260 feet
485	21 Mar., 1841	64 7	140 22 E	S. Ocean	900	34	30·8	27	James Ross	Near the pack no soundings. N of Adélie Land
486	„ „	„ „	„ „	„	1800	36·5	„	„	„	
487	„ „	„ „	„ „	„	2700	38	„	„	„	
488	„ „	„ „	„ „	„	3600	38·7	„	„	„	
489	13 Jan., 1773	64 30?	39 E ?	S Ocean	600	32	33·5	36	Cook	N W of Enderby Land
490	30 Dec., 1840	64 38	173 10 E	S Ocean	900	35·2	31	32·2	James Ross	Between New Zealand and South Victoria Land. Soundings in 9360 feet
491	„ „	„ „	„ „	„	1800	37·2	„	„	„	
492	„ „	„ „	„ „	„	2400	38·8	„	„	„	
493	„ „	„ „	„ „	„	3600	39·8	„	„	„	

TABLE II.—Southern Hemisphere (continued).

I	II. Date.	III. South Latitude	IV Longitude of Greenwich.	V. Sea	VI. Depth in feet.	VII. Temperature in degrees of Fahr. At depth	Surface	Air	VIII. Name of observer.	IX. Remarks.
494	6 Mar, 1841	64° 51′	164° 45′ E	S Ocean	3000	37°2	29°2	31°	James Ross	No soundings.
494a	12 Feb, 1840	64 57	112 16 E	S Ocean	1500	30 5		31	Wilkes	Near the ice-barrier No soundings.
495	3 Jan, 1842	66 34	156 22 W	Antarctic O	6300	39 6	28	31 1	James Ross	In the pack.
496	3 Mar, 1842	67 28	174 27 W	Ant. Ocean	900	34 2	33	32 3	James Ross	No soundings Not far from icebergs.
497	,, ,,	,, ,,	,, ,,	,, ,,	1800	35 5	,,	,,	,,	
498	,, ,,	,, ,,	,, ,,	,, ,,	2700	37 5	,,	,,	,,	
499	,, ,,	,, ,,	,, ,,	,, ,,	3600	38	,,	,,	,,	
500	7 Jan, 1841	68 17	175 21 E	Ant Ocean	900	37 5	28	28	James Ross	Not far from icebergs Approaching the Antarctic continent
501	,, ,,	,, ,,	,,	,, ,,	1800	38 2	,,	,,	,,	
502	,, ,,	,, ,,	,, ,,	,, ,,	2700	39 2	,,	,,	,,	
503	,, ,,	,, ,,	,, ,,	,, ,,	3600	39 8	,,	,,	,,	
504	2 Mar, 1841	68 27	167 42 E		2400	36	28 2	27	James Ross	Ditto No soundings.
505	3 Mar, 1843	68 32	12 49 W	Ant Ocean	900	33	30 8	29 4	James Ross	Between Louis Philippe Land and Enderby Land No soundings in 24,000 feet
506	,, ,,	,, ,,	,, ,,	,, ,,	1800	35 5	,,	,,	,,	
507	,, ,,	,, ,,	,, ,,	,, ,,	3600	38·7	,,	,,	,,	
508	,, ,,	,, ,,	,, ,,	,, ,,	4500	39 4	,,	,,	,,	
509	,, ,,	,, ,,	,, ,,	,, ,,	5400	39	,,	,,	,,	
510	,, ,,	,, ,,	,, ,,	,, ,,	6300	39 5	,,	,,	,,	
511	9 Feb, 1842	70 39	174 31 W	Ant Ocean	900	32 1	28	27 9	James Ross	To the N E of S Victoria Land Near the pack no soundings
512	,, ,,	,, ,,	,, ,,	,, ,,	1800	35	,,	,,	,,	
513	,, ,,	,, ,,	,, ,,	,, ,,	2700	35 8	,,	,,	,,	
514	,, ,,	,, ,,	,, ,,	,, ,,	3600	37·6	,,	,,	,,	
515	18 Jan, 1841	72 57	176 6 E	Ant Ocean	900	33 8	30	31	James Ross	In the parallel of Mt. Sabine, S Victoria no soundings.
516	,, ,,	,, ,,	,, ,,	,, ,,	1380	34 6	,,		,,	
517	15 Feb, 1842	75 6	172 56 E	Ant Ocean	1740	32	30	25 1	James Ross	Off S Victoria Ld. in soundings.
518	1 Feb, 1841	77 5	171 33 W	Ant. Ocean	900	33	32	27 ?	James Ross	Off the perpendicular ice-barrier Appearance of land beyond in soundings
519	,, ,,	,, ,,	,, ,,	,, ,,	1500	33 2	,,	,,	,,	
520	29 Jan, 1841	77 47	176 43 E	Ant Ocean	900	33	31	28	James Ross	13 miles off the ice-wall. Soundings in 2400 feet.
521	,, ,,	,, ,,	,, ,,	,, ,,	1800	34 2	,,	,,	,,	
522	23 Feb, 1842	77 49	162 36 W	Ant Ocean	1740	30 8	28 5	25	James Ross	Off the perpendicular ice-barrier Appearance of land beyond

ADDENDA —Omitted Observations of Capt KELLETT

Depth in feet	13 Dec, 1845—19° 10′ S, 77° 17′ W Temperature At depth	Surface	Air	20 Jan, 1846—0° 18′ S, 83° W Temperature At depth	Surface
60	66° F	68° F	65° F	75° F	76° F
120	65			70	,,
180	63	,,	,,	67	,,
300	60	,,	,,	65 5	,,
600	55	,,	,,	62 5	,,
1200	51	,,	,,	54	,,
1800	52	,,	,,	51	,,
2400	46	,,	,,	48	,,
3000	46	,,	,,	47	,,

Table III.—Submarine Temperatures of Inland Seas*.

The Mediterranean.

I.	II. Date.	III Position.	IV Depth in feet.	V. Temperature in degrees of Fahr At depth	Surface	Air	VI. Name of observer	VII Remarks
		Western Division						
1	8 Oct., 1780	Off Port Fino, near Genoa	944	55·8°	69°	66°	Saussure	Thermometers left down 12 hours.
2	16 „ „	Off Cape della Causa, near Nice	1918	55·8	68·5	„	„	
3	22 Mar, 1820	41° N lat. 5° 20 E long	3204	54·7	58·5	57·6	D'Urville	Between the coast of France and Straits of Gibraltar
4	23 „ „	41° N lat 2° 20 E long.	1602	54·7	57·1	58·1	„	
5	27 Apr, 1826	40° N lat 4° 50 E. long	1602	54·5	56·9	62·3	„	
6	27 „ „	Two miles N of Alboran	293	56·3	61·3	59·6	„	
7	5 May, „	Straits of Gibraltar	1068	54·2	64·1	65·3	„	
8	8 „ „	Five miles E of Ceuta	1335	57·4	63	63·6	„	
9	21 „ „	Anchorage of Algesiras	106*	56·5	59·2	63·8	„	
10	22 „ „	„ „	106*	61·3	60·5	65·9	„	
11	26 „ „	„ „	119*	58·3	60·8	60·8	„	
12	3 June „	„ „	112*	59·2	62·7	66·5	„	
13	26 June, 1831	Between Mahon and Algiers	6408	55·4	69·8	75·2	Bérard	Between the Balearic Isles and Algeria.
14	27 „ „	„ „	3204	55·4	73·7	74·3	„	
15	23 July, 1832	14 miles N E. of Bougie	3850	55·7	79·7	83·2	„	
16	9 Aug, „	10 miles N of Bougie	267	56·3	79·7	81·6	„	
17	23 „ „	8 miles E N E of Bougie	267	55·4	80·8	89·9	„	
18	23 Oct, „	40° 41 N lat. 2° 10 E. long	373	58·8	70·8	73·7	„	
19	„ „	„ „	213	61·8	72	75·2	„	
20	„ „	„ „	106	69	68	72	„	
21	15 Nov, 1831	Off Cape St. Martin	3204	55·4	67·1	60·7	„	
22	23 „ „	„ „	4005	55·4	58·3	59	„	
23	July, 1844	Between Marseilles and Algiers	3	73·4	74·5		Aimé	Mean of July (evening) observations The mean temperature of the air in July is 75°
24	„ „	„ „	33	68	„		„	
25	„ „	„ „	49	66·2	„		„	
26	„ „	„ „	65	65·5	„		„	
27	„ „	„ „	82	64·4	„		„	
28	„ „	„ „	98	63·5	„		„	
29	Mar, 1844	„ „	3	57·4	57·5		Aimé	Mean of March (evening) observations The mean temperature of the air in March is 58°·1
30	„ „	„ „	6½	57·2	„		„	
31	„ „	„ „	33	57	„		„	
32	„ „	„ „	46	56·8	„		„	
33	„ „	„ „	59	56·6	„		„	
34	„ „	„ „	72	56·6	„		„	
35	1841–1844	Between Marseilles and Algiers, exact position not specified	82	61·3	64·7	64·4	Aimé	These numbers give the mean annual temperatures resulting from the total of his observations.
36	„ „	„ „	164	58	„	„	„	
37	„ „	„ „	328	56·7	„	„	„	
38	„ „	„ „	656	55·4	„	„	„	
39	„ „	„ „	1148	54·6	„	„	„	

* A few observations of D'Urville, marked thus, have F affixed in the original Possibly this may mark "Fathoms," but in the absence of information the reduction is for "Brasses," which is the measure he otherwise used

TABLE III.—INLAND SEAS (continued).

The Mediterranean (continued).

I.	II. Date.	III. Position.	IV. Depth in feet	V. Temperature in degrees of Fahr At depth	Surface	Air	VI. Name of observer	VII. Remarks.
		WESTERN DIVISION (continued)						
40	9 May, 1857	38° 36′ N lat 13° 41′ E long	720	58°8	61°6	61°	Wüllerstorf.	
41	11 „ „	38° 51′ N lat 10° 36′ E long	750	57 2	60	62	„	
42	15 „ „	37° 56′ N lat 3° 47′ E long	750	56 4	62·2	64 5	„	
43	„ „	„ „	648	61 2	„	„	„	Between the Straits of Messina and Straits of Gibraltar
44	19 „ „	36° 2′ N lat. 4° 2′ W long	750	59·2	62 6	63 2	„	
45	24 „ „	36° 8′ N lat 5° 21′ W long	60	60 1	59 2	63 5	„	
46	30 „ „	36° 7′ N lat. 5° 22′ W long	270	58 4	61 5	64 8	„	
47	2 June, „	36° 33′ N lat 4° 34′ W long	72	56 4	57 6	66 2	„	
		EASTERN DIVISION						
48	4 May, 1857	39° 33′ N lat 18° 51′ E long	180	58 2	61	61	Wullerstorf	
49	„ „	„ „	300	60 1	„	„	„	Between the Ionian Islands and Sicily
50	5 „ „	38° 21′ N lat 16° 56′ E long	150	60 8	61 5	60 5	„	
51	July, 1845	Egina Gulf	12	82		88	Spratt	
52	„ „	„ „	60	78		„	„	
53	„ „	„ „	120	69		„	„	
54	„ „	„ „	210	62		„	„	A mile and a half from shore.
55	„ „	„ „	450	56		„	„	
56	„ „	„ „	780	55 5		„	„	
57	„ „	„ „	12	80		84	Spratt	
58	„ „	„ „	60	76		„	„	
59	„ „	„ „	120	69		„	„	
60	„ „	„ „	210	61		„	„	Three miles from shore
61	„ „	„ „	330	57		„	„	
62	„ „	„ „	1260	55 5		„	„	
62a	23 July, 1846	N Division of Archipelago	30	76		86	Spratt (*)	
62b	„ „	„ „	60	69		„	„	
62c	„ „	„ „	150	62		„	„	Between Eubœa and Skyros.
62d	„ „	„ „	300	58		„	„	
62e	„ „	„ „	600	55		„	„	
63	Aug., 1847	Off Nio	1080	55 5		86	Spratt	Four miles from shore
64	„ „	Off Andros	1200	55 5		„	„	Seven miles from shore
65	25 July, 1847	Grecian Archipelago	60	74	78	86	Spratt	
66	„ „	„ „	120	74	„	„	„	
67	„ „	„ „	360	64	„	„	„	Southern division of Greek archipelago.
68	„ „	„ „	540	64	„	„	„	
69	„ „	„ „	720	56	„	„	„	
70	20 Sept., 1852	Off Crete	60	72	75	76	Spratt	
71	„ „	„ „	300	59	„	„	„	North coast of Crete.
72	„ „	„ „	720	56	„	„	„	
73	14 June, 1860	Off Crete	60	68	73	80	Spratt	About 50 miles west of Casisotta, on the N.W. coast of Crete.
74	„ „	„ „	120	68	„	„	„	

Table III.—Inland Seas (continued).

The Mediterranean (continued).

I.	II. Date.	III. Position.	IV. Depth in feet.	V. Temperature in degrees of Fahr. At depth.	Surface.	Air	VI. Name of Observer.	VII. Remarks.
75.	14 June, 1860*	Off Crete	180	65°	73°	80°	Spratt	About 50 miles west of Cerizotta, on the N W coast of Crete.
76.	" "	" "	300	63	"	"	"	
77.	" "	" "	600	59¾	"	"	"	
78.	" "	" "	1200	59½	"	"	"	
79	" "	" "	7440	59¼	"	"	"	
80	25 Aug, 1860	Off East End of Rhodes	60	81	82	88	Spratt	About 2 or 3 miles from the coast
81	" "	" "	120	79½	"	"	"	
82	" "	" "	180	78½	"	"	"	
83	" "	" "	300	77	"	"	"	
84.	" "	" "	600	73	"	"	"	
85	May, 1861	Between Malta and Tripoli	1770	62	62	68	Spratt	
86	21 Feb, 1861	Gulf of Syrtis	120	62	61	64	Spratt	About 200 miles west of Benghazi
87	" "	" "	300	62	"		"	
88.	" "	" "	1740	62?	"		"	
89.	27 Feb, 1861	Gulf of Syrtis	300	61	60	56	Spratt	180 miles S E of Malta No soundings in 6000 feet
90	" "	" "	600	61½	"	"	"	
91	6 April, 1861	Arabs Gulf W of Alexandria	120	61½	62	68	Spratt	Near the coast
92.	" "	" "	1800	59¾	"	"	"	
93	April, 1861	Off the coast of Egypt	120	61½	63	65	Spratt	Off Alexandria
94.	" "	" "	1620	59½	"		"	
95	15 Nov, 1861	Off the coast of Egypt	180	71	73	69	Spratt	Off Arabs Tower, west of Alexandria
96	" "	" "	300	68	"	..	"	
97	" "	" "	480	64	"		"	
98.	" "	" "	600	62½	"		"	
99	15 Feb, 1861	65 miles from Malta	180	59½	60	57	Spratt (u)	S W of Malta.
100.	" "	" "	600	59¼	"		"	
101	15 Feb., 1861	55 miles S W of Malta	300	59	59½		Spratt (u)	
102	" "	" "	900	59	"		"	
103	15 Feb, 1861	150 miles S S W of Malta	120	60	60		Spratt (u)	Between Malta and Tripoli
104.	" "	" "	300	59½	"		"	
104a	11 June, 1860	150 miles E of Malta	60	72½	74	75	Spratt (u)	Between Malta and the Greek archipelago
105	" "	" "	120	69	"		"	
106.	" "	" "	180	63	"		"	
107	" "	" "	300	59¾	"		"	
108	" "	" "	600	58½	"		"	
109	" "	" "	7200	58¾	"		"	
110	17 Nov, 1853	Sea of Marmora	60	55¼	55½	61	Spratt (u)	Near entrance
111	" "	" "	300	54	"	"	"	N of Marmora Island
112	17 Nov, 1853	Sea of Marmora	60	55	56	60	Spratt (u)	10 miles distant from the preceding Soundings in 1320 feet
113.	" "	" "	300	54½	"	"	"	
114.	5 May, 1854	Black Sea, Bourgas Gulf	60	49	52	68	Spratt (u)	On W coast In soundings

* The observations of Admiral Spratt before 1860 were made on mud brought up from the bottom Those in and after 1860 were made with Six's self-registering thermometer.

Table III.—Inland Seas (continued).

Red Sea.

I.	II. Date.	III. North Latitude	IV. Longitude of Greenwich.	V. Sea	VI Depth in feet	VII. Temperature in degrees of Fahr. At depth	Surface.	Air	VIII. Name of observer.	IX. Remarks.
115	March and Apr., 1858	13° 19′	42° 53′ E	Red Sea	570	74°5	85°	°	Pullen (a)	Near Strait of Bab-el-Mandeb
116		15 18	41 43 E	" "	240	77	86		"	Near the Islands of Dhalak.
117	" "	16 59	40 5 E	" "	300	77	86		"	Between Ras Debeer (Nubia) and Ghumfooda, on the Arabian coast.
118	" "	17 49	40 2 E	" "	3342	70 5	80		"	
119	" "	18 3	38 57 E.	" "	1392	70 5	86		"	
120	" "	20 57	37 29 E	" "	1800	71	83 5		"	Off Jeddah
121	" "	22 1	38 16 E	" "	2552	71	78		"	
122	" "	23 30	36 58 E	" "	4068	70 5	77 5		"	Between Berenice and Yembo.
123	" "	27 33	Jubal Strait	" "	2892	70	72		"	Top of Red Sea.

Sea of Okhotsk.

I.	II.	III.	IV.	V.	VI	At depth	Surface.	Air	VIII.	IX.
124	May, 1804	46	144 E	Sea of Okhotsk	360	32	34 6	..	Horner	Near the N coast of Japan.
125	August	53	144 E	" "	480	30	55 8		Horner	Off the north end of Saghalien.
126	August	53	152 E	" "	84	44 6	46 4		Horner	Between the Island of Saghalien and the coast of Kamtschatka
127	"	"	"	" "	96	36 5	"		"	
128	"	"	"	" "	108	31 6	"		"	
129	"	"	"	" "	126	29 3	"		"	
130	"	"	"	" "	180	29	"		"	
131	"	"	"	" "	360	29	"		"	
132	"	"	"	" "	600	29	"		"	
133	"	"	"	" "	690	29	"		"	

Explanation of Map.

PLATE 65.

This Map is reduced, so far as relates to the hydrographical details, from the last edition of the Admiralty 'Chart of the World for Tracks.'

On this the observations recorded in Tables I., II., and III. are laid down according to the latitude or longitude given by the original observers A few corrections have been made in the Tables since the Map was engraved. In the case of these or any other discrepancies*, the Tables give the correct reading. In the Mediterranean only a portion of the numbers (without the initials) are given for want of space.

The numbers in the Map correspond with those in the Tables, and the name of the observer is indicated by initial letters as under. The name of the ship is added for convenience of reference: the fuller particulars and titles will be found in the text, § II, in the order of date.

A.	Armstrong	Voyage of the 'Investigator'	1850–54.
a.	Abel	On the Voyage of the 'Alceste'	1817
B.	Beechey	Voyage of the 'Blossom'	1825–28.
B.	Belcher	Voyage of the 'Sulphur'	1836–46.
,,	,,	Voyage of the 'Samarang'	1843–46.
Bl.	Blosseville	See D'Urville	1828.
Ba.	Bache	United States Coast Survey for 1854	1854.
C.	Cook	Voyage of the 'Resolution' and 'Adventure'	1772–75
c	Chimmo	See note, p. 610	1868.
cr.	Craven	See Bache	1855.
D.†	D'Urville	Voyage de 'L'Astrolabe'	1826–29.
d	Dayman	Voyage of the 'Rattlesnake'	1846–47.
Ds.	Dunsterville	See Maury	
E.	Ellis	On a Voyage to the Coast of Africa	1749.
F.	Franklin & Buchan	Voyage of the 'Dorothea' and 'Trent'	1818.
F.	FitzRoy	Voyage of the 'Adventure' and 'Beagle'	1826–36.
f.	Forster	Voyage of the 'Resolution'	1772–75
G.	Graah	Expedition to the East Coast of Greenland	1828.
H.	Horner	See Krusenstern's Voyage	1803–6.
I.	Irving	See Phipps's Voyage	1773.
K.	Kotzebue	Voyage of the 'Rurick'	1815–18.
,,	,,	Voyage of the 'Predpriatie' (see Lenz)	1823–26

* The whole group of observations to the west and north of Spitzbergen are placed rather too far (from $\frac{1}{2}^\circ$ to 1°) north

† D should have stood for Dayman and U for D'Urville, as it is, U stands for D'Urville in the north hemisphere and D in the south hemisphere.

K.	KELLETT	Voyage of the 'Herald'	1845–51.
k.	KÜNDSON	Voyage of the 'Queen'	1859.
Kr	KRUSENSTERN . . .	Voyage of the 'Neva' and 'Nadeshda' . . .	1803–6.
L.	EMIL. LENZ . . .	With KOTZEBUE on his 2nd Voyage	1823–26.
L.	ED LENZ	On Voyages in the 'Atcha'	1847–49.
M.	MARTINS & BRAVAIS .	Voyage de 'La Recherche'	1838.
Ma.	MAURY	Physical Geography of the Sea . . . . edit.	1857
P.	PARRY	Voyage of the 'Alexander'	1818.
,,	,, ...	Voyage of the 'Hecla' and 'Griper'	1819–20.
,,	,, ...	Voyage of the 'Fury' and 'Hecla'	1821–23.
,,	,, ...	Voyage of the 'Hecla'	1827.
P.	PULLEN . . .	On the Voyage of the 'Cyclops'	1857–59.
	PHIPPS (see IRVING).	Voyage toward the North Pole (the 'Racehorse')	1773.
p.	PÉRON	Voyage sur les Corvettes 'Le Géographe,' 'Le Naturaliste,' et 'Le Casuarina'	1800–4.
pr.	PRATT	On a Voyage to India	1840.
R.	JOHN ROSS . . .	Voyage of the 'Isabella'	1818.
R.	JAMES ROSS . .	Voyage of the 'Discovery' and 'Research' . .	1839–43.
Ro.	RODGERS	See MAURY	1855.
S.	SCORESBY . . .	Various Voyages (the 'Esk' and 'Baffin') . .	1810–22.
S.	SABINE	With Ross in 1818, and PARRY in 1819 . . .	1819.
Sh.	SHORTLAND . . .	On the Voyage of the 'Hydra'	1868.
T	DU PETIT-THOUARS .	Voyage de 'La Vénus'	1836–39.
U.	D'URVILLE . . .	Voyage de 'L'Astrolabe' (see D)	1826–29.
V	VAILLANT	Voyage de 'La Bonite'	1836–39.
W	WULLERSTORF . .	Voyage of the 'Novara'	1857–59.
W.	WAUCHOPE . . .	See notes, pp. 595 & 601	1816 & 1836.
Wi.	WILKES . . .	United States Exploring Expedition (the 'Vincennes' and 'Peacock')	1839–42.
Wa.	WALKER . .	On the Voyage of the 'Fox'	1858.

The other numbers in *italics* mark (in feet) the further depth to which some of the soundings have been carried. Where they have reached the bottom a stop ( ) is added; where, on the contrary, the soundings have not reached the bottom, the sign + is added

The many other voyages for scientific purposes sent out by the English, French, and American governments during the period here described contain many very numerous meteorological observations, but no observation on submarine temperatures, unless I have inadvertently overlooked any.

EXPLANATION OF SECTIONS.

PLATES 66, 67, & 68.

The position of the sections will be found on the Map, and the initials attached to the numbers have the same reference on both.

In the absence of observations in the direct line of section some of those at a short distance on either side are included.

The vertical lines indicate the position and depth of the temperature-soundings, and the figures in italics connected with them give the temperature at the surface and at depths in degrees of FAHRENHEIT. The other figures on the top line mark the degrees of latitude. The stronger figures in italics relate to the probable position of the bathymetrical isotherms generally.

The separate numbers at depths indicate the depth in feet to which soundings have been made in any latitude, the sign + showing that no bottom has been reached

All the observations used in the Sections have been subjected to correction for pressure, as adopted p. 612, viz by making a deduction of 1° FAHR. for every 1700 feet of depth, exclusive of the observations of LENZ, DU PETIT-THOUARS (such of them as are given in parentheses in the Tables), MARTINS, PULLEN (in part), and those of ROSS, PARRY, and SABINE of 1818–19, which are taken, for reasons before given, as recorded by the original observers It is possible that in some instances (as, for example, JAMES ROSS) a larger correction might be necessary, and that in the Antarctic seas the isotherm of 35° F. should be replaced by one of 33° or 32°*; but this will not much affect the correction for the more numerous observations at lesser depths

All the depths are given, for the sake of uniformity, without correction for angle of rope, as that could only possibly be known in but few cases The importance, however, of a correction for this also will be evident by reference to the large allowances which DU PETIT-THOUARS† has often thought it necessary to make in his soundings, the corrected readings being given between parentheses. Only in 21 cases does he record "the angle of the line from the vertical" as 0, in the other 38 cases he found it to vary from 10° to 67°; and he estimated the difference caused by the latter extreme case as equal to a reduction of the observed depth of 5872 feet to a corrected depth of 2296 feet. The want of information on this point is one reason for taking, as we have done, a minimum correction for pressure.

Where the observations are sufficiently numerous the bathymetrical isotherms are laid down in continuous lines. The dotted lines indicate the probable prolongation of the isotherms, on the supposition that there are no disturbing causes, but it must be borne in mind that the isotherms (the lower ones especially) are liable to rise with every

* Should some of the observations of the 'Challenger' be found to correspond in position with any of those recorded in these pages, they will furnish a measure whereby to correct these or those of other observers

† See also the corrected depths of LENZ (*antè*, p. 599) and of WAUCHOPE, 1816, and SABINE, 1822 (Tables)

important irregularity (banks, shoals, &c.) in the bed of the ocean, and the upper isotherms may be variously deflected by surface-drifts and currents.

It is probable that in some of these Sections (as, for example, in the North Pacific, Sect. 4, and in the South Atlantic, Sects. 1 & 2) the irregularities of curvature may be exaggerated, owing to the want of uniformity in the instruments used by the different observers, and by the necessity of using a general correction for all.

Very little was known before 1868 of the deep bed of the Atlantic The few indications of the ocean-bed given in the sections are taken from notices in the several voyages above recorded and from MAURY. In the higher north latitudes we have the soundings of ROSS, KANE, SCORESBY, and MARTINS. In section No. 2 the greater depths of SCORESBY are in the sea west of Spitzbergen, and the lesser ones of MARTINS between Spitzbergen and Norway, which accounts for the break in continuity of depth.

The position of the bathymetrical isotherms and the indications of the sea-bed are confined strictly to observations anterior to 1868.

XXII. *A Memoir on Prepotentials.* *By Professor* CAYLEY, *F.R.S.*

Received April 8,—Read June 10, 1875.

THE present Memoir relates to multiple integrals expressed in terms of the $(s+1)$ ultimately disappearing variables $(x \ldots z, w)$, and the same number of parameters $(a \ldots c, e)$, and being of the form

$$\int \frac{\varrho \, d\varpi}{\{(a-x)^2 \ldots + (c-z)^2 + (e-w)^2\}^{\frac{1}{2}s+q}},$$

where ϱ and $d\varpi$ depend only on the variables $(x \ldots z, w)$. Such an integral, in regard to the index $\frac{1}{2}s+q$, is said to be "prepotential," and in the particular case $q=-\frac{1}{2}$ to be "potential."

I use throughout the language of hyper-tridimensional geometry $(x \ldots z, w)$ and $(a \ldots c, e)$ are regarded as coordinates of points in $(s+1)$dimensional space, the former of them determining the position of an element $\varrho d\varpi$ of attracting matter, the latter being the attracted point; viz. we have a mass of matter $= \int \varrho d\varpi$ distributed in such manner that, $d\varpi$ being the element of $(s+1)$- or lower-dimensional volume at the point $(x \ldots z, w)$, the corresponding density is ϱ, a given function of $(x \ldots z, w)$, and that the element of mass $\varrho d\varpi$ exerts on the attracted point $(a \ldots c, e)$ a force inversely proportional to the $(s+2q+1)$th power of the distance $\{(a-x)^2 \ldots + (c-z)^2 + (e-w)^2\}^{\frac{1}{2}}$ The integration is extended so as to include the whole attracting mass $\int \varrho d\varpi$; and the integral is then said to represent the Prepotential of the mass in regard to the point $(a \ldots c, e)$. In the particular case $s=2$, $q=-\frac{1}{2}$, the force is as the inverse square of the distance, and the integral represents the Potential in the ordinary sense of the word

The element of volume $d\varpi$ is usually either the element of solid (spatial or $(s+1)$-dimensional) volume $dx \ldots dz\, dw$, or else the element of superficial (s-dimensional) volume dS. In particular, when the surface (s-dimensional locus) is the (s-dimensional) plane $w=0$, the superficial element dS is $=dx \ldots dz$. The cases of a less-than-s-dimensional volume are in the present memoir considered only incidentally It is scarcely necessary to remark that the notion of density is dependent on the dimensionality of the element of volume $d\varpi$. in passing from a spatial distribution, $\varrho\, dx \ldots dz\, dw$, to a superficial distribution, $\varrho\, dS$, we alter the signification of ϱ. In fact if, in order to connect the two, we imagine the spatial distribution as made over an indefinitely thin layer or stratum bounded by the surface, so that at any element dS of the surface the normal thickness is $d\nu$, where $d\nu$ is a function of the coordinates $(x \ldots z, w)$ of the element dS, the spatial element is $=d\nu\, dS$, and the element of mass $\varrho\, dx \ldots dz\, dw$ is $=\varrho\, d\nu\, dS$; and

then changing the signification of ϱ, so as to denote by it the product $\varrho\, d\nu$, the expression for the element of mass becomes $\varrho\, d\mathrm{S}$, which is the formula in the case of the superficial distribution.

The space or surface over which the distribution extends may be spoken of as the material space or surface; so that the density ϱ is not $=0$ for any finite portion of the material space or surface, and if the distribution be such that the density becomes $=0$ for any point or locus of the material space or surface, then such point or locus, considered as an infinitesimal portion of space or surface, may be excluded from and regarded as not belonging to the material space or surface. It is allowable, and frequently convenient, to regard ϱ as a discontinuous function, having its proper value within the material space or surface, and having its value $=0$ beyond these limits, and this being so, the integrations may be regarded as extending as far as we please beyond the material space or surface (but so always as to include the whole of the material space or surface)—for instance, in the case of a spatial distribution, over the whole $(s+1)$dimensional space; and in the case of a superficial distribution, over the whole of the s-dimensional surface of which the material surface is a part.

In all cases of surface-integrals it is, unless the contrary is expressly stated, assumed that the attracted point does not lie on the material surface; to make it do so is, in fact, a particular supposition. As to solid integrals, the cases where the attracted point is not, and is, in the material space may be regarded as cases of coordinate generality; or we may regard the latter one as the general case, deducing the former one from it by supposing the density at the attracted point to become $=0$.

The present memoir has chiefly reference to three principal cases, which I call A, C, D, and a special case, B, included both under A and C viz. these are:—

A. The prepotential-plane case; q general, but the surface is here the plane $w=0$, so that the integral is

$$\int \frac{\varrho\, dx \ldots dz}{\{(a-x)^2 \ldots + (c-z)^2 + e^2\}^{\frac{1}{2}s+q}}.$$

B. The potential-plane case, $q=-\tfrac{1}{2}$, and the surface the plane $w=0$, so that the integral is

$$\int \frac{\varrho\, dx \ldots dz}{\{(a-x)^2 \ldots + (c-z)^2 + e^2\}^{\frac{1}{2}s-\frac{1}{2}}}.$$

C. The potential-surface case; $q=-\tfrac{1}{2}$, the surface arbitrary, so that the integral is

$$\int \frac{\varrho\, d\mathrm{S}}{\{(a-x)^2 \ldots + (c-z)^2 + (e-w)^2\}^{\frac{1}{2}s-\frac{1}{2}}}.$$

D. The potential-solid case; $q=-\tfrac{1}{2}$, and the integral is

$$\int \frac{\varrho\, dx \ldots dz\, dw}{(a-x)^2 \ldots + (c-z)^2 + (e-w)^2\}^{\frac{1}{2}s-\frac{1}{2}}}.$$

It is, in fact, only the prepotential-plane case which is connected with the partial differential equation

$$\left(\frac{d^2}{da^2}\cdots+\frac{d^2}{dc^2}+\frac{d^2}{de^2}+\frac{2q+1}{e}\,\frac{d}{de}\right)\mathrm{V}=0,$$

considered in GREEN's memoir 'On the Attractions of Ellipsoids' (1835), and called here "the prepotential equation." For this equation is satisfied by the function

$$\frac{1}{\{a^2\ldots+c^2+e^2\}^{\frac{1}{2}s+q}},$$

and therefore also by

$$\frac{1}{\{(a-x)^2\ldots+(c-z)^2+e^2\}^{\frac{1}{2}s+q}},$$

and consequently by the integral

$$\int\frac{\rho\,dx\ldots dz}{\{(a-x)^2\ldots+(c-z)^2+e^2\}^{\frac{1}{2}s+q}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(A)}$$

that is by the prepotential-plane integral; but the equation is *not* satisfied by the value

$$\frac{1}{\{(a-x)^2\ldots+(c-z)^2+(e-w)^2\}^{\frac{1}{2}s+q}},$$

nor, therefore, by the prepotential-solid, or general superficial, integral.

But if $q=-\frac{1}{2}$, then, instead of the prepotential equation, we have "the potential equation"

$$\left(\frac{d^2}{da^2}\cdots+\frac{d^2}{dc^2}+\frac{d^2}{de^2}\right)\mathrm{V}=0,$$

and this is satisfied by

$$\frac{1}{\{a^2\ldots+c^2+e^2\}^{\frac{1}{2}s-\frac{1}{2}}},$$

and therefore also by

$$\frac{1}{\{(a-x)^2\ldots+(c-z)^2+(e-w)^2\}^{\frac{1}{2}s-\frac{1}{2}}}.$$

Hence it is satisfied by

$$\int\frac{\rho\,dx\ldots dz\,dw}{\{(a-x)^2\ldots+(c-z)^2+(e-w)^2\}^{\frac{1}{2}s-\frac{1}{2}}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(D)}$$

the potential-solid integral, *provided that the point* $(a\ldots c,\,e)$ *does not lie within the material space*: I would rather say that the integral does *not* satisfy the equation, but of this more hereafter; and it is satisfied by

$$\int\frac{\rho\,d\mathrm{S}}{\{(a-x)^2\ldots+(c-z)^2+(e-w)^2\}^{\frac{1}{2}s-\frac{1}{2}}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(C)}$$

the potential-surface integral. The potential-plane integral (B), as a particular case of (C), of course also satisfies the equation.

Each of the four cases give rise to what may be called a distribution-theorem; viz. given V a function of $(a\ldots c,\,e)$ satisfying certain prescribed conditions, but otherwise arbitrary, then the form of the theorem is that there exists and that we can find an expres-

sion for ρ, the density or distribution of matter over the space or surface to which the theorem relates, such that the corresponding integral V has its given value, viz. in A and B there exists such a distribution over the plane $w=0$, in C such a distribution over a given surface, and in D such a distribution in space. The establishment, and exhibition in connexion with each other, of these four distribution-theorems is the principal object of the present memoir, but the memoir contains other investigations which have presented themselves to me in treating the question. It is to be noticed that the theorem A belongs to GREEN, being in fact the fundamental theorem of his memoir of 1835, already referred to. Theorem C, in the particular case of tridimensional space, belongs also to him, being given in his 'Essay on the Application of Mathematical Analysis to the theories of Electricity and Magnetism' (Nottingham, 1828), being partially rediscovered by GAUSS in the year 1840, and theorem D, in the same case of tridimensional space, to LEJEUNE-DIRICHLET: see his memoir "Sur un moyen général de vérifier l'expression du potentiel relatif à une masse quelconque homogène ou hétérogène," *Crelle*, t. xxxii. pp. 80–84 (1840). I refer more particularly to these and other researches by GAUSS, JACOBI, and others in an Annex to the present memoir.

On the Prepotential Surface-integral.—Art. Nos. 1 to 18

1 In what immediately follows we require

$$V=\int\frac{dx\ ..dz}{(x^2..\ +z^2+e^2)^{\frac{1}{2}s+q}},$$

limiting condition $x^2..\ +z^2=R^2$, the prepotential of a uniform (s-coordinal) circular disk*, radius R, in regard to a point $(0\ldots0, e)$ on the axis; and in particular the value is required in the case where the distance e (taken to be always positive) is indefinitely small in regard to the radius R.

Writing $x=r\xi\ldots z=r\zeta$, where the s new variables $\xi\ldots\zeta$ are such that $\xi^2\ldots+\zeta^2=1$, the integral becomes

$$\int\frac{r^{s-1}dr\,dS}{(r^2+e^2)^{\frac{1}{2}s+q}},\quad =\int dS\int_0^R\frac{r^{s-1}dr}{(r^2+e^2)^{\frac{1}{2}s+q}},$$

where dS is the element of surface of the s-dimensional unit-sphere $\xi^2\ldots+\zeta^2=1$; the integral $\int dS$ denotes the entire surface of this sphere, which (see Annex I.) is $=\frac{2(\Gamma\frac{1}{2})^s}{\Gamma\frac{1}{2}s}$. The other factor,

$$\int_0^R\frac{r^{s-1}dr}{(r^2+e^2)^{\frac{1}{2}s+q}},$$

is the r-integral of Annex II.

* It is to be throughout borne in mind that $x\ ..z$ denotes a set of s coordinates, $x\ldots z, w$ a set of $s+1$ coordinates, the adjective coordinal refers to the number of coordinates which enter into the equation; thus, $x^2\ \cdot+z^2+w^2=f^2$ is an $(s+1)$coordinal sphere (observe that the surface of such a sphere is s-dimensional), $x^2..\ +z^2=f^2$, according as we tacitly associate with it the condition $w=0$, or w arbitrary, is an s-coordinal circle, or cylinder, the surface of such circle or cylinder being s-dimensional, but the circumference of the circle $(s-1)$dimensional, or if we attend only to the s-dimensional space constituted by the plane $w=0$, the locus may be considered as an s-coordinal sphere, its surface being $(s-1)$dimensional.

2. We now consider the prepotential-surface integral

$$V=\int\frac{\rho\, dS}{\{(a-x)^2\ldots+(c-z)^2+(e-w)^2\}^{\frac{1}{2}s+q}}.$$

As already mentioned, it is only a particular case of this, the prepotential-plane integral, which is specially discussed; but at present I consider the general case, for the purpose of establishing a theorem in relation thereto. The surface (s-dimensional surface) S is any given surface whatever.

Let the attracted point P be situate indefinitely near to the surface, on the normal thereto at a point N, say the normal distance NP is $=e$*; and let this point N be taken as the centre of an indefinitely small circular (s-dimensional) disk or segment (of the surface), the radius of which R, although indefinitely small, is indefinitely large in comparison with the normal distance e. I proceed to determine the prepotential of the disk; for this purpose, transforming to new axes, the origin being at N and the axes of $x\ldots z$ in the tangent-plane at N, then the coordinates of the attracted point P will be $(0\ldots0,\ e)$, and the expression for the prepotential of the disk will be

$$V=\int\frac{\rho\, dx\ldots dz}{\{x^2\ldots+z^2+\ {}^2\}^{\frac{1}{2}s+q}},$$

where the limits are given by $x^2\ldots+z^2<R^2$.

Suppose for a moment that the density at the point N is $=\rho'$, then the density throughout the disk may be taken $=\rho'$, and the integral becomes

$$V=\rho'\int\frac{dx\ldots dz}{\{x^2\ldots+z^2+e^2\}^{\frac{1}{2}s+q}},$$

where instead of ρ' I write ρ, viz. ρ now denotes the density at the point N. Making this change, then (by what precedes) the value is

$$=\rho\cdot\frac{2(\Gamma\frac{1}{2})^s}{\Gamma(\frac{1}{2}s)}\cdot\int_0^R\frac{r^{s-1}dr}{\{r^2+e^2\}^{\frac{1}{2}s+q}}.$$

$q=$*Positive*.—Nos. 3 to 7.

3. I consider first the case where q is positive. The value is here

$$=\rho\,\frac{2(\Gamma\frac{1}{2})^s}{\Gamma(\frac{1}{2}s)}\,\frac{1}{2e^{2q}}\left\{\frac{\Gamma\frac{1}{2}s\,\Gamma q}{\Gamma(\frac{1}{2}s+q)}-\int_0^{\frac{e^2}{R^2}}\frac{x^{q-1}dx}{(1+x)^{\frac{1}{2}s+q}}\right\};$$

or since $\frac{e}{R}$ is indefinitely small, the x-integral may be neglected, and the value is

$$=\frac{1}{e^{2q}}\,\rho\,\frac{(\Gamma\frac{1}{2})^s\,\Gamma q}{\Gamma(\frac{1}{2}s+q)}.$$

Observe that this value is independent of R, and that the expression is thus the same as if (instead of the disk) we had taken the whole of the infinite tangent-plane, the

* e is positive; in afterwards writing $e=0$, we mean by 0 the limit of an indefinitely small positive quantity

density at every point thereof being $=\rho$. It is proper to remark that the neglected terms are of the orders

$$\frac{1}{s^{2q}}\left\{\left(\frac{s}{\mathrm{R}}\right)^{2q},\ \left(\frac{s}{\mathrm{R}}\right)^{2q+2},\ \&c.\right\};$$

so that the complete value multiplied by s^{2q} is equal to the constant $\rho\frac{(\Gamma\frac{1}{2})^s\Gamma q}{\Gamma(\frac{1}{2}s+q)}$ + terms of the orders $\left(\frac{s}{\mathrm{R}}\right)^{2q}$, $\left(\frac{s}{\mathrm{R}}\right)^{2q+2}$, &c.

4. Let us now consider the prepotential of the remaining portion of the surface; every part thereof is at a distance from P exceeding, in fact far exceeding, R; so that imagining the whole mass $\int\rho\, d\mathrm{S}$ to be collected at the distance R, the prepotential of the remaining portion of the surface is less than

$$\frac{\int\rho\, d\mathrm{S}}{\mathrm{R}^{s+2q}};$$

viz. we have thus, in the case where the mass $\int\rho\, d\mathrm{S}$ is finite, a superior limit to the prepotential of the remaining portion of the surface. This will be indefinitely small in comparison with the prepotential of the disk, provided only s^{2q} is indefinitely small compared with R^{s+2q}, that is s indefinitely small in comparison with $\mathrm{R}^{1+\frac{s}{2q}}$. The proof assumes that the mass $\int\rho\, d\mathrm{S}$ is finite; but considering the very rough manner in which the limit $\frac{\int\rho\, d\mathrm{S}}{\mathrm{R}^{s+2q}}$ was obtained, it can scarcely be doubted that, if not universally, at least for very general laws of distribution, even when $\int\rho\, d\mathrm{S}$ is infinite, the same thing is true; viz. that by taking s sufficiently small in regard to R, we can make the prepotential of the remaining portion of the surface vanish in comparison with that of the disk. But without entering into the question I assume that the prepotential of the remaining portion does thus vanish, the prepotential of the whole surface in regard to the indefinitely near point P is thus equal to the prepotential of the disk; viz. its value is

$$=\frac{1}{s^{2q}}\rho\frac{(\Gamma\frac{1}{2})^s\Gamma q}{\Gamma(\frac{1}{2}s+q)},$$

which, observe, is infinite for a point P on the surface.

5. Considering the prepotential V of an arbitrary point $(a\ldots c, e)$ as a given function of $(a\ldots c, e)$ the coordinates of this point, and taking $(x\ldots z, w)$ for the coordinates of the point N, which is, in fact, an arbitrary point on the surface, then the value of V at the point P indefinitely near to N will be $=$W, if W denote the same function of $(x\ldots z, w)$ that V is of $(a\ldots c, e)$. The result just obtained is therefore

$$\mathrm{W}=\frac{1}{s^{2q}}\rho\frac{(\Gamma\frac{1}{2})^s\Gamma q}{\Gamma(\frac{1}{2}s+q)},\ (s=0),$$

or, what is the same thing,

$$\rho=\frac{\Gamma(\frac{1}{2}s+q)}{(\Gamma\frac{1}{2})^s\Gamma q}\left(s^{2q}\mathrm{W}\right)_{s=0}.$$

As to this, remark that V is not an arbitrary function of $(a \ldots c, e)$: *non constat* that there is any distribution of matter, and still less that there is any distribution of matter on the surface, which will produce at the point $(a \ldots c, e)$, that is at every point whatever, a prepotential the value of which shall be a function assumed at pleasure of the coordinates $(a \ldots c, e)$. But suppose that V, the given function of $(a \ldots c, e)$, is such that there does exist a corresponding distribution of matter on the surface (viz. that V satisfies the conditions, whatever they are, required in order that this may be the case), then the foregoing formula determines the distribution, viz. it gives the expression of ρ, that is, the density at any point of the surface.

6. The theorem may be presented in a somewhat different form; regarding the prepotential as a function of the normal distance u, its derived function in regard to u is

$$=-\frac{2q}{u^{2q+1}}\rho\frac{(\Gamma\tfrac{1}{2})^s\Gamma q}{\Gamma(\tfrac{1}{2}s+q)}, \text{ that is}$$

$$=-\frac{1}{u^{2q+1}}\rho\frac{2(\Gamma\tfrac{1}{2})^s\Gamma(q+1)}{\Gamma(\tfrac{1}{2}s+q)};$$

and we thus have

$$\frac{dW}{du}=-\frac{1}{u^{2q+1}}\rho\frac{2(\Gamma\tfrac{1}{2})^s\Gamma(q+1)}{\Gamma(\tfrac{1}{2}s+q)}, \quad (u=0),$$

or, what is the same thing,

$$\rho=-\frac{\Gamma(\tfrac{1}{2}s+q)}{2(\Gamma\tfrac{1}{2})^s\Gamma(q+1)}\left(u^{2q+1}\frac{dW}{du}\right)_{u=0},$$

where, however, W being given as a function of $(x \ldots z, w)$, the notation $\frac{dW}{du}$ requires explanation. Taking $\cos\alpha \ldots \cos\gamma$ to be the inclinations of the normal at N, in the direction NP in which the distance u is measured, to the positive parts of the axes of $(x \ldots z)$, viz. these cosines denote the values of

$$\frac{dS}{dx} \cdots \frac{dS}{dz},$$

each taken with the same sign $+$ or $-$, and divided by the square root of the sum of the squares of the last-mentioned quantities, then the meaning is

$$\frac{dW}{du}=\frac{dW}{dx}\cos\alpha \ldots +\frac{dW}{dz}\cos\gamma.$$

7. The surface S may be the plane $w=0$, viz. we have then the prepotential-plane integral

$$V=\int\frac{\rho\,dx \ldots dz}{\{(a-x)^2 \ldots +(c-z)^2+e^2\}^{\frac{1}{2}s+q}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (A)$$

where e (like u) is positive. In afterwards writing $e=0$, we mean by 0 the limit of an indefinitely small positive quantity.

The foregoing distribution-formulæ then become

$$\rho=\frac{\Gamma(\tfrac{1}{2}s+q)}{(\Gamma\tfrac{1}{2})^s\Gamma q}\left(e^{2q}W\right)_{e=0}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (A)$$

and

$$\rho = -\frac{\Gamma(\tfrac{1}{2}s+q)}{2(\Gamma\tfrac{1}{2})^s\Gamma(q+1)}\left(e^{2q+1}\frac{dW}{de}\right)_{e=0}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (A^*)$$

which will be used in the sequel.

It will be remembered that in the preceding investigation it has been assumed that q is positive, the limiting case $q=0$ being excluded†.

$q=-\frac{1}{2}$.—Nos. 8 to 13.

8. I pass to the case $q=-\frac{1}{2}$, viz. we here have the potential-surface integral

$$V=\int\frac{\rho\, dS}{\{(a-x)^2 \ldots + (c-z)^2+(e-w)^2\}^{\frac{1}{2}s-\frac{1}{2}}}: \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (C)$$

it will be seen that the results present themselves under a remarkably different form.

The potential of the disk is, as before,

$$\rho\,\frac{2(\Gamma\tfrac{1}{2})^s}{\Gamma\tfrac{1}{2}s}\int\frac{r^{s-1}\,dr}{(r^2+\varepsilon^2)^{\frac{1}{2}s-\frac{1}{2}}},$$

where ρ here denotes the density at the point N; and the value of the r-integral

$$=R\left(1+\text{terms in } \frac{\varepsilon^2}{R^2}, \frac{\varepsilon^4}{R^4}, \ldots\right)-\varepsilon\,.\,\frac{\Gamma\tfrac{1}{2}s\,\Gamma\tfrac{1}{2}}{\Gamma(\tfrac{1}{2}s-\tfrac{1}{2})}.$$

Observe that this is indefinitely small, and remains so for a point P on the surface; the potential of the remaining portion of the surface (for a point P near to or on the surface) is finite, that is, neither indefinitely large nor indefinitely small, and it varies continuously as the attracted point passes through the disk (or aperture in the material surface now under consideration); hence the potential of the whole surface is finite for an attracted point P on the surface, and it varies continuously as P passes through the surface.

It will be noticed that there is in this case a term in V independent of ε; and it is on this account necessary, instead of the potential, to consider its derived function in regard to ε, viz. neglecting the indefinitely small terms which contain powers of $\frac{\varepsilon}{R}$, I write

$$\frac{dV}{d\varepsilon}=-\frac{2(\Gamma\tfrac{1}{2})^{s+1}}{\Gamma(\tfrac{1}{2}s-\tfrac{1}{2})}\,\rho.$$

The corresponding term arising from the potential of the other portion of the surface, viz the derived function of the potential in regard to ε, is not indefinitely small; and calling it Q, the formula for the whole surface becomes

$$\frac{dV}{d\varepsilon}=Q-\frac{2(\Gamma\tfrac{1}{2})^{s+1}}{\Gamma(\tfrac{1}{2}s-\tfrac{1}{2})}\,\rho.$$

† This is, as regards q, the case throughout; a limiting value, if not expressly stated to be included, is always excluded.

9. I consider positions of the point P on the two opposite sides of the point N, say at the normal distances ν', ν'', these being positive distances measured in opposite directions from the point N. The function V, which represents the potential of the surface in regard to the point P, is or may be a different function of the coordinates $(a\ldots c, e)$ of the point P, according as the point is situate on the one side or the other of the surface (as to this more presently) I represent it in the one case by V', and in the other case by V''; and in further explanation state that ν' is measured *into* the space to which V' refers, ν'' *into* that to which V'' refers, and I say that the formulæ belonging to the two positions of the point P are

$$\frac{d\mathrm{W}'}{d\nu'}=\mathrm{Q}'-\frac{2(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s-\frac{1}{2})}\rho,$$

$$\frac{d\mathrm{W}''}{d\nu''}=\mathrm{Q}''-\frac{2(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s-\frac{1}{2})}\rho,$$

where, instead of V', V'', I have written W', W'' to denote that the coordinates, as well of P' as of P'', are taken to be the values $(x\ldots z, w)$ which belong to the point N. The symbols denote

$$\frac{d\mathrm{W}'}{d\nu'}=\frac{d\mathrm{W}'}{dx}\cos\alpha'\ldots+\frac{d\mathrm{W}''}{dz}\cos\gamma',$$

$$\frac{d\mathrm{W}''}{d\nu''}=\frac{d\mathrm{W}''}{dx}\cos\alpha''\ldots+\frac{d\mathrm{W}''}{dz}\cos\gamma'',$$

where $(\cos\alpha'\ldots\cos\gamma')$ and $(\cos\alpha''\ldots\cos\gamma'')$ are the cosine inclinations of the normal distances ν', ν'' to the positive parts of the axes of $(x\ldots z)$; since these distances are measured in opposite directions, we have $\cos\alpha''=-\cos\alpha'\ldots\cos\gamma''=-\cos\gamma'$. If we imagine a curve through N cutting the surface at right angles, or, what is the same thing, an element of the curve coinciding in direction with the normal element P'NP'', and if s denote the distance of N from a fixed point of the curve, and for the point P' s becomes $s+\delta' s$, while for the point P'' it becomes $s-\delta'' s$, or, what is the same thing, if s increase in the direction of NP' and decrease in that of NP'', then if any function Θ of the coordinates $(x\ldots z, w)$ of N be regarded as a function of s, we have

$$\frac{d\Theta}{ds}=\frac{d\Theta}{d\nu'},\quad \frac{d\Theta}{ds}=-\frac{d\Theta}{d\nu''}.$$

10. In particular, let Θ denote the potential of the remaining portion of the surface, that is, of the whole surface exclusive of the disk; the curve last spoken of is a curve which does not pass through the material surface, viz. the portion to which Θ has reference, and there is no discontinuity in the value of Θ as we pass along this curve through the point N. We have $\mathrm{Q}'=$value of $\frac{d\Theta}{d\nu'}$ at the point P', and $\mathrm{Q}''=$value of $\frac{d\Theta}{d\nu''}$ at the point P''; and the two points P', P'' coming to coincide together at the point

N, we have then

$$Q'=\frac{d\Theta}{ds'},\quad =\frac{d\Theta}{ds},$$

$$Q''=\frac{d\Theta}{ds''},\quad =-\frac{d\Theta}{ds}.$$

We have in like manner $\frac{dW'}{ds'}=\frac{dW'}{ds}$, $\frac{dW''}{ds''}=-\frac{dW''}{ds}$; and the equations obtained above may be written

$$\frac{dW'}{ds}=\frac{d\Theta}{ds}-\frac{2(\Gamma\tfrac{1}{2})^{s+1}}{\Gamma(\tfrac{1}{2}s-\tfrac{1}{2})}\rho,$$

$$\frac{dW''}{ds}=\frac{d\Theta}{ds}+\frac{2(\Gamma\tfrac{1}{2})^{s+1}}{\Gamma(\tfrac{1}{2}s-\tfrac{1}{2})}\rho,$$

in which form they show that as the attracted point passes through the surface from the position P′ on the one side to P″ on the other, there is an abrupt change in the value of $\frac{dW}{ds}$, or say of $\frac{dV}{ds}$, the first derived function of the potential in regard to the orthotomic arc s, that is in the rate of increase of V in the passage of the attracted point normally to the surface. It is obvious that if the attracted point traverses the surface obliquely instead of normally, viz. if the arc s cuts the surface obliquely, there is the like abrupt change in the value of $\frac{dV}{ds}$

Reverting to the original form of the two equations, and attending to the relation $Q'+Q''=0$, we obtain

$$\frac{dW'}{ds'}+\frac{dW''}{ds''}=\frac{-4(\Gamma\tfrac{1}{2})^{s+1}}{\Gamma(\tfrac{1}{2}s-\tfrac{1}{2})}\rho,$$

or, what is the same thing,

$$\rho=-\frac{\Gamma(\tfrac{1}{2}s-\tfrac{1}{2})}{4(\Gamma\tfrac{1}{2})^{s+1}}\left(\frac{dW'}{ds'}+\frac{dW''}{ds''}\right) \quad \text{(C)}$$

11. I recall the signification of the symbols:—V′, V″ are the potentials, it may be different functions of the coordinates $(a\,.\,.\,c, e)$ of the attracted point, for positions of this point on the two sides of the surface (as to this more presently), and W′, W″ are what V′, V″ respectively become when the coordinates $(a\ldots c, e)$ are replaced by $(x\ldots z, w)$, the coordinates of a point N on the surface. The explanation of the symbols $\frac{dW'}{ds'}$, $\frac{dW''}{ds''}$ is given a little above; ρ denotes the density at the point $(x\ldots z, w)$.

12. The like remarks arise as with regard to the former distribution theorem (A); the functions V′, V″ cannot be assumed at pleasure; *non constat* that there is any distribution in space, and still less any distribution on the surface, which would give such values to the potential of a point $(a\,.\,.\,c, e)$ on the two sides of the surface respectively; but assuming that the functions V′, V″ are such that they do arise from a distribution on the surface, or say that they satisfy all the conditions, whatever they are, required in

order that this may be so, then the formula determines the distribution, viz. it gives the value of ρ, the density at a point $(x, \ldots z, w)$ of the surface.

13. In the case where the surface is the plane $w=0$, viz. in the case of the potential-plane integral,

$$V=\int\frac{\rho\, dx \ldots dz}{\{(a-x)^2 \ldots +(c-z)^2+e^2\}^{\frac{1}{2}s-\frac{1}{2}}} \quad . \quad . \quad . \quad . \quad . \quad (B)$$

(e assumed to be positive); then, since every thing is symmetrical on the two sides of the plane, V' and V'' are the same functions of $(a \ldots c, e)$, say they are each$=V$; W', W'' are each of them the same function, say they are each $=W$, of $(x \ldots z, e)$ that V is of $(a \ldots c, e)$, and the distribution-formula becomes

$$\rho=-\frac{\Gamma(\frac{1}{2}s-\frac{1}{2})}{2(\Gamma\frac{1}{2})^{s+1}}\left(\frac{dW}{de}\right)_{e=0}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (B)$$

viz. this is also what one of the prepotential-plane formulæ becomes on writing therein $q=-\frac{1}{2}$.

$q=0$, *or Negative.*—Nos. 14 to 18.

14. Consider the case $q=0$. The prepotential of the disk is

$$\rho \cdot \frac{2(\Gamma\frac{1}{2})^s}{\Gamma\frac{1}{2}s}(\log R+N-\log \epsilon \ldots);$$

and to get rid of the constant term we must consider the derived function in regard to ϵ, viz. this is

$$=-\rho\frac{2(\Gamma\frac{1}{2})^s}{\Gamma\frac{1}{2}s}\;\frac{1}{\epsilon},$$

and we have thus for the whole surface

$$\frac{dV}{d\epsilon}=Q-\rho\frac{2(\Gamma\frac{1}{2})^s}{\Gamma\frac{1}{2}s}\frac{1}{\epsilon},$$

where Q, which relates to the remaining portion of the surface, is finite, we have thence, writing, as before, W in place of V,

$$\epsilon\frac{dW}{d\epsilon}=-\rho\frac{2(\Gamma\frac{1}{2})^s}{\Gamma\frac{1}{2}s},$$

or say

$$\rho=-\frac{\Gamma\frac{1}{2}s}{2(\Gamma\frac{1}{2})^s}\left(\epsilon\frac{dW}{d\epsilon}\right)_{\epsilon=0}.$$

15. Consider the case q negative, but $-q<\frac{1}{2}$. The prepotential of the disk is here

$$=\rho\frac{2(\Gamma\frac{1}{2})^s}{\Gamma\frac{1}{2}s}\rho\left\{\frac{R^{-2q}}{-2q}+\tfrac{1}{2}\epsilon^{-2q}\frac{\Gamma\frac{1}{2}s\,\Gamma q}{\Gamma(\frac{1}{2}s+q)}+\ldots\right\},$$

and to get rid of the first term we must consider the derived function in regard to ϵ, viz. this is

$$-\epsilon^{-2q-1}\rho\frac{2(\Gamma\frac{1}{2})^s\,\Gamma(q+1)}{\Gamma(\frac{1}{2}s+q)};$$

whence for the potential of the whole surface

$$\frac{dV}{du}=Q-u^{-2q-1}\,\varrho\,\frac{2(\Gamma\frac{1}{2})^s\,\Gamma(q+1)}{\Gamma(\frac{1}{2}s+q)},$$

where Q, the part relating to the remaining portion of the surface, is finite. Multiplying by u^{2q+1} (where the index $2q+1$ is positive), the term in Q disappears; and writing, as before, W in place of V, this is

$$u^{2q+1}\frac{dW}{du}=-\varrho\,\frac{2(\Gamma\frac{1}{2})^s\,\Gamma(q+1)}{\Gamma\frac{1}{2}s+q},$$

or say

$$\varrho=-\frac{\Gamma(\frac{1}{2}s+q)}{2(\Gamma\frac{1}{2})^s\,\Gamma(q+1)}\left(u^{2q+1}\frac{dW}{du}\right)_{u=0};$$

viz. we thus see that the formula (A*) originally obtained for the case q positive extends to the case $q=0$, and $q=-$, but $-q<\frac{1}{2}$; but, as already seen, it does not extend to the limiting case $q=\ \frac{1}{2}$.

16. If q be negative and between $-\frac{1}{2}$ and -1, we have in like manner a formula

$$\frac{dV}{du}=Q-\varrho\,\frac{2(\Gamma\frac{1}{2})^s\,\Gamma(q+1)}{\Gamma(\frac{1}{2}s+q)}\,u^{-2q-1};$$

but here $2q+1$ being negative, the term $u^{2q-1}Q$ does not disappear: the formula has to be treated in the same way as for $q=-\frac{1}{2}$, and we arrive at

$$\left\{u'^{2q+1}\frac{dW'}{du'}+u''^{2q+1}\frac{dW''}{du''}\right\}=-\frac{4(\Gamma\frac{1}{2})^s\,\Gamma(q+1)}{\Gamma(\frac{1}{2}s+q),}\,\varrho;$$

viz. the formula is of the same form as for the potential case $q=-\frac{1}{2}$ Observe that the formula does not hold good in the limiting case $q=-1$.

17. We have, in fact, here the potential of the disk

$$=\frac{2(\Gamma\frac{1}{2})^s}{\Gamma(\frac{1}{2}s)}\,\varrho\left\{\frac{R^2}{2}-u^2\log u\,\frac{\Gamma\frac{1}{2}s}{\Gamma(\frac{1}{2}s-1)}\right\};$$

whence

$$\frac{dV}{du}=Q-\frac{2(\Gamma\frac{1}{2})^s}{\Gamma(\frac{1}{2}s-1)}\,\varrho(2u\log u),$$

since in the complete differential coefficient $u+2u\log u$ the term u vanishes in comparison with $2u\log u$; and then, proceeding as before, we find

$$\frac{1}{u'\log u'}\frac{dW'}{du'}+\frac{1}{u''\log u''}\frac{dW''}{du''}=\frac{-8(\Gamma\frac{1}{2})^s}{\Gamma(\frac{1}{2}s-1)}\,\varrho;$$

but I have not particularly examined this formula.

18. If q be negative and >-1 (that is, $-q>1$), then the prepotential for the disk is

$$=\varrho\,\frac{(\Gamma\frac{1}{2})^s}{\Gamma\frac{1}{2}s}\left\{\frac{R^{-2q}}{-2q}+\frac{\frac{1}{2}s+q}{1}\,\frac{R^{-2q-2}}{-2q-2}\cdot u^2\ldots+Ku^{-2q}\right\};$$

and it would seem that in order to obtain a result it would be necessary to proceed to a derived function higher than the first; but I have not examined the case.

Continuity of the Prepotential-surface Integral.—Art. Nos. 19 to 25.

19. I again consider the prepotential-surface integral

$$\int \frac{\rho dS}{\{(a-x)^2 \;..+(c-z)^2+(e-w)^2\}^{\frac{1}{2}s-q}}$$

in regard to a point $(a \ldots c, e)$ not on the surface; q is either positive or negative, as afterwards mentioned.

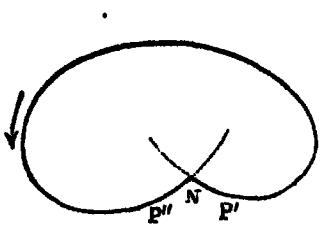
The integral or prepotential and all its derived functions, first, second, &c. *ad infinitum*, in regard to each or all or any of the coordinates $(a \ldots c, e)$ are all finite. This is certainly the case when the mass $\int \rho dS$ is finite, and possibly in other cases also; but to fix the ideas we may assume that the mass is finite. And the prepotential and its derived functions vary continuously with the position of the attracted point $(a \ldots c, e)$, so long as this point in its course does not traverse the material surface. For greater clearness we may consider the point as moving along a continuous curve (one-dimensional locus), which curve, or the part of it under consideration. does not meet the surface; and the meaning is that the prepotential and each of its derived functions varies continuously as the point $(a \ldots c, e)$ passes continuously along the curve

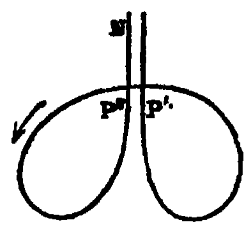
20. Consider a "region," that is, a portion of space any point of which can be by a continuous curve not meeting the material surface connected with any other point of the region. It is a legitimate inference, from what just precedes, that the prepotential is, for any point $(a \ldots c, e)$ whatever within the region, one and the same function of the coordinates $(a \ldots c, e)$, viz the theorem, rightly understood, is true, but the theorem gives rise to a difficulty, and needs explanation.

Consider, for instance, a closed surface made up of two segments, the attracting matter being distributed in any manner over the whole surface (as a particular case $s+1=3$, a uniform spherical shell made up of two hemispheres), then, as regards the first segment (now taken as the material surface), there is no division into regions, but the whole of the $(s+1)$dimensional space is one region, wherefore the prepotential of the first segment is one and the same function of the coordinates $(a .. c, e)$ of the attracted point for any position whatever of this point But in like manner the prepotential of the second segment is one and the same function of the coordinates $(a \ldots c, e)$ for any position whatever of the attracted point And the prepotential of the whole surface, being the sum of the prepotentials of the two segments, is consequently one and the same function of the coordinates $(a \ldots c, e)$ of the attracted point for any position whatever of this point; viz. it is the same function for a point in the region inside the closed surface and for a point in the outside region. That this is not in general the case we know from the particular case, $s+1=3$, of a uniform spherical shell referred to above.

21. Consider in general an unclosed surface or segment, with matter distributed over it in any manner; and imagine a closed curve or circuit cutting the segment once, and let the attracted point $(a \ldots c, e)$ move continuously along the circuit. We may consider the circuit as corresponding to (in ordinary tridimensional space) a plane curve of

equal periphery, the corresponding points on the circuit and the plane curve being points at equal distances s along the curves from fixed points on the two curves respectively; and then treating the plane curve as the base of a cylinder, we may represent the potential as a length or ordinate, $V=y$, measured upwards from the point on the plane curve along the generating line of the cylinder, in such wise that the upper extremity of the length or ordinate y traces out on the cylinder a curve, say the prepotential curve, which represents the march of the prepotential. The attracted point may, for greater convenience, be represented as a point *on* the prepotential curve, viz. by the upper instead of the lower extremity of the length or ordinate y; and the ordinate, or height of this point above the base of the cylinder, then represents the value of the prepotential The before-mentioned continuity-theorem is that the prepotential curve corresponding to any portion (of the circuit) which does not meet the material surface is a continuous curve, viz. that there is no abrupt change of value either in the ordinate $y(=V)$ of the prepotential curve, or in the first or any other of the derived functions $\frac{dy}{ds}, \frac{d^2y}{ds^2}$, &c. We have thus (in each of the two figures) a continuous curve as we pass



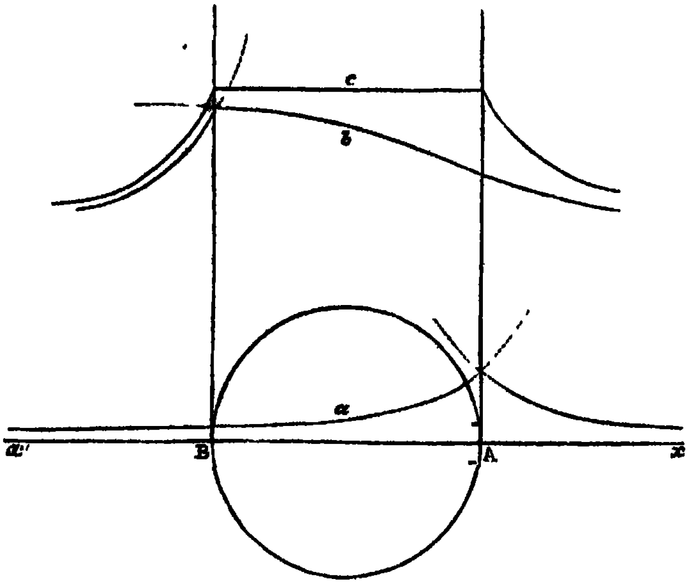


(in the direction of the arrow) from a point P′ on one side of the segment to a point P″ on the other side of the segment, but this continuity does not exist in regard to the remaining part, from P″ to P′, of the prepotential curve corresponding to the portion (of the circuit) which traverses the material surface.

22. I consider first the case $q=-\frac{1}{2}$ (see the left-hand figure)· the prepotential is here a potential. At the point N, which corresponds to the passage through the material surface, then, as was seen, the ordinate y (=the Potential V) remains finite and continuous; but there is an abrupt change in the value of $\frac{dy}{ds}$, that is, in the direction of the curve: the point N is really a node with two branches crossing at this point, as shown in the figure, but the dotted continuations have only an analytical existence, and do not represent values of the potential. And by means of this branch-to-branch discontinuity at the point N, we escape from the foregoing conclusion as to the continuity of the potential on the passage of the attracted point through a closed surface.

23. To show how this is I will for greater clearness examine the case $(s+1)=3$, in ordinary tridimensional space, of the uniform spherical shell attracting according to the inverse square of the distance; instead of dividing the shell into hemispheres, I divide it by a plane into any two segments (see the figure, wherein A, B represent the

centres of the two segments respectively, and where for graphical convenience the segment A is taken to be small.



We may consider the attracted point as moving along the axis xx', viz. the two extremities may be regarded as meeting at infinity, or we may outside the sphere bend the line round, so as to produce a closed circuit. We are only concerned with what happens at the intersections with the spherical surface. The ordinates represent the potentials, viz. the curves are a, b, c for the segments A, B, and the whole spherical surface respectively. Practically, we construct the curves c, a, and deduce the curve b by taking for its ordinate the difference of the other two ordinates. The curve c is, as we know, a discontinuous curve, composed of a horizontal line and two hyperbolic branches; the curve a can be laid down approximately by treating the segment A as a plane circular disk; it is of the form shown in the figure, having a node at the point corresponding to A. [In the case where the segment A is actually a plane disk, the curve is made up of portions of branches of two hyperbolas; but taking the segment A as being what it is, the segment of a spherical surface, the curve is a single curve, having a node as mentioned above.] And from the curves c and a, deducing the curve b, we see that this is a curve without any discontinuity corresponding to the passage of the attracted point through A (but with an abrupt change of direction or node corresponding to the passage through B). And conversely, using the curves a, b to determine the curve c, we see how, on the passage of the attracted point at A into the interior of the sphere, in consequence of the branch-to-branch discontinuity of the curve a, the curve c, obtained by combination of the two curves, undergoes a change of law, passing abruptly from a hyperbolic to a rectilinear form, and how similarly on the passage of the attracted point at B from the interior to the exterior of the sphere, in consequence of the branch-to-branch discontinuity of the curve b, the curve c again undergoes a change of law, abruptly reverting to the hyperbolic form.

24. In the case q positive the prepotential curve is as shown by the right-hand figure in p. 688, viz. the ordinate is here infinite at the point N corresponding to the passage through the surface, the value of the derived function changes between + infinity and − infinity, and there is thus a discontinuity of value in the derived function. It would seem that when q is fractional this occasions a change of law on passage through the surface, but that there is no change of law when q is integral.

In illustration, consider the closed surface as made up of an infinitesimal circular disk, as before, and of a residual portion, the potential of the disk on an indefinitely near point is found as before, and the prepotential of the whole surface is

$$=\frac{1}{s^{2q}}\,\varrho\,\frac{(\Gamma\frac{1}{2})^s\Gamma q}{\Gamma(\frac{1}{2}s+q)}+\mathrm{V}_1,$$

where V_1, the prepotential of the remaining portion of the surface, is a function which varies (and its derived functions vary) continuously as the attracted point traverses the disk. To fix the ideas we may take the origin at the centre of the disk, and the axis of e as coinciding with the normal, so that s, which is always positive, is $=\pm e$, and the expression for the prepotential at a point $(a\ldots c,\, e)$ on the normal through the centre of the disk is

$$=\frac{1}{(\pm e)^{2q}}\cdot\varrho\,\frac{(\Gamma\frac{1}{2})^s.\,\Gamma q}{\Gamma(\frac{1}{2}s+q)}+\mathrm{V}_1,$$

viz. when q is fractional there is the discontinuity of law, inasmuch as the term changes from $\frac{1}{(+e)^{2q}}$ to $\frac{1}{(-e)^{2q}}$; but when q is integral this discontinuity disappears. The like considerations, using of course the proper formula for the attraction of the disk, would apply to the case $q=0$ or negative.

25 Or again, we might use the formulæ which belong to the case of a uniform $(s+1)$-coordinal spherical shell (see Annex No. III.), viz. we decompose the surface as follows,

surface = disk + residue of surface;

and then, considering a spherical shell touching the surface at the point in question (so that the disk is in fact an element common to the surface and the spherical shell), and being of a uniform density equal to that of the disk, we have

disk = spherical shell − residue of spherical shell,

and consequently

surface = spherical shell − residue of spherical shell + residue of surface,

and then, considering the attracted point as passing through the disk, it does not pass through either of the two residues, and there is not any discontinuity, as regards the prepotentials of these residues respectively; there is consequently, as regards the prepotential of the surface, the same discontinuity that there is as regards the prepotential of the spherical shell. But I do not further consider the question from this point of view.

The Potential Solid Integral.—Art. No. 26.

26. We have further to consider the prepotential (and in particular the potential) of a material space; to fix the ideas, consider for the moment the case of a distribution over the space included within a closed surface, the exterior density being zero, and the interior density being, suppose for the moment, constant; we consider the discontinuity which takes place as the attracting point passes from the exterior space through the bounding surface into the interior material space. We may imagine the interior space divided into indefinitely thin shells by a series of closed surfaces similar, if we please, to the bounding surface, and we may conceive the matter included between any two consecutive surfaces as concentrated on the exterior of the two surfaces, so as to give rise to a series of consecutive material surfaces; the quantity of such matter is infinitesimal, and the density of each of the material surfaces is therefore also infinitesimal As the attracted point comes from the external space to pass through the first of the material surfaces—suppose, to fix the ideas, it moves continuously along a curve the arc of which measured from a fixed point is $=s$—there is in the value of V (or, as the case may be, in the values of its derived functions $\frac{dV}{ds}$, &c) the discontinuity due to the passage through the material surface; and the like as the attracted point passes through the different material surfaces respectively. Take the case of a potential, $q=-\frac{1}{2}$, then, if the surface-density were finite, there would be no finite change in the value of V, but there would be a finite change in the value of $\frac{dV}{ds}$, as it is, the changes are to be multiplied by the infinitesimal density, say ρ, of the material surface, there is consequently no finite change in the value of the first derived function, but there is, or may be, a finite change in the value of $\frac{d^2V}{ds^2}$ and the higher derived functions But there is in V an infinitesimal change corresponding to the passage through the successive material surfaces respectively; that is, as the attracted point enters into the material space there is a change in the law of V considered as a function of the coordinates $(a \ldots c, e)$ of the attracted point, but by what precedes this change of law takes place without any abrupt change of value either of V or of its first derived function, which derived function may be considered as representing the derived function in regard to any one of the coordinates $a \ldots c, e$ The suppositions that the density outside the bounding surface was zero and inside it constant, were made for simplicity only, and were not essential, it is enough if the density, changing abruptly at the bounding surface, varies continuously in the material space within the bounding surface*. The

* It is, indeed, enough if the density varies continuously within the bounding surface in the neighbourhood of the point of passage through the surface, but the condition may without loss of generality be stated as in the text, it being understood that for each abrupt change of density within the bounding surface we must consider the attracted point as passing through a new bounding surface, and have regard to the resulting discontinuity.

conclusion is that V′, V″ being the values at points within and without the bounding surface, V′ and V″ are in general different functions of the coordinates $(a \ldots c, e)$ of the attracting point; but that *at* the surface we have not only $V'=V''$, but that the first derived functions are also equal, viz. that we have

$$\frac{dV'}{da}=\frac{dV''}{da}, \ldots \frac{dV'}{dc}=\frac{dV''}{dc}, \; \frac{dV'}{de}=\frac{dV''}{de}$$

27. In the general case of a Potential,

$$V=\int\frac{\rho\, dx \ldots dz\, dw}{\{(a-x)^2 \ldots +(c-z)^2+(e-w)^2\}^{\frac{1}{2}s-\frac{1}{2}}};$$

if ρ does not vanish at the attracted point $(a \ldots c, e)$, but has there a value ρ' different from zero, we may consider the attracting $(s+1)$dimensional mass as made up of an indefinitely small sphere, radius ϵ and density ρ', which includes within it the attracted point, and of a remaining portion external to the attracted point. Writing ∇ to denote $\frac{d^2}{da^2} \ldots +\frac{d^2}{dc^2}+\frac{d^2}{de^2}$, then, as regards the potential of the sphere, we have $\nabla V=-\frac{4(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s-\frac{1}{2})}\rho'$ (see Annex III. No. 67), and as regards the remaining portion $\nabla V=0$; hence, as regards the whole attracting mass, ∇V has the first-mentioned value, that is we have

$$\left(\frac{d^2}{da^2} \ldots +\frac{d^2}{dc^2}+\frac{d^2}{de^2}\right)V=-\frac{4(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s-\frac{1}{2})}\rho',$$

where ρ' is the same function of the coordinates $(a \ldots c, e)$ that ρ is of $(x \ldots z, w)$; viz. the potential of an attracting mass distributed not on a surface, but over a portion of space, *does not satisfy the potential equation*

$$\left(\frac{d^2}{da^2} \ldots +\frac{d^2}{dc^2}+\frac{d^2}{de^2}\right)V=0,$$

but it satisfies the foregoing equation, which only agrees with the potential equation in regard to a point $(a \ldots c, e)$ outside the material space, and for which, therefore, ρ' is $=0$.

The equation may be written

$$\rho'=-\frac{\Gamma(\frac{1}{2}s-\frac{1}{2})}{4(\Gamma\frac{1}{2})^{s+1}}\left(\frac{d^2}{da^2} \ldots +\frac{d^2}{dc^2}+\frac{d^2}{de^2}\right)V;$$

or, considering V as a given function of $(a \ldots c, e)$, in general a discontinuous function but subject to certain conditions as afterwards mentioned, and taking W the same function of $(x \ldots z, w)$ that V is of $(a \ldots c, e)$, then we have

$$\rho=-\frac{\Gamma(\frac{1}{2}s-\frac{1}{2})}{4(\Gamma\frac{1}{2})^{s+1}}\left(\frac{d^2}{dx^2} \ldots +\frac{d^2}{dz^2}+\frac{d^2}{dw^2}\right)W, \quad \ldots \ldots \quad \text{(D)}$$

viz. this equation determines ϱ as a function, in general a discontinuous function, of $(x \ldots z, w)$ such that the corresponding integral

$$V=\int \frac{\varrho\, dx \ldots dz\, dw}{\{(a-x)^2 \ldots +(c-z)^2+(e-w)^2\}^{\frac{1}{2}s+q}}$$

may be the given function of the coordinates $(a \ldots c, e)$. The equation is, in fact, the distribution-theorem D.

28. It is to be observed that the given function of $(a \ldots c, e)$ must satisfy certain conditions as to value at infinity and continuity, but it is not (as in the distribution-theorems A, B, and C it is) required to satisfy a partial differential equation; the function, except as regards the conditions as to value at infinity and continuity, is absolutely arbitrary.

The potential (assuming that the matter which gives rise to it lies wholly within a finite closed surface) must vanish for points at an infinite distance, or more accurately it must for indefinitely large values of $a^2 \ldots +c^2+e^2$ be of the form, Constant $\div$ by $(a^2 \ldots +c^2+e^2)^{\frac{1}{2}s-\frac{1}{2}}$. It may be a discontinuous function; for instance outside a given closed surface it may be one function, and inside the same surface a different function of the coordinates $(a \ldots c, e)$, viz. this may happen in consequence of an abrupt change of the density of the attracting matter on the one and the other side of the given closed surface, but not in any other manner; and, happening in this manner, then V', V'' being the values for points within and without the surface respectively, it has been seen to be necessary that, at the surface, not only $V'=V''$, but also $\frac{dV'}{da}=\frac{dV''}{da} \ldots \frac{dV'}{dc}=\frac{dV''}{dc}, \frac{dV'}{de}=\frac{dV''}{de}$. Subject to these conditions as to value at infinity and continuity, V may be any function whatever of the coordinates $(a \ldots c, e)$, and then taking W, the same function of $(x \ldots z, w)$, the foregoing equation determines ϱ, viz. determines it to be $=0$ for those parts of space which do not belong to the material space, and to have its proper value as a function of $(x \ldots z, w)$ for the remaining or material space.

The Prepotential Plane Theorem A.—Art. Nos. 29 to 36.

29. We have seen that if there exists on the plane $w=0$ a distribution of matter producing at the point $(a \ldots c, e)$ a given prepotential V (viz. V is to be regarded as a given function of $(a \ldots c, e)$), then that the distribution or density ϱ is given by a determinate formula, but it was remarked that the prepotential V cannot be a function assumed at pleasure; it must be a function satisfying certain conditions. One of these is the condition of continuity, the function V and all its derived functions must vary continuously as we pass, without traversing the material plane, from any given point to any other given point. But it is sufficient to attend to points on one side of the plane, say the upperside, or that for which e is positive; and since any such point is accessible from any other such point by a path which does not meet the plane, it is sufficient to say that the function V must vary continuously for a passage by such path from any such point to any such point; the function V must therefore be one and the same

function (and that a continuous one in value) for all values of the coordinates $(a \ldots c)$ and positive values of the coordinate e.

If, moreover, we assume that the distribution which corresponds to the given potential V is a distribution of a finite mass $\int \rho\, dx \ldots dz$ over a finite portion of the plane $w=0$, viz. over a portion or area such that the distance of a point within the area from a fixed point, or say from the origin $(a \ldots c) = (0 \ldots 0)$, is always finite; this being so, we have the further condition that the prepotential V must for indefinitely large values of all or any of the coordinates $(a \ldots c, e)$ reduce itself to the form

$$(\int \rho\, dx \ldots dz) \div (a^2 \ldots + c^2 + e^2)^{\frac{1}{2}s+q}.$$

The assumptions upon which this last condition is obtained are perhaps unnecessary; instead of the condition in the foregoing form we, in fact, use only the condition that the prepotential vanishes for a point at infinity, that is when all or any one or more of the coordinates $(a \ldots c, e)$ are or is infinite.

Again, as we have seen, the prepotential V must satisfy the prepotential equation

$$\left(\frac{d^2}{da^2} \ldots + \frac{d^2}{dc^2} + \frac{d^2}{de^2} + \frac{e}{2q+1}\,\frac{d}{de}\right)\mathrm{V}=0.$$

These conditions satisfied, to the given prepotential V, there corresponds on the plane $w=0$, a distribution given by the foregoing formula, and which will be a distribution over a finite portion of the plane, as already mentioned.

30. The proof depends upon properties of the prepotential equation,

$$\left(\frac{d^2}{dx^2} \ldots + \frac{d^2}{dz^2} + \frac{d^2}{de^2} + \frac{2q+1}{e}\,\frac{d}{de}\right)\mathrm{W}=0,$$

or, what is the same thing,

$$\frac{d}{dx}\left(e^{2q+1}\frac{d\mathrm{W}}{dx}\right) \ldots + \frac{d}{dz}\left(e^{2q+1}\frac{d\mathrm{W}}{dz}\right) + \frac{d}{de}\left(e^{2q+1}\frac{d\mathrm{W}}{de}\right)=0,$$

say, for shortness, $\square\mathrm{W}=0$

Consider, in general, the integral

$$\int dx \ldots dz\, de\, e^{2q+1}\left\{\left(\frac{d\mathrm{W}}{dx}\right)^2 \ldots + \left(\frac{d\mathrm{W}}{dz}\right)^2 + \left(\frac{d\mathrm{W}}{de}\right)^2\right\}$$

taken over a closed surface S lying altogether on the positive side of the plane $e=0$, the function W being in the first instance arbitrary.

Writing the integral under the form

$$\int dx \ldots dz\, de \left(e^{2q+1}\frac{d\mathrm{W}}{dx}\cdot\frac{d\mathrm{W}}{dx} \ldots + e^{2q+1}\frac{d\mathrm{W}}{dz}\cdot\frac{d\mathrm{W}}{dz} + e^{2q+1}\frac{d\mathrm{W}}{de}\cdot\frac{d\mathrm{W}}{de}\right),$$

we reduce the several terms by an integration by parts as follows:—

$$\text{term in } \frac{dW}{dx} \text{ is } = \int dy \ldots dz\, de\, W e^{2q+1} \frac{dW}{dx} - \int dx \ldots dz\, de\, W \frac{d}{dx}\left(e^{2q+1}\frac{dW}{dx}\right),$$

$$\vdots$$

$$\frac{dW}{dz} \text{ is } = \int dx \ldots\ldots de\, W e^{2q+1} \frac{dW}{dz} - \int dx \ldots dz\, de\, W \frac{d}{dz}\left(e^{2q+1}\frac{dW}{dz}\right),$$

$$\frac{dW}{de} \text{ is } = \int dx \ldots\ldots dz\, W e^{2q+1} \frac{dW}{de} - \int dx \ldots dz \quad W \frac{d}{de}\left(e^{2q+1}\frac{dW}{de}\right).$$

Write dS to denote an element of surface at the point $(x \ldots z, e)$; and taking $\alpha \ldots \gamma, \delta$ to denote the inclinations of the interior normal at that point to the positive axes of coordinates, we have

$$dy \ldots dz\, de = -dS \cos\alpha,$$

$$\vdots$$

$$dx \ldots\ldots de = -dS \cos\gamma,$$

$$dx \ldots \; . dz = -dS \cos\delta;$$

and the first terms are together

$$= -\int e^{2q+1} W \left(\frac{dW}{dx}\cos\alpha \ldots + \frac{dW}{dz}\cos\gamma + \frac{dW}{de}\cos\delta\right) dS,$$

W here denoting the value at the surface, and the integration being extended over the whole of the closed surface: this may also be written

$$= -\int e^{2q+1} W \frac{dW}{d\nu}\, dS,$$

where ν denotes an element of the internal normal.

The second terms are together

$$= -\int dx \ldots dz\, de\, W \left\{\frac{d}{dx}\left(e^{2q+1}\frac{dW}{dx}\right) \ldots + \frac{d}{dz}\left(e^{2q+1}\frac{dW}{dz}\right) + \frac{d}{de}\left(e^{2q+1}\frac{dW}{de}\right)\right\} = -\int dx \,.\, dz\, de W \square W.$$

We have consequently

$$\int dx \quad . dz\, de\, e^{2q+1} \left\{\left(\frac{dW}{dx}\right)^2 \ldots + \left(\frac{dW}{dz}\right)^2 + \left(\frac{dW}{de}\right)^2\right\}$$

$$= -\int e^{2q+1} W \frac{dW}{d\nu}\, dS - \int dx \ldots dz\, de\, e^{2q+1} W \square W.$$

31. The second term vanishes if W satisfies the prepotential equation $\square W = 0$; and this being so, if also $W=0$ for all points of the closed surface S, then the first term also vanishes, and we therefore have

$$\int dx \ldots dz\, de\, e^{2q+1} \left\{\left(\frac{dW}{dx}\right)^2 \ldots + \left(\frac{dW}{dz}\right)^2 + \left(\frac{dW}{de}\right)^2\right\} = 0,$$

where the integration extends over the whole space included within the closed surface; whence, W being a real function,

$$\frac{dW}{dx} = 0, \; \ldots \; \frac{dW}{dz} = 0, \; \frac{dW}{de} = 0,$$

for all points within the closed surface; consequently, since W vanishes at the surface, $W=0$ for all points within the closed surface.

32. Considering W as satisfying the equation $\square W=0$, we may imagine the closed surface to become larger and larger, and ultimately infinite, at the same time flattening itself out into coincidence with the plane $e=0$, so that it comes to include the whole space above the plane $e=0$, say the surface breaks up into the surface positive infinity and the infinite plane $e=0$.

The integral $\int e^{2q+1} W \frac{dW}{d\mathfrak{n}} dS$ separates itself into two parts, the first relating to the surface positive infinity, and which vanishes if $W=0$ at infinity (that is, if all or any of the coordinates $x \ldots z, e$ are infinite), the second relating to the plane $e=0$ is $\int W\left(e^{2q+1}\frac{dW}{de}\right) dx \ldots dz$, W here denoting its value at the plane, that is when $e=0$, and the integral being extended over the whole plane. The theorem thus becomes

$$\int dx \ldots dz\, de\,.\, e^{2q+1}\left\{\left(\frac{dW}{dx}\right)^2 \ldots + \left(\frac{dW}{dz}\right)^2 + \left(\frac{dW}{de}\right)^2\right\}$$
$$= -\int W\left(e^{2q+1}\frac{dW}{de}\right) dx \ldots dz$$

Hence also if $W=0$ at all points of the plane $e=0$, the right-hand side vanishes, and we have

$$\int dx \ldots dz\, de\, e^{2q+1}\left\{\left(\frac{dW}{dx}\right)^2 \ldots + \left(\frac{dW}{dz}\right)^2 + \left(\frac{dW}{de}\right)^2\right\}=0.$$

Consequently $\frac{dW}{dx}=0 \ldots \frac{dW}{dz}=0$, $\frac{dW}{de}=0$, for all points whatever of positive space; and therefore also $W=0$ for all points whatever of positive space.

33. Take next U, W, each of them a function of $(x \ldots z, e)$, and consider the integral

$$\int dx \ldots dz\, de\, e^{2q+1}\left(\frac{dU}{dx}\frac{dW}{dx} \ldots + \frac{dU}{dz}\frac{dW}{dz} + \frac{dU}{de}\frac{dW}{de}\right),$$

taken over the space within a closed surface S, treating this in a similar manner, we find it to be

$$= -\int e^{2q+1} W \frac{dU}{d\mathfrak{n}} dS - \int dx \ldots dz\, de\, e^{2q+1} W \square U,$$

where the integration extends over the whole of the closed surface S; and by parity of reasoning it is also

$$= -\int e^{2q+1} U \frac{dW}{d\mathfrak{n}} dS - \int dx \ldots dz\, de\, e^{2q+1} U \square W,$$

with the same limits of integration; that is, we have

$$\int e^{2q+1} W \frac{dU}{d\mathfrak{n}} dS + \int dx \ldots dz\, de\,.\, e^{2q+1} W \square U = \int e^{2q+1} U \frac{dW}{d\mathfrak{n}} dS + \int dx \ldots dz\, de\,.\, e^{2q+1} U \square W,$$

which, if U, W each satisfy the prepotential equation, becomes

$$\int e^{2q+1}\mathrm{W}\frac{d\mathrm{U}}{du}\,d\mathrm{S}=\int e^{2q+1}\mathrm{U}\frac{d\mathrm{W}}{du}\,d\mathrm{S}.$$

And if we now take the closed surface S to be the surface positive infinity, together with the plane $e=0$, then, provided only U and V vanish at infinity, for each integral the portion belonging to the surface positive infinity vanishes, and there remains only the portion belonging to the plane $e=0$, we have therefore

$$\int e^{2q+1}\mathrm{W}\frac{d\mathrm{U}}{de}\,dx\ldots dz=\int e^{2q+1}\mathrm{U}\frac{d\mathrm{W}}{de}\,dx\ldots dz,$$

where the functions U, W have each of them the value belonging to the plane $e=0$, viz. in U, W considered as given functions of $(x\ldots z,\,e)$ we regard e as a positive quantity ultimately put $=0$, and where the integrations extend each of them over the whole infinite plane

34. Assume

$$\mathrm{U}=\frac{1}{\{(a-x)^2\ldots+(c-z)^2+e^2\}^{\frac{1}{2}s+q}},$$

an expression which, regarded as a function of $(x\ldots z,\,e)$, satisfies the prepotential equation in regard to these variables, and which vanishes at infinity when all or any of these coordinates $(x\ldots z,\,e)$ are infinite.

We have

$$\frac{d\mathrm{U}}{de}=\frac{-2(\frac{1}{2}s+q)e}{\{(a-x)^2\ldots+(c-z)^2+e^2\}^{\frac{1}{2}s+q+1}};$$

and we have consequently

$$\int\mathrm{W}\frac{-2(\frac{1}{2}s+q)e^{2q+2}}{\{(a-x)^2\ldots+(c-z)^2+e^2\}^{\frac{1}{2}s+q+1}}\,dx\ldots dz=\int\left(e^{2q+1}\frac{d\mathrm{W}}{de}\right)\frac{dx\ldots dz}{\{(a-x)^2\ldots+(c-z)^2+e^2\}^{\frac{1}{2}s+q}},$$

where it will be recollected that e is ultimately $=0$; to mark this we may for W write W_0.

Attend to the left-hand side; take V_0 the same function of $a\ldots c,\,e=0$, that W_0 is of $x\ldots z,\,e=0$; then first writing the expression in the form

$$\mathrm{V}_0\int\frac{-2(\frac{1}{2}s+q)e^{2q+2}dx\ldots dz}{\{(a-x)^2\ldots+(c-z)^2+e^2\}^{\frac{1}{2}s+q+1}},$$

write $x=a+e\xi\ldots z=c+e\zeta$, the expression becomes

$$=\mathrm{V}_0\int\frac{-2(\frac{1}{2}s+q)e^{2q+2}.\,e^s d\xi\ldots d\zeta}{\{e^2(1+\xi^2\ldots+\zeta^2)\}^{\frac{1}{2}s+q+1}},\quad=-2(\tfrac{1}{2}s+q)\mathrm{V}_0\int\frac{d\xi\ldots d\zeta}{\{1+\xi^2\ldots+\zeta^2\}^{\frac{1}{2}s+q+1}},$$

where the integral is to be taken from $-\infty$ to $+\infty$ for each of the new variables $\xi\ldots\zeta$.

Writing $\xi=r\alpha\ldots\zeta=r\gamma$, where $\alpha^2\ldots+\gamma^2=1$, we have $d\xi\ldots d\zeta=r^{s-1}dr\,d\mathrm{S}$ also $\xi^2\ldots+\zeta^2=r^2$, and the integral is

$$=\int\frac{r^{s-1}dr\,d\mathrm{S}}{(1+r^2)^{\frac{1}{2}s+q+1}},\quad=\int d\mathrm{S}\int_0^\infty\frac{r^{s-1}dr}{(1+r^2)^{\frac{1}{2}s+q+1}},$$

where $\int dS$ denotes the surface of the s-coordinal unit sphere $\alpha^2\ldots+\gamma^2=1$, and the r-integral is to be taken from $r=0$ to $r=\infty$; the values of the two factors thus are

$$\int dS=\frac{2(\Gamma\tfrac{1}{2})^s}{\Gamma(\tfrac{1}{2}s)},\text{ and }\int\frac{r^{s-1}dr}{(1+r^2)^{\frac{1}{2}s+q+1}}=\frac{\tfrac{1}{2}\Gamma\tfrac{1}{2}s\Gamma(q+1)}{\Gamma(\tfrac{1}{2}s+q+1)}.$$

Hence the expression in question is

$$-2(\tfrac{1}{2}s+q)V_0\frac{2(\Gamma\tfrac{1}{2})^s}{\Gamma\tfrac{1}{2}s}\frac{\tfrac{1}{2}\Gamma\tfrac{1}{2}s\Gamma(q+1)}{\Gamma(\tfrac{1}{2}s+q+1)},\ =\frac{-2(\Gamma\tfrac{1}{2})^s\Gamma(q+1)}{\Gamma(\tfrac{1}{2}s+q)}V_0,$$

and we have

$$\int\left(e^{2q+1}\frac{dW}{de}\right)_0\frac{dx\ldots dz}{\{(a-x)^2\ldots+(c-z)^2+e^2\}^{\frac{1}{2}s+q}}=\frac{-2(\Gamma\tfrac{1}{2})^s\Gamma(q+1)}{\Gamma(\tfrac{1}{2}s+q)}V_0;$$

or, what is the same thing,

$$V_0=\int\frac{\dfrac{-\Gamma(\tfrac{1}{2}s+q)}{2(\Gamma\tfrac{1}{2})^s\Gamma(q+1)}\left(e^{2q+1}\dfrac{dW}{de}\right)_0}{\{(a-x)^2\ldots+(c-z)^2+e^2\}^{\frac{1}{2}s+q}}.$$

35. Take now V a function of $(a\ldots c, e)$ satisfying the prepotential equation in regard to these variables, always finite, and vanishing at infinity, and let W be the same function of $(x\ldots z, e)$, W therefore satisfying the prepotential equation in regard to the last-mentioned variables, and consider the function

$$V-\int\frac{-\dfrac{\Gamma(\tfrac{1}{2}s+q)}{2(\Gamma\tfrac{1}{2})^s\Gamma(q+1)}\left(e^{2q+1}\dfrac{dW}{de}\right)_0dx\ldots dz}{\{(a-x)^2\ldots+(c-z)^2+e^2\}^{\frac{1}{2}s+q}},$$

where the integral is taken over the infinite plane $e=0$, then this function (V − the integral) satisfies the prepotential equation (for each term separately satisfies it), is always finite, and it vanishes at infinity. It also, as has just been seen, vanishes for any point whatever of the plane $e=0$. Consequently it vanishes for all points whatever of positive space. Or, what is the same thing, if we write

$$V=\int\frac{\rho\,dx\ldots dz}{\{(a-x)^2\ldots+(c-z)^2+e^2\}^{\frac{1}{2}s+q}},\quad\ldots\ldots\ldots\quad\text{(A)}$$

where ρ is a function of $(x\ldots z)$, and the integral is taken over the whole infinite plane, then if V is a function of $(a\ldots c, e)$ satisfying the above conditions, there exists a corresponding value of ρ; viz. taking W the same function of $(x\ldots z, e)$ which V is of $(a\ldots c, e)$, the value of ρ is

$$\rho=-\frac{\Gamma(\tfrac{1}{2}s+q)}{2(\Gamma\tfrac{1}{2})^s\Gamma(q+1)}\left(e^{2q+1}\frac{dW}{de}\right)_0,\quad\ldots\ldots\ldots\ldots\quad\text{(A)}$$

where e is to be put $=0$ in the function $e^{2q+1}\frac{dW}{de}$. This is the prepotential-plane theorem; viz. taking for the prepotential in regard to a given point $(a,\ldots c, e)$ a function of $(a\ldots c, e)$ satisfying the prescribed conditions, but otherwise arbitrary, there exists on the plane $e=0$ a distribution ρ given by the last-mentioned formula.

36. It is assumed in the proof that $2q+1$ is positive or zero; viz. q is positive, or if negative then $-q \not> \frac{1}{2}$; the limiting case $q=-\frac{1}{2}$ is included.

It is to be remarked that by what precedes, if q be positive (but excluding the case $q=0$) the density ϱ is given by the equivalent more simple formula

$$\varrho=\frac{\Gamma(\frac{1}{2}s+q)}{(\Gamma\frac{1}{2})^s\Gamma q}(e^{2q}\text{W})_e.$$

The foregoing proof is substantially that given in GREEN's memoir on the Attraction of Ellipsoids; it will be observed that the proof only imposes upon V the condition of vanishing at infinity, without obliging it to assume for large values of $(a\ldots c, e)$ the form $\frac{\text{M}}{\{a^2\ldots+c^2+e^2\}^{\frac{1}{2}s+q}}$.

The Potential-surface Theorem C.—Art. Nos 37 to 42.

37. In the case $q=-\frac{1}{2}$, writing here $\nabla=\frac{d^2}{dx^2}\ldots+\frac{d^2}{dz^2}+\frac{d^2}{de^2}$, we have precisely, as in the general case,

$$\int \text{W}\frac{d\text{U}}{d\nu}\,d\text{S}+\int dx\ldots dz\,de\,\text{W}\nabla\text{U}=\int \text{U}\frac{d\text{W}}{d\nu}\,d\text{S}+\int dx\ldots dz\,de\,\text{U}\nabla\text{W};$$

and if the functions U, W satisfy the equations $\nabla\text{U}=0$, $\nabla\text{W}=0$, then (subject to the exception presently referred to) the second terms on the two sides respectively each of them vanish.

But, instead of taking the surface to be the surface positive infinity together with the plane $e=0$, we now leave it an arbitrary closed surface, and for greater symmetry of notation write w in place of e; and we suppose that the functions U and W, or one of them, may become infinite at points within the closed surface; on this last account the second terms do not in every case vanish.

38. Suppose, for instance, that U at a point indefinitely near the point $(a\ldots c, e)$ within the surface becomes

$$=\frac{1}{\{(x-a)^2\ldots+(z-c)^2+(w-e)^2\}^{\frac{1}{2}s-\frac{1}{2}}};$$

then if V be the value of W at the point $(a\ldots c, e)$, we have

$$\int dx\ldots dz\,dw\,\text{W}\nabla\text{U}=\text{V}\int dx\ldots dz\,dw\,\nabla\text{U};$$

and since $\nabla\text{U}=0$, except at the point in question, the integral may be taken over any portion of space surrounding this point, for instance, over the space included within the sphere, radius R, having the point $(a\ldots c, e)$ for its centre; or taking the origin at this point, we have to find $\int dx\ldots dz\,dw\,\nabla\text{U}$, where

$$\text{U}=\frac{1}{\{x^2\ldots+z^2+w^2\}^{\frac{1}{2}s-\frac{1}{2}}},$$

and the integration extends over the space within the sphere $x^2\ldots+z^2+w^2=\text{R}^2$.

39. This may be accomplished most easily by means of a particular case of the last-mentioned theorem; viz. writing $W=1$, we have

$$\int \frac{dU}{d\nu}\,dS+\int dx\ldots dz\,dw\,\nabla U=0,$$

or the required value is $=-\int \frac{dU}{d\nu}\,dS$ over the surface of the last-mentioned sphere. We have, if for a moment $r^2=x^2\ldots+z^2+w^2$,

$$\frac{dU}{d\nu}=-\left(\frac{x}{r}\frac{d}{dx}\ldots+\frac{z}{r}\frac{d}{dz}+\frac{w}{r}\frac{d}{dw}\right)U,\ =-\left(\frac{x}{r}\frac{d}{dx}\ldots+\frac{z}{r}\frac{d}{dz}+\frac{w}{r}\frac{d}{dw}\right)r\,.\frac{dU}{dr},\ =-\frac{dU}{dr},$$

that is, $\frac{dU}{d\nu}=\frac{s-1}{r^s},\ =\frac{s-1}{R^s}$, and hence

$$\int \frac{dU}{d\nu}\,dS=\frac{s-1}{R^s}\int dS,$$

where $\int dS$ is the whole surface of the sphere $x^2\ldots+z^2+w^2=R^2$, viz. it is $=R^s$ into the surface of the unit-sphere $x^2\ldots+z^2+w^2=1$. This spherical surface, say

$$\int d\Sigma \text{ is } =\frac{2(\Gamma\frac{1}{2})^{s+1}}{\Gamma\frac{1}{2}(s+1)},\ =\frac{4(\Gamma\frac{1}{2})^{s+1}}{(s-1)\Gamma\frac{1}{2}(s-1)},$$

and we have thus $\int \frac{dU}{d\nu}\,dS=\frac{4(\Gamma\frac{1}{2})^{s+1}}{\Gamma\frac{1}{2}(s-1)}$, and consequently

$$\int dx\ldots dz\,dw\,\nabla U=-\frac{4(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s-\frac{1}{2})}.$$

40. Treating in like manner the case where W at a point indefinitely near the point $(a,\ldots c, e)$ within the surface becomes

$$=\frac{1}{\{(x-a)^2\ \ldots+(z-c)^2+(w-e)^2\}^{\frac{1}{2}s-\frac{1}{2}}},$$

and writing T to denote the same function of $(a,\ldots c, e)$ that U is of $(x\ldots z, w)$, we have, instead of the foregoing, the more general theorem

$$\int W\frac{dU}{d\nu}\,dS+\int dx\,.\ dz\,dw\,W\nabla U-\frac{4(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s-\frac{1}{2})}\,V$$
$$=\int U\frac{dW}{d\nu}\,dS+\int dx\ldots dz\,dw\,U\nabla W-\frac{4(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s-\frac{1}{2})}\,T,$$

where in the two solid integrals respectively we exclude from consideration the space in the immediate neighbourhood of the two critical points $(a\ldots c, e)$ and $(a\ldots c, e)$ respectively.

Suppose that W is always finite within the surface, and that U is finite except at the point $(a\ldots c, e)$, and moreover that U, W are such that $\nabla U=0$, $\nabla W=0$, then the equation becomes

$$\int W\frac{dU}{d\nu}\,dS-\frac{4(\Gamma\frac{1}{2})^{s+1}}{\Gamma\frac{1}{2}s-\frac{1}{2}}V=\int U\frac{dW}{d\nu}\,dS.$$

In particular this equation holds good if U is $=\dfrac{1}{\{(a-x)^2\ldots+(e-w)^2\}^{\frac{1}{2}s-\frac{1}{2}}}$.

41. Imagine now on the surface S a distribution $\rho d\mathrm{S}$ producing at a point $(a'\ldots c', e')$ within the surface a potential V′, and at a point $(a''\ldots c'', e'')$ without the surface a potential V″; where, by what precedes, V″ is in general not the same function of $(a''\ldots c'', e'')$ that V′ is of $(a'\ldots c', e')$.

It is further assumed that at a point $(a\ldots c, e)$ on the surface we have $\mathrm{V}'=\mathrm{V}''$.

that V′, or any of its derived functions, are not infinite for any point $(a'\ldots c', e')$ within the surface:

that V″, or any of its derived functions, are not infinite for any point $(a''\ldots c'', e'')$ without the surface.

and that $\mathrm{V}''=0$ for any point at infinity.

Consider V′ as a given function of $(a\ldots c, e)$; and take W′ the same function of $(x\ldots z, w)$. Then if, as before,

$$\mathrm{U}=\frac{1}{\{(a-x)^2\ldots+(c-z)^2+(e-w)^2\}^{\frac{1}{2}s-\frac{1}{2}}},$$

$$\left(\frac{d^2}{dx^2}\cdots+\frac{d^2}{de^2}+\frac{d^2}{dw^2}\right)\mathrm{U}=0,$$

then

$$\int \mathrm{U}\,\frac{d\mathrm{W}'}{d\nu'}\,d\mathrm{S}=\int \mathrm{W}'\,\frac{d\mathrm{U}}{d\nu'}\,d\mathrm{S}-\frac{4(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s-\frac{1}{2})}\,\mathrm{V}'.$$

Similarly, considering V″ as a given function of $(a\ldots c, e)$ and take W″ the same function of $(x\ldots z, e)$. Then, by considering the space outside the surface S, or say between this surface and infinity, and observing that U does not become infinite for any point in this space, we have

$$\int \mathrm{U}\,\frac{d\mathrm{W}''}{d\nu''}\,d\mathrm{S}=\int \mathrm{W}''\frac{d\mathrm{U}}{d\nu''}\,d\mathrm{S};$$

and adding these two equations, we have

$$\int \mathrm{U}\left(\frac{d\mathrm{W}'}{d\nu'}+\frac{d\mathrm{W}''}{d\nu''}\right)d\mathrm{S}=\int\left(\mathrm{W}'\frac{d\mathrm{U}}{d\nu'}+\mathrm{W}''\frac{d\mathrm{U}}{d\nu''}\right)d\mathrm{S}-\frac{4(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s-\frac{1}{2})}\,\mathrm{V}'.$$

But in this equation the functions W′ and W″ each of them belong to a point $(x\ldots z, w)$ on the surface, and we have at the surface $\mathrm{W}'=\mathrm{W}''$, $=\mathrm{W}$ suppose; the term on the right-hand side thus is $\int \mathrm{W}\left(\frac{d\mathrm{U}}{d\nu'}+\frac{d\mathrm{U}}{d\nu''}\right)d\mathrm{S}$, which vanishes in virtue of $\frac{d\mathrm{U}}{d\nu'}+\frac{d\mathrm{U}}{d\nu''}=0$; and the equation thus becomes

$$\int \mathrm{U}\left(\frac{d\mathrm{W}'}{d\nu'}+\frac{d\mathrm{W}''}{d\nu''}\right)d\mathrm{S}=-\frac{4(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s-\frac{1}{2})}\mathrm{V}':$$

that is, the point $(a\ldots c, e)$ being interior, we have

$$\mathrm{V}'=\int\frac{-\Gamma(\frac{1}{2}s-\frac{1}{2})}{4(\Gamma\frac{1}{2})^{s+1}}\left(\frac{d\mathrm{W}'}{d\nu'}+\frac{d\mathrm{W}''}{d\nu''}\right)\frac{d\mathrm{S}}{\{(a-x)^2\ldots+(c-z)^2+(e-w)^2\}^{\frac{1}{2}s-\frac{1}{2}}}.$$

In exactly the same way if $(a\ldots c, e)$ be an exterior point, then we have

$$\int \mathrm{U}\frac{d\mathrm{W}'}{d\nu'}\,d\mathrm{S}=\int \mathrm{W}'\frac{d\mathrm{U}}{d\nu'}\,d\mathrm{S},$$

$$\int \mathrm{U}\frac{d\mathrm{W}''}{d\nu''}\,d\mathrm{S}=\int \mathrm{W}''\frac{d\mathrm{U}}{d\nu''}\,d\mathrm{S}-\frac{4(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s-\frac{1}{2})}\mathrm{V}'';$$

and adding, and omitting the terms which vanish,

$$\int \mathrm{U}\left(\frac{d\mathrm{W}'}{d\nu'}+\frac{d\mathrm{W}''}{d\nu''}\right)d\mathrm{S}=-\frac{4(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s-\frac{1}{2})}\mathrm{V}'',$$

that is,

$$\mathrm{V}''=\int\frac{-\Gamma(\frac{1}{2}s-\frac{1}{2})}{4(\Gamma\frac{1}{2})^{s+1}}\left(\frac{d\mathrm{W}'}{d\nu'}+\frac{d\mathrm{W}''}{d\nu''}\right)\frac{d\mathrm{S}}{\{(a-x)^2\ldots(c-z)^2+(e-w)^2\}^{\frac{1}{2}s-\frac{1}{2}}}.$$

42. Comparing the two results with

$$\mathrm{V}=\int\frac{\rho\,d\mathrm{S}}{\{(a-x)^2\ldots+(c-z)^2+(e-w)^2\}^{\frac{1}{2}s-\frac{1}{2}}},$$

we see that V′, V″ satisfying the foregoing conditions, there exists a distribution ρ on the surface, producing the potentials V′ and V″ at an interior point and an exterior point respectively, the value of ρ in fact being

$$\rho=-\frac{\Gamma(\frac{1}{2}s-\frac{1}{2})}{4(\Gamma\frac{1}{2})^{s+1}}\left(\frac{d\mathrm{W}'}{d\nu'}+\frac{d\mathrm{W}''}{d\nu''}\right),\quad\ldots\ldots\ldots\ldots\quad (\mathrm{C})$$

where W′, W″ are respectively the same functions of $(x\ldots z, w)$ that V′, V″ are of $(a\ldots c, e)$.

The Potential-solid Theorem D.—Art. No. 43.

43. We have as before (No. 40),

$$\int \mathrm{W}\frac{d\mathrm{U}}{d\nu}\,d\mathrm{S}+\int dx\ldots dz\,dw\,\mathrm{W}\nabla\mathrm{U}-\frac{4(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s-\frac{1}{2})}\mathrm{V}$$
$$=\int \mathrm{U}\frac{d\mathrm{W}}{d\nu}d\mathrm{S}+\int dx\ldots dz\,dw\,\mathrm{U}\nabla\mathrm{W}-\frac{4(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s-\frac{1}{2})}\mathrm{T},$$

where, assuming first that W is not infinite for any point $(x\ldots z, w)$ whatever, we have no term in T; and taking next $\mathrm{U}=\frac{1}{\{(a-x)^2\ldots+(c-z)^2+(e-w)^2\}^{\frac{1}{2}s-\frac{1}{2}}}$ as before, we have $\nabla\mathrm{U}=0$; the equation thus becomes

$$\int \mathrm{W}\frac{d\mathrm{U}}{d\nu}\,d\mathrm{S}-\int \mathrm{U}\frac{d\mathrm{W}}{d\nu}\,d\mathrm{S}-\frac{4(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s-\frac{1}{2})}\mathrm{V}=\int dx\ldots dz\,dw\,\mathrm{U}\nabla\mathrm{W},$$

where W may be a discontinuous function of the coordinates $(x\ldots z, w)$, provided only there is no abrupt change in the value either of W or of any of its first derived functions $\frac{d\mathrm{W}}{dx}\cdots\frac{d\mathrm{W}}{dz},\frac{d\mathrm{W}}{dw}$, viz. it may be any function which can represent the potential of a solid mass on an attracted point $(x\ldots z, w)$; the resulting value of $\nabla\mathrm{W}$ is of course discon-

tinuous. Taking, then, for the closed surface S the boundary of infinite space, U and W each vanish at this boundary, and the equation becomes

$$-\frac{(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s-\frac{1}{2})}\mathrm{V}=\int dx\ldots dz\,dw\,\mathrm{U}\nabla\mathrm{W},$$

viz. substituting for U its value, and comparing with

$$\mathrm{V}=\int\frac{\rho\,dx\ldots dz\,dw}{\{(a-x)^2\ldots+(c-z)^2+(e-w)^2\}^{\frac{1}{2}s-\frac{1}{2}}},$$

where the integral in the first instance extends to the whole of infinite space, but the limits may be ultimately restricted by ρ being $=0$, we see that the value of ρ is

$$\rho=-\frac{\Gamma(\frac{1}{2}s-\frac{1}{2})}{(\Gamma\frac{1}{2})^{s+1}}\left(\frac{d^2}{dx^2}\cdots+\frac{d^2}{dz^2}+\frac{d^2}{dw^2}\right)\mathrm{W},$$

W being the same function of $(x\ldots z, w)$ that V is of $(a\ldots c, e)$, which is the theorem D.

Examples of the foregoing Theorems.—Art. Nos 44 to 49.

44. It will be remarked, as regards all the theorems, that we do not start with known limits; we start with V a function of $(a\ldots c, e)$, the coordinates of the attracted point, satisfying certain prescribed conditions, and we thence find ρ, a function of the coordinates $(x\ldots z)$ or $(x\ldots z, w)$, as the case may be, which function is found to be $=0$ for values of $(x\ldots z)$ or $(x\ldots z, w)$ lying beyond certain limits, and to have a determinate non-evanescent value for values of $(x\ldots z)$ or $(x\ldots z, w)$ lying within these limits, and we thus, as a result, obtain these limits for the limits of the multiple integral V.

45. Thus in theorem A, in the example where the limiting equation is ultimately found to be $x^2\ldots+z^2=f^2$, we start with V a certain function of $a^2\ldots+c^2$ ($=\varkappa^2$ suppose) and e^2, viz. V is a function of these quantities through θ, which denotes the positive root of the equation

$$\frac{\varkappa^2}{f^2+\theta}+\frac{e^2}{\theta}=1,$$

the value in fact being $\mathrm{V}=\int_\theta^\infty t^{-q-1}(t+f^2)^{-\frac{1}{2}s}dt$, and the resulting value of ρ is found to be $=0$ for values of $(x\ldots z)$ for which $x^2\ldots+z^2>f^2$. Hence V denotes an integral

$$\int\frac{\rho\,dx\ldots dz}{\{(a-x)^2\ldots+(c-z)^2+e^2\}^{\frac{1}{2}s+q}},$$

the limiting equation being $x^2\ldots+z^2=f^2$, say this is the s-coordinal sphere.

And similarly, in the examples where the limiting equation is ultimately found to be $\frac{x^2}{f^2}\ldots+\frac{z^2}{h^2}=1$, we start with V a certain function of $a,\ldots c, e$ through θ (or directly and through θ), where θ denotes the positive root of the equation

$$\frac{a^2}{f^2+\theta}\cdots+\frac{c^2}{h^2+\theta}+\frac{e^2}{\theta}=1,$$

and the resulting value of ϱ is found to be $=0$ for values of $(x\ldots z)$ for which

$$\frac{x^2}{f^2}\ldots+\frac{z^2}{h^2}>1.$$

Hence V denotes an integral,

$$\int\frac{\varrho\, dx\ldots dz}{\{(a-x)^2\ldots+(c-z)^2+e^2\}^{\frac{1}{2}s+q}},$$

the limiting equation being $\frac{x^2}{f^2}\ldots+\frac{z^2}{h^2}=1$, say this is the s-coordinal ellipsoid. It is clear that this includes the before-mentioned case of the s-coordinal sphere; but it is, on account of the more simple form of the θ-equation, worth while to work out directly an example for the sphere.

46. Three examples are worked out in Annex IV.; the results are as follows:—

First, θ defined for the sphere as above, $q+1$ positive;

$$V=\int\frac{\left(1-\frac{x^2\ldots+z^2}{f^2}\right)^q dx\ldots dz}{\{(a-x)^2\ldots+(c-z)^2+e^2\}^{\frac{1}{2}s+q}}$$

over the sphere $x^2\ldots+y^2=f^2$,

$$=\frac{(\Gamma\frac{1}{2})^s\Gamma(q+1)}{\Gamma(\frac{1}{2}s+q)}f^s\int_\theta^\infty t^{-q-1}(t+f^2)^{-\frac{1}{2}s}\,dt.$$

This is included in the next-mentioned example for the ellipsoid.

Secondly, θ defined for the ellipsoid as above; $q+1$ positive;

$$V=\int\frac{\left(1-\frac{x^2}{f^2}\ldots-\frac{z^2}{h^2}\right)^q dx\ldots dz}{\{(a-x)^2\ldots+(c-z)^2+e^2\}^{\frac{1}{2}s+q}}$$

over the ellipsoid $\frac{x^2}{f^2}\ldots+\frac{z^2}{h^2}=1$,

$$=\frac{(\Gamma\frac{1}{2})^s\Gamma(q+1)}{\Gamma(\frac{1}{2}s+q)}(f\ldots h)\int_\theta^\infty t^{-q-1}(t+f^2\ldots t+h^2)^{-\frac{1}{2}}\,dt.$$

This result is included in the next-mentioned example; but the proof for the general value of m is not directly applicable to the value $m=0$ for the case in question.

Thirdly, θ and the ellipsoid as above; $q+1$ positive; $m=0$ or positive, and apparently in other cases,

$$V=\int\frac{\left(1+\frac{x^2}{f^2}\ldots-\frac{z^2}{h^2}\right)^{q+m} dx\ldots dz}{\{(a-x)^2\ldots+(c-z)^2+e^2\}^{\frac{1}{2}s+q}}$$

over the ellipsoid as above,

$$=\frac{(\Gamma\frac{1}{2})^s\Gamma(1+q+m)}{\Gamma(\frac{1}{2}s+q)\Gamma(1+m)}(f\ldots h)\int_\theta^\infty\left(1-\frac{a^2}{f^2+\theta}\ldots-\frac{c^2}{h^2+\theta}-\frac{e^2}{\theta}\right)^m t^{-q-1}(t+f^2\ldots t+h^2)^{-\frac{1}{2}}dt.$$

And we have in Annex V. a fourth example; here θ and the ellipsoid are as above: the result involves the Greenian functions.

47. We may in the foregoing results write $e=0$; the results, writing therein $s+1$ for s, and in the new forms taking $(a \ldots c, e)$ and $(x \ldots z, w)$ for the two sets of coordinates respectively, also writing $q-\frac{1}{2}$ for q, would give integrals of the form

$$\int \frac{\rho dx \ldots dz dw}{\{(a-x)^2 \ldots + (c-z)^2 + (e-w)^2\}^{\frac{1}{2}s+q}}$$

for the $(s+1)$coordinal sphere and ellipsoid $x^2 \ldots + z^2 + w^2 = f^2$ and $\frac{x^2}{f^2} \ldots + \frac{z^2}{h^2} + \frac{w^2}{k^2} = 1$; say these are prepotential solid integrals; and then, writing $q=-\frac{1}{2}$, we should obtain potential solid integrals, such as are also given by the theorem D. The change can be made if necessary; but it is more convenient to retain the results in their original forms, as relating to the s-coordinal sphere and ellipsoid.

There are two cases, according as the attracted point $(a \ldots c)$ is external or internal.

For the sphere.—For an external point $\kappa^2 > f^2$; writing $e=0$, the equation $\frac{\kappa^2}{f^2+\theta}=1$ has a positive root, viz. this is $\theta = \kappa^2 - f^2$; and θ will have, or it may be replaced by, this value $\kappa^2 - f^2$: for an internal point $\kappa^2 < f^2$, as e approaches zero, the positive root of the original equation gradually diminishes and becomes ultimately $=0$, viz. in the formulæ θ is to be replaced by this value 0.

For the ellipsoid.—For an external point $\frac{a^2}{f^2} \ldots + \frac{c^2}{h^2} > 1$; writing $e=0$, the equation $\frac{a^2}{\theta+f^2} \ldots + \frac{c^2}{\theta+h^2} = 1$ has a positive root, and θ will denote this positive root: for an internal point $\frac{a^2}{f^2} \ldots + \frac{c^2}{h^2} < 1$, as e approaches zero the positive root of the original equation gradually diminishes and becomes ultimately $=0$, viz. in the formulæ θ is to be replaced by this value 0.

The resulting formulæ for the sphere $x^2 \ldots + z^2 = f^2$ may be compared with formulæ for the spherical shell, Annex VI., and each set with formulæ obtained by direct integration in Annex III.

We may in any of the formulæ write $q=-\frac{1}{2}$, and so obtain examples of theorem B.

48. As regards theorem C, we might in like manner obtain examples of potentials relating to the surfaces of the $(s+1)$coordinal sphere $x^2 \ldots + z^2 + w^2 = f^2$, and ellipsoid $\frac{x^2}{f^2} \ldots + \frac{z^2}{h^2} + \frac{w^2}{k^2} = 1$, or say to spherical and ellipsoidal shells; but I have confined myself to the sphere. We have to assume values V′ and V″ belonging to the cases of an internal and an external point respectively, and thence to obtain a value ϱ, or distribution over the spherical surface, which shall produce these potentials respectively. The result (see Annex VI.) is

$$\int \frac{dS}{\{(a-x)^2 + \ldots + (c-z)^2 + (e-w)^2\}^{\frac{1}{2}s-\frac{1}{2}}}$$

over the surface of the $(s+1)$coordinal sphere $x^2 \ldots + z^2 + w^2 = f^2$,

$$=\frac{2(\Gamma\tfrac{1}{2})^{s+1}f^s}{\Gamma(\tfrac{1}{2}s+\tfrac{1}{2})}\frac{1}{\varkappa^{s-1}} \text{ for exterior point } \varkappa > f$$

and

$$=\frac{2(\Gamma\tfrac{1}{2})^{s+1}f^s}{\Gamma(\tfrac{1}{2}s+\tfrac{1}{2})}\frac{1}{f^{s-1}} \text{ for interior point } \varkappa < f,$$

where $\varkappa^2 = a^2 \ldots + c^2 + e^2$. Observe that for the interior point the potential is a mere constant multiple of f.

The same Annex VI. contains the case of the s-coordinal cylinder $x^2 \ldots + z^2 = f^2$, which is peculiar in that the cylinder is not a finite closed surface, but the theorem C is found to extend to it.

49. As regards theorem D, we might in like manner obtain potentials relating to the $(s+1)$coordinal sphere $x^2 \ldots + z^2 + w^2 = f^2$ and ellipsoid $\frac{x^2}{f^2} \ldots + \frac{z^2}{h^2} + \frac{w^2}{k^2} = 1$; but I confine myself to the case of the sphere (see Annex VII.). We here assume values V′ and V″ belonging to an internal and an external point respectively, and thence obtain a value ρ, or distribution over the whole $(s+1)$dimensional space, which density is found to be $=0$ for points outside the sphere. The result obtained is

$$\mathrm{V} = \int \frac{dx \ldots dz\, dw}{\{(a-x)^2 \ldots + (c-z)^2 + (e-w)^2\}^{\frac{1}{2}s-\frac{1}{2}}}$$

over $(s+1)$coordinal sphere $x^2 \ldots + z^2 + w^2 = f^2$,

$$=\frac{(\Gamma\tfrac{1}{2}s)^{s+1}}{\Gamma(\tfrac{1}{2}s+\tfrac{3}{2})}\cdot\frac{f^{s+1}}{\varkappa^{s-1}} \text{ for exterior point } \varkappa > f$$

$$=\frac{(\Gamma\tfrac{1}{2}s)^{s+1}}{\Gamma(\tfrac{1}{2}s+\tfrac{3}{2})}\{(\tfrac{1}{2}s+\tfrac{1}{2})f^2 - (\tfrac{1}{2}s-\tfrac{1}{2})\varkappa^2\} \text{ for interior point } \varkappa < f,$$

where $\varkappa^2 = a^2 \ldots + c^2 + e^2$

The remaining Annexes VIII. and IX. have no immediate reference to the theorems A, B, C, D, which are the principal objects of the memoir. The subjects to which they relate will be seen from the headings and introductory paragraphs.

ANNEX I. *Surface and Volume of Sphere* $x^2 \ldots + z^2 + w^2 = f^2$.—Nos. 51 & 52.

51. We require in $(s+1)$dimensional space, $\int dx \ldots dz\, dw$, the volume of the sphere $x^2 \ldots + z^2 + w^2 = f^2$, and $\int d\mathrm{S}$, the surface of the same sphere.

Writing $x = f\sqrt{\xi} \ldots z = f\sqrt{\zeta}$, $w = f\sqrt{\omega}$, we have

$$dx \ldots dz\, dw = \frac{1}{2^{s+1}} f^{s+1} \xi^{-\frac{1}{2}} \ldots \zeta^{-\frac{1}{2}} \omega^{-\frac{1}{2}} d\xi \ldots d\zeta\, d\omega,$$

with the limiting condition $\xi \ldots + \zeta + \omega = 1$; but in order to take account as well of the negative as the positive values of $x \ldots z, w$, we must multiply by 2^{s+1}. The value is therefore

$$= f^{s+1} \int \xi^{-\frac{1}{2}} \ldots \zeta^{-\frac{1}{2}} \omega^{-\frac{1}{2}} d\xi \ldots d\zeta\, d\omega,$$

extended to all positive values of $\xi\ldots\zeta,\omega$, such that $\xi\ldots+\zeta+\omega<1$; and we obtain this by a known theorem, viz.

$$\text{Volume of }(s+1)\text{dimensional sphere}=f^{s+1}\frac{(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s+\frac{3}{2})}.$$

Writing $x=f\xi\ldots z=f\zeta$, $w=f\omega$, we obtain $d\mathrm{S}=f^s\,d\Sigma$, where $d\Sigma$ is the element of surface of the unit-sphere $\xi^2\ldots+\zeta^2+\omega^2=1$; we have element of volume $d\xi\,.\,.\,d\zeta\,d\omega=r^s\,dr\,d\Sigma$, where r is to be taken from 0 to 1, and thence

$$\int d\xi\ldots d\zeta\,d\omega=\int_0^1 r^s\,dr\,.\int d\Sigma=\frac{1}{s+1}\int d\Sigma,$$

that is,

$$\int d\Sigma=(s+1)\int d\xi\,.\,.\,d\zeta\,d\omega,\;=2(\tfrac{1}{2}s+\tfrac{1}{2})\frac{(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s+\frac{3}{2})}=\frac{2(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s+\frac{1}{2})};$$

consequently $\int d\mathrm{S}=\text{surface of }(s+1)\text{dimensional sphere}=f^s\,\dfrac{2(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s+\frac{1}{2})}$.

52. Writing $s-1$ for s, we have

$$\text{Volume of }(s-1)\text{dimensional sphere}=f^s\,\frac{(\Gamma\frac{1}{2})^s}{\Gamma(\frac{1}{2}s+1)},$$

$$\text{Surface of}\qquad\text{do.}\qquad=f^{s-1}\,\frac{2(\Gamma\frac{1}{2})^s}{\Gamma(\frac{1}{2}s)},$$

which forms are sometimes convenient.

Writing in the first forms $s+1=3$, or in the second forms $s=3$, we find in ordinary space

$$\text{Volume of sphere}=f^3\,\frac{(\Gamma\frac{1}{2})^3}{\Gamma(\frac{5}{2})}=f^3\,\frac{\pi^{\frac{3}{2}}}{\frac{3}{2}\,\frac{1}{2}\,\sqrt{\pi}},\;=\frac{4\pi f^3}{3},$$

and

$$\text{Surface of sphere}=f^2\,\frac{2(\Gamma\frac{1}{2})^3}{\Gamma\frac{3}{2}}=f^2\,\frac{2\pi^{\frac{3}{2}}}{\frac{1}{2}\sqrt{\pi}},\qquad=4\pi f^2,$$

as they should be.

ANNEX II *The Integral* $\int_0^{\mathrm{R}}\frac{r^{s-1}dr}{(r^2+e^2)^{\frac{1}{2}s+q}}$.—Nos. 53 to 63.

53. The integral in question (which occurs *antè*, No. 2) may also be considered as arising from a prepotential integral in tridimensional space; the prepotential of an element of mass dm is taken to be $=\frac{dm}{d^{s+2q}}$, where d is the distance of the element from the attracted point P. Hence if the element of mass be an element of the plane $z=0$, coordinates (x, y), ρ being the density, and if the attracted point be situate in the axis of z at a distance e from the origin, the prepotential is

$$\mathrm{V}=\int\frac{\rho\,dx\,dy}{(x^2+y^2+e^2)^{\frac{1}{2}s+q}}.$$

For convenience it is assumed throughout that e is positive.

Suppose that the attracting body is a circular disk radius R, having the origin for its centre (viz. that bounded by the curve $x^2+y^2=R^2$); then writing $x=r\cos\theta, y=r\sin\theta$, we have

$$V=\int\frac{\rho\, r\, dr\, d\theta}{(r^2+e^2)^{\frac{1}{2}s+q}},$$

which, if ρ is a function of r only, is

$$=2\pi\int\frac{\rho r\, dr}{(r^2+e^2)^{\frac{1}{2}s+q}};$$

and in particular if $\rho=r^{s-2}$, then the value is

$$=2\pi\int\frac{r^{s-1}\, dr}{(r^2+e^2)^{\frac{1}{2}s+q}},$$

the integral in regard to r being taken from $r=0$ to $r=R$. It is assumed that $s-1$ is not negative, viz. it is positive or (it may be) zero. I consider the integral

$$\int_0^R\frac{r^{s-1}\, dr}{(r^2+e^2)^{\frac{1}{2}s+q}},$$

which I call the r-integral, more particularly in the case where e is small in comparison with R. It is to be observed that e not being $=0$, and R being finite, the integral contains no infinite element, and is therefore finite, whether q is positive, negative, or zero.

54. Writing $r=e\sqrt{v}$, the integral is

$$=\tfrac{1}{2}e^{-2q}\int\frac{v^{\frac{1}{2}s-1}\, dv}{(1+v)^{\frac{1}{2}s+q}},$$

the limits being $\frac{R^2}{e^2}$ and 0.

In the case where q is positive this is

$$=\tfrac{1}{2}e^{-2q}\left(\int_0^\infty-\int_{\frac{R^2}{e^2}}^\infty\right)\frac{v^{\frac{1}{2}s-1}\, dv}{(1+v)^{\frac{1}{2}s+q}};$$

viz. the first term of this is

$$=\tfrac{1}{2}e^{-2q}\frac{\Gamma\tfrac{1}{2}s\,.\,\Gamma q}{\Gamma(\tfrac{1}{2}s+q)},$$

and the second term is a term expansible in a series containing the powers $2q$, $2q+2$, &c of the small quantity $\frac{e^2}{R^2}$, as appears by effecting therein the substitution $v=\frac{1}{x}$; viz. the value of the entire integral is by this means found to be

$$=\tfrac{1}{2}e^{-2q}\left\{\frac{\Gamma\tfrac{1}{2}s\,.\,\Gamma q}{\Gamma(\tfrac{1}{2}s+q)}-\int_0^{\frac{e^2}{R^2}}\frac{x^{q-1}\, dx}{(1+x)^{\frac{1}{2}s+q}}\right\}.$$

55. In the case where q is $=0$, or negative, the formula fails by reason that the element $\frac{v^{\frac{1}{2}s-1}\, dv}{(1+v)^{\frac{1}{2}s+q}}$ of the integrals $\int_0^\infty$, $\int_{\frac{R^2}{e^2}}^\infty$ becomes infinite for indefinitely large values of v. Recurring to the original form $\int_0^R\frac{r^{s-1}\, dr}{(r^2+e^2)^{\frac{1}{2}s+q}}$, it is to be observed that the integral has a

finite value when $e=0$; and it might therefore at first sight be imagined that the factor $(r^2+e^2)^{-\frac{1}{2}s-q}$ might be expanded in ascending powers of e^2, and the value of the integral consequently obtained as a series of positive powers of e^2. But the series thus obtained is of the form $e^{2k}\int_0^R r^{-2q-2k-1}\,dr$, where $2q$ being positive, the exponent $-2q-2k-1$ is for a sufficiently small value of k at first positive, or if negative less than -1, and the value of the integral is finite; but as k increases the exponent becomes negative, and equal or greater than -1, and the value of the integral is then infinite. The inference is that the series *commences* in the form $A+Be^2+Ce^4\ldots$, but that we come at last when q is fractional to a term of the form Ke^{-2q}, and when q is $=0$, or integral, to a term of the form $Ke^{-2q}\log e$, the process giving the coefficients A, B, C .., so long as the exponent of the corresponding term $e^0, e^2, e^4\ldots$ is less than $-2q$ (in particular $q=0$, there is a term $k\log e$, *and the expansion-process does not give any term of the result*), and the failure of the series after this point being indicated by the values of the subsequent coefficients coming out $=\infty$.

56. In illustration, we may consider any of the cases in which the integral can be obtained in finite terms. For instance,

$s=2$, $q=-\frac{3}{2}$,

$$\text{Integral is }\int r(r^2+e^2)^{\frac{1}{2}}\,dr,=\tfrac{1}{3}(r^2+e^2)^{\frac{3}{2}},\text{ from 0 to R},$$
$$=\tfrac{1}{3}(R^2+e^2)^{\frac{3}{2}}-\tfrac{1}{3}e^3;$$

viz. expanding in ascending powers of e this is

$$=\tfrac{1}{3}R^3+\tfrac{1}{2}Re^2\ldots-\tfrac{1}{3}e^3,$$

or we have here a term in e^3. And so,

$s=1$, $q=-2$,

$$\text{Integral is }\int(r^2+e^2)^{\frac{3}{2}}dr,=(\tfrac{1}{4}r^3+\tfrac{5}{8}e^2)r\sqrt{r^2+e^2}+\tfrac{3}{8}e^4\log\left(r+\sqrt{r^2+e^2}\right),\text{ from 0 to R},$$
$$=(\tfrac{1}{4}R^3+\tfrac{5}{8}e^2)R\sqrt{R^2+e^2}+\tfrac{3}{8}e^4\log\frac{R+\sqrt{R^2+e^2}}{e},$$

viz. expanding in ascending powers of e this is

$$=\tfrac{1}{4}R^4+\tfrac{3}{4}R^2e^2\ldots+\tfrac{3}{8}e^4\log\frac{R}{e}\,{}^*,$$

or we have here a term in $e^4\log e$.

57. Returning to the form

$$\tfrac{1}{2}e^{-2q}\int_0^{\frac{R^2}{e^2}}\frac{v^{\frac{1}{2}s-1}dv}{(1+v)^{\frac{1}{2}s+q}},$$

and writing herein $v=\frac{1-x}{x}$, or, what is the same thing, $x=\frac{1}{1+v}$, and for shortness

* Term is $\frac{3}{8}e^4\log\frac{2R}{e}$, $=\frac{3}{8}e^4\left(\log\frac{R}{e}+\log 2\right)$, which, $\frac{R}{e}$ being large, is reduced to $\frac{3}{8}e^4\log\frac{R}{e}$

$X=\frac{e^2}{e^2+R^2}, =\frac{1}{1+\frac{R^2}{e^2}}$, the value is

$$=\tfrac{1}{2}e^{-2q}\int_X^1 x^{q-1}(1-x)^{\frac{1}{2}s-1}dx,$$

where observe that $q-1$ is 0 or negative, but X being a positive quantity less than 1, the function $x^{q-1}(1-x)^{\frac{1}{2}s-1}$ is finite for the whole extent of the integration.

58. If $q=0$, this is

$$=\tfrac{1}{2}\int_X^1 \frac{1-\{1-(1-x)^{\frac{1}{2}s-1}\}dx}{x}$$

$$=\tfrac{1}{2}\log X-\tfrac{1}{2}\int_X^1 \frac{\{1-(1-x)^{\frac{1}{2}s-1}\}dx}{x}$$

$$=\tfrac{1}{2}\log\sqrt{1+\frac{R^2}{e^2}}-\tfrac{1}{2}\int_0^1 \frac{\{1-(1-x)^{\frac{1}{2}s-1}\}dx}{x}+\tfrac{1}{2}\int_0^X \frac{\{1-(1-x)^{\frac{1}{2}s-1}\}dx}{x},$$

where observe that in virtue of the change made from $\frac{1}{x}(1-x)^{\frac{1}{2}s-1}$ to $\frac{1}{x}\{1-(1-x)^{\frac{1}{2}s-1}\}$ (a function which becomes infinite, to one which does not become infinite, for $x=0$), it has become allowable in place of $\int_X^1$ to write $\int_0^1-\int_0^X$.

When e is small, the integral which is the third term of the foregoing expression is obviously a quantity of the order e^2; the first term is $\tfrac{1}{2}\left(\log\frac{R}{e}+\log\sqrt{1+\frac{e^2}{R^2}}\right)$, which, neglecting terms in e^2, is $=\tfrac{1}{2}\log\frac{R}{e}$, and hence the approximate value of the r-integral $\int_0^R \frac{r^{s-1}dr}{(r^2+e^2)^{\frac{1}{2}s}}$ is

$$=\log\frac{R}{e}-\tfrac{1}{2}\int_0^1 dx\,\frac{1-(1-x)^{\frac{1}{2}s-1}}{x},$$

or, what is the same thing, it is

$$=\log\frac{R}{e}-\tfrac{1}{2}\int_0^1 dy\,\frac{1-y^{\frac{1}{2}s-1}}{1-y},$$

where the integral in this expression is a mere numerical constant, which, when $\tfrac{1}{2}s-1$ is a positive integer, has the value

$$\tfrac{1}{1}+\tfrac{1}{2}\ldots+\frac{1}{\frac{1}{2}s-1};$$

and neglecting this in comparison with the logarithmic term, the approximate value is

$$=\log\frac{R}{e}.$$

59. I consider also the case $q=-\frac{1}{2}$; the integral is here

$$\begin{aligned}
&\tfrac{1}{2}e\int_{\mathrm{X}}^{1} x^{-\frac{3}{2}}(1-x)^{\frac{1}{2}s-1}dx\\
&=\tfrac{1}{2}e\int_{\mathrm{X}}^{1} x^{-\frac{3}{2}}(1-\{1-(1-x)^{\frac{1}{2}s-1}\})dx\\
&=e(\mathrm{X}^{-\frac{1}{2}}-1)+\tfrac{1}{2}e\int_{\mathrm{X}}^{1} x^{-\frac{3}{2}}\{1-(1-x)^{\frac{1}{2}s-1}\}\,dx\,;
\end{aligned}$$

and the first term of this being $=\sqrt{e^2+\mathrm{R}^2}-e$, this is consequently

$$=\sqrt{\mathrm{R}^2+e^2}+\tfrac{1}{2}e\int_0^{\mathrm{X}} x^{-\frac{3}{2}}\{1-(1-x)^{\frac{1}{2}s-1}\}\,dx-e\Big(1+\tfrac{1}{2}\int_0^1 x^{-\frac{3}{2}}\{1-(1-x)^{\frac{1}{2}s-1}\}dx\Big).$$

As regards the second term of this we have

$$-2x^{-\frac{1}{2}}\{1-(1-x)^{\frac{1}{2}s-1}\}+2(\tfrac{1}{2}s-1)\int x^{-\frac{1}{2}}(1-x)^{\frac{1}{2}s-2}dx=\int x^{-\frac{3}{2}}\{1-(1-x)^{\frac{1}{2}s-1}\}dx\,;$$

or taking each term between the limits 1, 0,

$$-2+2(\tfrac{1}{2}s-1)\frac{\Gamma\frac{1}{2}\Gamma(\frac{1}{2}s-1)}{\Gamma(\frac{1}{2}s-\frac{1}{2})}=\int_0^1 x^{-\frac{3}{2}}\{1-(1-x)^{\frac{1}{2}s-1}\}dx\,;$$

viz. this integral has the value

$$-2+\frac{2\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s-\frac{1}{2})}\,;$$

and the value of the r-integral $\displaystyle\int_0^{\mathrm{R}}\frac{r^{s-1}dr}{(r^2+e^2)^{\frac{1}{2}s-\frac{1}{2}}}$ is consequently

$$=\sqrt{\mathrm{R}^2+e^2}+\tfrac{1}{2}e\int_0^{\mathrm{X}} x^{-\frac{3}{2}}\{1-(1-x)^{\frac{1}{2}s-1}\}dx-e\frac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s-\frac{1}{2})},$$

which is of the form

$$\mathrm{R}\left\{1+\text{terms in }\frac{e^2}{\mathrm{R}^2},\frac{e^4}{\mathrm{R}^4},\ldots\right\}-e\frac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s-\frac{1}{2})},$$

say the approximate value is

$$\mathrm{R}-e\frac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s-\frac{1}{2})},$$

where the first term R is the term $\displaystyle\int_0^{\mathrm{R}} dr$, given by the expansion in ascending powers of e^2; the second term is the term in e^{-2q}. And observe that term is the value of

$$\tfrac{1}{2}e\int_0^1 x^{-\frac{3}{2}}(1-x)^{\frac{1}{2}s-1}dx,$$

calculated by means of the ordinary formula for a Eulerian integral (which formula, on account of the negative exponent $-\frac{3}{2}$, is not really applicable, the value of the integral being $=\infty$) on the assumption that the Γ of a negative q is interpreted in accordance with the equation $\Gamma(q+1)=q\Gamma q$; viz. the value thus calculated is

$$=\tfrac{1}{2}e\frac{\Gamma(-\frac{1}{2})\Gamma(\frac{1}{2}s)}{\Gamma(\frac{1}{2}s-\frac{1}{2})},\ =-e\frac{\Gamma\frac{1}{2}\Gamma\frac{1}{2}s}{\Gamma(\frac{1}{2}s-\frac{1}{2})}$$

on the assumption $\Gamma\frac{1}{2}=-\frac{1}{2}\Gamma(-\frac{1}{2})$; and this agrees with the foregoing value.

60. It is now easy to see in general how the foregoing transformed value $\frac{1}{2}e^{-sq}\int_{\mathrm{X}}^{1}x^{q-1}(1-x)^{\frac{1}{2}s-1}dx$, where q is negative and fractional, gives at once the value of the term in e^{-sq}. Observe that in the integral x is always between 1 and $\mathrm{X}\ (=\frac{e^2}{e^2+\mathrm{R}^2}$, a positive quantity less than 1), the function to be integrated never becomes infinite. Imagine for a moment an integral $\int_{\mathrm{X}}^{1}x^{\alpha}dx$, where α is positive or negative. We may conventionally write this $=\int_{0}^{1}x^{\alpha}dx-\int_{0}^{\mathrm{X}}x^{\alpha}dx$, understanding the first symbol to mean $\frac{1^{1+\alpha}}{1+\alpha}$, and the second to mean $\frac{\mathrm{X}^{1+\alpha}}{1+\alpha}$, they of course properly mean $\frac{1^{1+\alpha}-0^{1+\alpha}}{1+\alpha}$ and $\frac{\mathrm{X}^{1+\alpha}-0^{1+\alpha}}{1+\alpha}$; but the terms in $0^{1+\alpha}$, whether zero or infinite, destroy each other, the original form $\int_{\mathrm{X}}^{1}x^{\alpha}dx$, in fact, showing that no such terms can appear in the result.

In accordance with the convention we write

$$\int_{\mathrm{X}}^{1}x^{q-1}(1-x)^{\frac{1}{2}s-1}dx=\int_{0}^{1}x^{q-1}(1-x)^{\frac{1}{2}s-1}dx-\int_{0}^{\mathrm{X}}x^{q-1}(1-x)^{\frac{1}{2}s-1}dx\,;$$

and it follows that the term in e^{-sq} is

$$\tfrac{1}{2}e^{-sq}\int_{0}^{1}x^{q-1}(1-x)^{\frac{1}{2}s-1}dx,$$

this last expression (wherein q, it will be remembered, is a negative fraction) being understood according to the convention; and so understanding it the value of the term is

$$=\tfrac{1}{2}e^{-sq}\frac{\Gamma\frac{1}{2}s\Gamma q}{\Gamma(\frac{1}{2}s+q)},$$

where the Γ of the negative q is to be interpreted in accordance with the equation $\Gamma(q+1)=q\Gamma q$; viz. we have $\Gamma q=\frac{1}{q}\Gamma(q+1),\ =\frac{1}{q(q+1)}\Gamma(q+2)$, &c., so as to make the argument of the Γ positive. Observe that under this convention we have

$$\Gamma q\Gamma(1-q)=\frac{\Gamma^2(\frac{1}{2})}{\sin q\pi},\text{ or the term is }\tfrac{1}{2}e^{-sq}\,.\,\frac{\Gamma^2(\frac{1}{2})}{\sin q\pi}\,\frac{\Gamma\frac{1}{2}s}{\Gamma(\frac{1}{2}s+q)\Gamma(1-q)}.$$

61. An example in which $\frac{1}{2}s-1$ is integral will make the process clearer, and will serve instead of a general proof. Suppose $q=-\frac{1}{7}$, $\frac{1}{2}s-1=4$, the expression

$$\int_{0}^{1}x^{-\frac{8}{7}}(1-x)^4\,dx=\int_{0}^{1}(x^{-\frac{8}{7}}-4x^{-\frac{1}{7}}+6x^{\frac{6}{7}}-4x^{\frac{13}{7}}+x^{\frac{20}{7}})\,dx$$

is used to denote the value

$$-7-\tfrac{14}{3}+\tfrac{42}{13}-\tfrac{7}{5}+\tfrac{7}{27}$$

$$=7(-1-\tfrac{2}{3}+\tfrac{6}{13}-\tfrac{1}{5}+\tfrac{1}{27}),\ =7(-\tfrac{44}{27}-\tfrac{1}{5}+\tfrac{6}{13}),\ =\frac{-7\,.\,2401}{5\,.\,13\,.\,27},\ =\frac{-7^5}{5\,.\,13\,.\,27}.$$

But we have

$$\frac{\Gamma\tfrac{1}{2}s\,\Gamma q}{\Gamma(\tfrac{1}{2}s+q)}=\frac{\Gamma 5\,\Gamma(-\tfrac{1}{7})}{\Gamma(5-\tfrac{1}{7})}=\frac{24\,\Gamma(-\tfrac{1}{7})}{\tfrac{27}{7}\,.\,\tfrac{20}{7}\,.\,\tfrac{13}{7}\,.\,\tfrac{6}{7}\,.\,\tfrac{-1}{7}\,\Gamma(-\tfrac{1}{7})}=\frac{-7^5}{5\,.\,13\,.\,27},$$

agreeing with the former value.

62. The case of a negative integer is more simple, to find the logarithmic term of

$$\tfrac{1}{2}e^{-2q}\int_{\mathrm{X}}^{1}x^{q-1}(1-x)^{\frac{1}{2}s-1}\,dx,$$

we have only to expand the factor $(1-x)^{\frac{1}{2}s-1}$ so as to obtain the term involving x^{-q}; we have thus the term

$$\tfrac{1}{2}e^{-2q}\int_{\mathrm{X}}^{1}x^{q-1}(-)^q\,\frac{\Gamma\tfrac{1}{2}s}{\Gamma(1-q)\,\Gamma(\tfrac{1}{2}s+q)}\,x^{-q}\,dx$$

$$=\tfrac{1}{2}(-)^q\,e^{-2q}\frac{\Gamma\tfrac{1}{2}s}{\Gamma(1-q)\,\Gamma(\tfrac{1}{2}s+q)}\log\frac{1}{\mathrm{X}},$$

where $\log\dfrac{1}{\mathrm{X}}=\log\left(1+\dfrac{\mathrm{R}^2}{e^2}\right)$, $=2\log\dfrac{\mathrm{R}}{e}+2\log\sqrt{1+\dfrac{e^2}{\mathrm{R}^2}}$, so that neglecting the terms in $\dfrac{e^2}{\mathrm{R}^2}$ &c this is $=2\log\dfrac{\mathrm{R}}{e}$, and the term in question is

$$=(-)^q\,e^{-2q}\frac{\Gamma\tfrac{1}{2}s}{\Gamma(1-q)\,\Gamma(\tfrac{1}{2}s+q)}\log\frac{\mathrm{R}}{e}.$$

The general conclusion is that q being negative, the r-integral

$$\int_0^{\mathrm{R}}\frac{r^{s-1}\,dr}{(r^2+e^2)^{\frac{1}{2}s+q}}$$

has for its value a series proceeding in powers of e^2, and which up to a certain point is equal to the series obtained by expanding in ascending powers of e^2 and integrating each term separately, viz. the series to the point in question is

$$\frac{\mathrm{R}^{-2q}}{-2q}-\frac{\tfrac{1}{2}s+q}{1}\,\frac{\mathrm{R}^{-2q-2}}{-2q-2}\,e^2+\frac{\tfrac{1}{2}s+q\,.\,\tfrac{1}{2}s+q+1}{1\,.\,2}\,\frac{\mathrm{R}^{-2q-4}}{-2q-4}\,e^4\ldots,$$

continued so long as the exponent of e is less than $-2q$, together with a term $\mathrm{K}e^{-2q}$ when q is fractional, and $\mathrm{K}e^{-2q}\log\dfrac{\mathrm{R}}{e}$ when q is integral, viz. q fractional this term is

$$=\tfrac{1}{2}e^{-2q}\,\frac{\Gamma\tfrac{1}{2}s\,\Gamma q}{\Gamma(\tfrac{1}{2}s+q)},\ =\tfrac{1}{2}e^{-2q}\,\frac{\Gamma^2\tfrac{1}{2}}{\sin q\pi}\,\frac{\Gamma\tfrac{1}{2}s}{\Gamma(\tfrac{1}{2}s+q)\,\Gamma(1-q)},$$

and q integral, it is

$$=(-)^q\,e^{-2q}\frac{\Gamma\tfrac{1}{2}s}{\Gamma(1-q)\,\Gamma(\tfrac{1}{2}s+q)}\log\frac{\mathrm{R}}{e}.$$

63. It has been tacitly assumed that $\tfrac{1}{2}s+q$ is positive; but the formulæ hold good if $\tfrac{1}{2}s+q$ is $=0$ or negative. Suppose $\tfrac{1}{2}s+q$ is 0 or a negative integer, then $\Gamma(\tfrac{1}{2}s+q)=\infty$, and the special term involving e^{-2q} or $e^{-2q}\log e$ vanishes; in fact in this case the r-integral is

$$=\int_0^{\mathrm{R}}r^{\frac{1}{2}s-1}(r^2+e^2)^{-(\frac{1}{2}s+q)}\,dr,$$

where $(r^2+e^2)^{-(\frac{1}{2}s+q)}$ has for its value a finite series, and the integral is therefore equal to a finite series $A+Be^2+Ce^4+\&c.$ If $\frac{1}{2}s+q$ be fractional, then the Γ of the negative quantity $\frac{1}{2}s+q$ must be understood as above, or, what is the same thing, we may, instead of $\Gamma(\frac{1}{2}s+q)$, write $\frac{(\Gamma\frac{1}{2})^2}{\sin(\frac{1}{2}s+q)\pi\,\Gamma(1-q-\frac{1}{2}s)}$; thus, q being integral, the exceptional term is

$$=(-)^q e^{-2q}\frac{\Gamma\frac{1}{2}s\sin(\frac{1}{2}s+q)\pi}{(\Gamma\frac{1}{2})^2\,\Gamma(1-q)}\frac{\Gamma(1-q-\frac{1}{2}s)}{}\log\frac{R}{e};$$

for instance, $s=1$, $q=-2$, the term is

$$\tfrac{1}{2}e^4\frac{\Gamma\frac{1}{2}\sin(-\frac{3}{2}\pi)\,\Gamma\frac{5}{2}}{(\Gamma\frac{1}{2})^2\,.\,\Gamma3}\log\frac{R}{e};$$

or, since $\Gamma\frac{5}{2}=\frac{3}{2}.\frac{1}{2}\Gamma\frac{1}{2}$, and $\Gamma3=2$, the term is $+\frac{3}{8}e^4\log\frac{R}{e}$, agreeing with a preceding result.

ANNEX III. *Prepotentials of Uniform Spherical Shell and Solid Sphere.*—Nos. 64 to 92.

64. The prepotentials in question depend ultimately upon two integrals, which also arise, as will presently appear, from prepotential problems in two-dimensional space, and which are for convenience termed the ring-integral and the disk-integral respectively. The analytical investigation in regard to these, depending as it does on a transformation of a function allied with the hypergeometric series, is I think interesting.

65. Consider first the prepotential of a uniform $(s+1)$dimensional spherical shell. This is

$$V=\int\frac{dS}{\{(a-x)^2\ldots+(c-z)^2+(e-w)^2\}^{\frac{1}{2}s+q}},$$

the equation of the surface being $x^2\ldots+z^2+w^2=f^2$, and there are the two cases of an internal point, $a^2\ldots+c^2+e^2<f^2$, and an external point, $a^2\ldots+c^2+e^2>f^2$.

The value is a function of $a^2\ldots+c^2+e^2$, say this is $=\kappa^2$; and taking the axes so that the coordinates of the attracted point are $(0\ldots0,\kappa)$, the integral is

$$=\int\frac{dS}{\{x^2\ldots+z^2+(\kappa-w)^2\}^{\frac{1}{2}s+q}},$$

where the equation of the surface is still $x^2\ldots+z^2+w^2=f^2$. Writing $x=f\xi\ldots z=f\zeta$, $w=f\omega$, where $\xi^2\ldots+\zeta^2+\omega^2=1$, we have $dS=\frac{f^s\,d\xi\ldots d\zeta}{\omega}$, or the integral is

$$=f^s\int\frac{d\xi\ldots d\zeta}{\omega(f^2-2\kappa f\omega+\kappa^2)^{\frac{1}{2}s+q}}.$$

Assume $\xi=px,\ldots\zeta=pz$, where $x^2\ldots+z^2=1$; then $p^2+\omega^2=1$. Moreover, $d\xi\ldots d\zeta$, $=p^{s-1}\,dp\,d\Sigma$, where $d\Sigma$ is the element of surface of the s-dimensional unit-sphere $x^2\ldots+z^2=1$; or for p, substituting its value $\sqrt{1-\omega^2}$, we have $dp=\frac{-\omega d\omega}{\sqrt{1-\omega^2}}$; and thence

$d\xi \ldots d\zeta = -(1-\omega^2)^{\frac{1}{2}s-1}\,\omega\, d\omega\, d\Sigma$. The integral as regards p is from $p=-1$ to $+1$, or as regards ω from 1 to -1; whence reversing the sign the integral will be from $\omega=-1$ to $+1$; and the required integral is thus

$$=f^s\int_{-1}^{1}\frac{(1-\omega^2)^{\frac{1}{2}s-1}\,d\omega\, d\Sigma}{(f^2-2\kappa f\omega+\kappa^2)^{\frac{1}{2}s+q}}=f^s\int d\Sigma\int_{-1}^{1}\frac{(1-\omega^2)^{\frac{1}{2}s-1}\,d\omega}{(f^2-2\kappa f\omega+\kappa^2)^{\frac{1}{2}s+q}},$$

where $\int d\Sigma$ is the surface of the s-dimensional unit-sphere (see Annex I.), $=\frac{2(\Gamma\frac{1}{2})^s}{\Gamma\frac{1}{2}s}$, and for greater convenience transforming the second factor by writing therein $\omega=\cos\theta$, the required integral is $=\frac{(\Gamma\frac{1}{2})^s}{\Gamma(\frac{1}{2}s)}$ into

$$2f^s\int_0^{\pi}\frac{\sin^{s-1}\theta\, d\theta}{(f^2-2\kappa f\cos\theta+\kappa^2)^{\frac{1}{2}s+q}},$$

which last expression $\left(\text{including the factor } 2f^s, \text{but without the factor } \frac{(\Gamma\frac{1}{2})^s}{(\Gamma\frac{1}{2}s)}\right)$ is the ring-integral discussed in the present Annex. It may be remarked that the value can be at once obtained in the particular case $s=2$, which belongs to tridimensional space, viz. we then have

$$\begin{aligned}V&=2\pi f^2\int_0^{\pi}\frac{\sin\theta\, d\theta}{(f^2-2\kappa f\cos\theta+\kappa^2)^{q+1}}\\&=\frac{2\pi f^2}{2\kappa f q}(f^2-2\kappa f\cos\theta+\kappa^2)^{-q}\\&=\frac{\pi f}{\kappa q}\{(f-\kappa)^{-2q}-(f+\kappa)^{-2q}\},\end{aligned}$$

which agrees with a result given, 'Mécanique Céleste,' Book XII. Chap. II.

66. Consider next the prepotential of the uniform solid $(s+1)$dimensional sphere,

$$V=\int\frac{dx\ldots dz dw}{\{(a-x)^2 \ldots +(c-z)^2+(e-w)^2\}^{\frac{1}{2}s+q}},$$

equation of surface $x^2\ldots+z^2+w^2=f^2$, and the two cases of an internal point $\kappa<f$, and an external point $\kappa>f$ $(a^2 \ldots +c^2+e^2=\kappa^2$ as before).

Transforming so that the coordinates of the attracted point are $0\ldots 0, \kappa$, the integral is

$$=\int\frac{dx \ldots dz dw}{\{x^2\ldots+z^2+(\kappa-w)^2\}^{\frac{1}{2}s+q}},$$

where the equation is still $x^2 \ldots +z^2+w^2=f^2$. Writing here $x=r\xi \ldots z=r\zeta$, where $\xi^2\ldots+\zeta^2=1$, we have $dx \ldots dz=r^{s-1}dr d\Sigma$, where $d\Sigma$ is an element of surface of the s-dimensional unit-sphere $\xi^2\ldots+\zeta^2=1$; the integral is therefore

$$\begin{aligned}&=\int\frac{r^{s-1}dr d\Sigma dw}{\{r^2+(\kappa-w)^2\}^{\frac{1}{2}s+q}}\\&=\int d\Sigma\int\frac{r^{s-1}dr dw}{\{r^2+(\kappa-w)^2\}^{\frac{1}{2}s+q}},\end{aligned}$$

where, as regards r and w, the integration extends over the circle $r^2+w^2=f^2$. The value

of the first factor (see Annex I.) is $=\frac{2(\Gamma\frac{1}{2})^s}{\Gamma\frac{1}{2}s}$; and writing y, x in place of r, w respectively, the integral is $=\frac{2(\Gamma\frac{1}{2})^s}{\Gamma\frac{1}{2}s}$ into

$$\int\frac{y^{s-1}\,dx\,dy}{\{(x-\varkappa)^2+y^2\}^{\frac{1}{2}s+q}}$$

over the circle $x^2+y^2=f^2$; viz. this last expression $\left(\text{without the factor } \frac{2(\Gamma\frac{1}{2})^s}{\Gamma(\frac{1}{2}s)}\right)$ is the disk-integral discussed in the present Annex.

67. We find for the value in regard to an internal point $\varkappa<f$,

$$V=\frac{(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s+q)\Gamma(\frac{1}{2}-q)}f^{s+1}\int_0^\infty(t+f^2-\varkappa^2)^{\frac{1}{2}-q}t^{-\frac{1}{2}-q}(t+f^2)^{-\frac{1}{2}s+q-1}dt,$$

which in the particular case $q=-\frac{1}{2}$ is

$$=\frac{(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s-\frac{1}{2})}f^{s+1}\int_0^\infty(t+f^2-\varkappa^2)(t+f^2)^{-\frac{1}{2}s-\frac{3}{2}}dt\,;$$

viz. the integral in t is here

$$=\int_0^\infty\{(t+f^2)^{-\frac{1}{2}s-\frac{1}{2}}-\varkappa^2(t+f^2)^{-\frac{1}{2}s-\frac{3}{2}}\},\ =\frac{1}{f^{s+1}}\left(\frac{f^2}{\frac{1}{2}s-\frac{1}{2}}-\frac{\varkappa^2}{\frac{1}{2}s+\frac{1}{2}}\right),$$

or we have

$$V=\frac{(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s-\frac{1}{2})}\left(\frac{f^2}{\frac{1}{2}s-\frac{1}{2}}-\frac{\varkappa^2}{\frac{1}{2}s+\frac{1}{2}}\right).$$

It may be added that in regard to an external point $\varkappa>f$, the value is

$$V=\frac{(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s+q)\Gamma(\frac{1}{2}-q)}f^{s+1}\cdot\int_{\varkappa^2-f^2}^\infty(t+f^2-\varkappa^2)^{\frac{1}{2}-q}t^{-\frac{1}{2}-q}(t+f^2)^{-\frac{1}{2}s+q-1}dt,$$

which in the same case $q=-\frac{1}{2}$ is

$$=\frac{(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s-\frac{1}{2})}f^{s+1}\cdot\int_{\varkappa^2-f^2}^\infty(t+f^2-\varkappa^2)(t+f^2)^{-\frac{1}{2}s-\frac{3}{2}}dt,$$

where the t-integral is

$$=\int_{\varkappa^2-f^2}^\infty\{(t+f^2)^{-\frac{1}{2}s-\frac{1}{2}}-\varkappa^2(t+f^2)^{-\frac{1}{2}s-\frac{3}{2}}\}dt,\ =\frac{\varkappa^{-s+1}}{\frac{1}{2}s-\frac{1}{2}}-\frac{\varkappa^2\cdot\varkappa^{-s-1}}{\frac{1}{2}s+\frac{1}{2}},\ =\frac{\varkappa^{-s+1}}{\frac{1}{2}s-\frac{1}{2}\cdot\frac{1}{2}s+\frac{1}{2}}\,;$$

and the value of V is therefore

$$=\frac{(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s+\frac{1}{2})}\frac{f^{s+1}}{\varkappa^{s-1}}.$$

Recurring to the case of the internal point, then writing $\nabla=\frac{d^2}{da^2}\cdots+\frac{d^2}{dc^2}+\frac{d^2}{de^2}$, and observing that $\nabla(\varkappa^2)=4(\frac{1}{2}s+\frac{1}{2})$, we have

$$\nabla V=-\frac{4(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s-\frac{1}{2})}$$

(in particular for ordinary space $s+1=3$, or the value is $\frac{-4\pi^{\frac{3}{2}}}{\sqrt{\pi}}$, $=-4\pi$, which is right).

68. The integrals referred to as the ring-integral and the disk-integral arise also from the following integrals in two-dimensional space, viz. these are

$$\int \frac{y^{s-1}\,dS}{\{(x-\varkappa)^2+y^2\}^{\frac{1}{2}s+q}},\quad \int \frac{y^{s-1}\,dxdy}{\{(x-\varkappa)^2+y^2\}^{\frac{1}{2}s+q}},$$

in the first of which dS denotes an element of arc of the circle $x^2+y^2=f^2$, the integration being extended over the whole circumference, and in the second the integration extends over the circle $x^2+y^2=f^2$; y^{s-1} is written for shortness instead of $(y^2)^{\frac{1}{2}(s-1)}$, viz. this is considered as always positive, whether y is positive or negative; it is moreover assumed that $s-1$ is zero or positive.

Writing in the first integral $x=f\cos\theta$, $y=f\sin\theta$, the value is

$$=f^s \int \frac{(\sin\theta)^{s-1}\,d\theta}{(f^2-2\varkappa f\cos\theta+\varkappa^2)^{\frac{1}{2}s+q}};$$

viz. this represents the prepotential of the circumference of the circle, density varying as $(\sin\theta)^{s-1}$, in regard to a point $x=\varkappa$, $y=0$ in the plane of the circle; and similarly the second integral represents the prepotential of the circular disk, density of the element at the point $(x, y)=y^{s-1}$, in regard to the same point $x=\varkappa$, $y=0$, it being in each case assumed that the prepotential of an element of mass $\rho d\varpi$ upon a point at distance d is $=\frac{\rho d\varpi}{d^{s+2q}}$.

69. In the case of the circumference, it is assumed that the attracted point is not on the circumference, $\varkappa$ not $=f$, and the function under the integral sign, and therefore the integral itself, is in every case finite. In the case of the circle, if $\varkappa$ be an interior point, then if $2q-1$ be $=0$ or positive, the element at the attracted point becomes infinite, but to avoid this we consider not the potential of the whole circle, but the potential of the circle *less* an indefinitely small circle radius ϵ having the attracted point for its centre; which being so, the element under the integral sign, and consequently the integral itself, remains finite.

It is to be remarked that the two integrals are connected with each other; viz. the circle of the second integral being divided in rings by means of a system of circles concentric with the bounding circle $x^2+y^2=f^2$, then the prepotential of each ring or annulus is determined by an integral such as the first integral, or, analytically, writing in the second integral $x=r\cos\theta$, $y=r\sin\theta$, and therefore $dxdy=rdrd\theta$, the second integral is

$$=\int r^s\,dr \int \frac{(\sin\theta)^{s-1}\,d\theta}{(r^2+\varkappa^2-2\varkappa r\cos\theta)^{\frac{1}{2}s+q}},$$

viz. the integral in regard to θ is here the same function of r, $\varkappa$ that the first integral is of f, $\varkappa$, and the integration in regard to r is of course to be taken from $r=0$ to $r=f$. But the θ-integral is not in its original form such a function of r as to render possible the integration in regard to r; and I, in fact, obtain the second integral by a different and in some respects a better process.

70. Consider first the ring-integral, which writing therein as above $x=f\cos\theta$,

$y=f\sin\theta$, and multiplying by 2 in order that the integral, instead of being taken from 0 to 2π, may be taken from 0 to π, becomes

$$=2f^s\int\frac{(\sin\theta)^{s-1}\,d\theta}{(f^2-2\varkappa f\cos\theta+\varkappa^2)^{\frac{1}{2}s+q}}.$$

Write $\cos\frac{1}{2}\theta=\sqrt{x}$; then $\sin\frac{1}{2}\theta=\sqrt{1-x}$, $\sin\theta=2x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}$; $d\theta=-x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}}dx$; $\cos\theta=-1+2x$, $\theta=0$ gives $x=1$, $\theta=\pi$ gives $x=0$, and the integral is

$$=2^{s-1}f^s\int_0^1\frac{x^{\frac{1}{2}s-1}(1-x)^{\frac{1}{2}s-1}\,dx}{\{(f+\varkappa)^2-4\varkappa fx\}^{\frac{1}{2}s+q}},$$

$$=\frac{2^{s-1}f^s}{(f+\varkappa)^{s+2q}}\int_0^1\frac{x^{\frac{1}{2}s-1}(1-x)^{\frac{1}{2}s-1}\,dx}{(1-ux)^{\frac{1}{2}s+q}},$$

if for shortness $u=\frac{4\varkappa f}{(\varkappa+f)^2}$ (obviously $u<1$).

The integral in x is here an integral belonging to the general form

$$\Pi(\alpha,\beta,\gamma,u)=\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}(1-ux)^{-\gamma}dx,$$

viz. we have

$$\text{Ring-integral }=\frac{2^{s-1}f^s}{(f+\varkappa)^{s+2q}}\,\Pi(\tfrac{1}{2}s,\tfrac{1}{2}s,\tfrac{1}{2}s+q,u).$$

71 The general function $\Pi(\alpha,\beta,\gamma,u)$ is

$$\Pi(\alpha,\beta,\gamma,u)=\frac{\Gamma\alpha\Gamma\beta}{\Gamma(\alpha+\beta)}\,F(\alpha,\gamma,\alpha+\beta,u),$$

or, what is the same thing,

$$F(\alpha,\beta,\gamma,u)=\frac{\Gamma\gamma}{\Gamma\alpha\Gamma(\gamma-\alpha)}\,\Pi(\alpha,\gamma-\alpha,\beta,u),$$

and consequently transformable by means of various theorems for the transformation of the hypergeometric series, in particular the theorems

$$F(\alpha,\beta,\gamma,u)=F(\beta,\alpha,\gamma,u),$$
$$F(\alpha,\beta,\gamma,u)=(1-u)^{\gamma-\alpha-\beta}F(\gamma-\alpha,\gamma-\beta,\gamma,u);$$

and if $v=\left(\frac{1-\sqrt{1-u}}{1+\sqrt{1-u}}\right)^2$, or, what is the same thing, $u=\frac{4\sqrt{v}}{(1+\sqrt{v})^2}$, then

$$F(\alpha,\beta,2\beta,u)=(1+\sqrt{v})^{2\alpha}F(\alpha,\alpha-\beta+\tfrac{1}{2},v);$$

in verification observe that if $u=1$ then also $v=1$, and that with these values, calculating each side by means of the formula

$$F(\alpha,\beta,\gamma,1)=\frac{\Gamma\gamma\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}\qquad\Bigg|\qquad\Pi(\alpha,\beta,\gamma,1)=\frac{\Gamma\alpha\Gamma(\beta-\gamma)}{\Gamma(\alpha+\beta-\gamma)},$$

the resulting equation, $F(\alpha,\beta,2\beta,1)=2^{2\alpha}F(\alpha,\alpha-\beta+\frac{1}{2},\beta+\frac{1}{2},1)$, becomes

$$\frac{\Gamma2\beta\Gamma(\beta-\alpha)}{\Gamma(2\beta-\alpha)\Gamma\beta}=2^{2\alpha}\,\frac{\Gamma(\beta+\frac{1}{2})\Gamma(2\beta-2\alpha)}{\Gamma(2\beta-\alpha)\Gamma(\beta-\alpha+\frac{1}{2})},$$

that is

$$\frac{\Gamma 2\beta}{\Gamma\beta\Gamma(\beta+\frac{1}{2})}=2^{2\alpha}\frac{\Gamma(2\beta-2\alpha)}{\Gamma(\beta-\alpha)\Gamma(\beta-\alpha+\frac{1}{2})},$$

which is true, in virtue of the relation $\frac{\Gamma 2x\Gamma\frac{1}{2}}{\Gamma x\Gamma(x+\frac{1}{2})}=2^{2x-1}$.

72. The foregoing formulæ, and in particular the formula which I have written $F(\alpha, \beta, 2\beta, u)=(1+\sqrt{v})^{2\alpha}F(\alpha, \alpha-\beta+\frac{1}{2}, \beta+\frac{1}{2}, v)$, are taken from KUMMER's Memoir, "Ueber die hypergeometrische Reihe," *Crelle*, t. xv. (1836), viz. the formula in question is under a slightly different form, his formula (41) p. 76; the formula (43), p. 77, is intended to be equivalent thereto; but there is an error of transcription, $2\alpha-2\beta+1$, in place of $\beta+\frac{1}{2}$, which makes the formula (43) erroneous.

It may be remarked as to the formulæ generally, that although very probably $\Pi(\alpha, \beta, \gamma, u)$ may denote a proper function of u, whatever be the values of the indices (α, β, γ), and the various transformation-theorems hold good accordingly (the Γ-function of a negative argument being interpreted in the usual manner by means of the equation $\Gamma x=\frac{1}{x}\Gamma(1+x), =\frac{1}{x(x+1)}\Gamma(2+x)$ &c.), yet that the function $\Pi(\alpha, \beta, \gamma, u)$, used as denoting the definite integral $\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}(1-ux)^{-\gamma}dx$, has no meaning except in the case where α and β are each of them positive.

In what follows we obtain for the ring-integral and the disk-integral various expressions in terms of Π-functions, which are afterwards transformed into t-integrals with a superior limit ∞ and inferior limit 0, or κ^2-f^2, but for values of the variable index, q lying beyond certain limits, the indices α and β, or one of them, of the Π-function will become negative, viz. the integral represented by the Π-function, or, what is the same thing, the t-integral, will cease to have a determinate value, and at the same time, or usually so, the argument or arguments of one or more of the Γ-functions will become negative. It is quite possible that in such cases the results are not without meaning, and that an interpretation for them might be found, but they have not any obvious interpretation, and we must in the first instance consider them as inapplicable.

73. We require further properties of the Π-functions. Starting with the foregoing equation,

$$F(\alpha, \beta, 2\beta, u)=(1+\sqrt{v})^{2\alpha}F(\alpha, \alpha-\beta+\tfrac{1}{2}, \beta+\tfrac{1}{2}, v),$$

each side may be expressed in a fourfold form:—

	=
$F(\alpha, \beta, 2\beta, u)$	$(1+\sqrt{v})^{2\alpha}F(\alpha, \alpha-\beta+\frac{1}{2}, \beta+\frac{1}{2}, v)$
$=F(\beta, \alpha, 2\beta, u)$	$=(1+\sqrt{v})^{2\alpha}F(\alpha-\beta+\frac{1}{2}, \alpha, \beta+\frac{1}{2}, v)$
$=(1-u)^{\beta-\alpha}F(2\beta-\alpha, \beta, 2\beta, u)$	$=(1+\sqrt{v})^{2\alpha}(1-v)^{2\beta-2\alpha}F(\beta-\alpha+\frac{1}{2}, 2\beta-\alpha, \beta+\frac{1}{2}, v)$
$=(1-u)^{\beta-\alpha}F(\alpha, 2\beta-\alpha, 2\beta, u)$	$=(1+\sqrt{v})^{2\alpha}(1-v)^{2\beta-2\alpha}F(2\beta-\alpha, \beta-\alpha+\frac{1}{2}, \beta+\frac{1}{2}, v),$

where, instead of $(1+\sqrt{v})^{2\alpha}(1-v)^{2\beta-2\alpha}$, it is proper to write $(1+\sqrt{v})^{2\beta}(1-\sqrt{v})^{2\beta-2\alpha}$; and then to each form applying the transformation

$$F(\alpha,\beta,\gamma,u)=\frac{\Gamma\gamma}{\Gamma\alpha\Gamma(\gamma-\alpha)}\,\Pi(\alpha,\gamma-\alpha,\beta,u),$$

we have

$$\begin{aligned}
&\frac{\Gamma 2\beta}{\Gamma\alpha\Gamma(2\beta-\alpha)}\,\Pi(\alpha,2\beta-\alpha,\beta,u)\\
&=\frac{\Gamma 2\beta}{\Gamma\beta\,\Gamma\beta}\,\Pi(\beta,\beta,\alpha,u)\\
&=(1-u)^{\beta-\alpha}\frac{\Gamma 2\beta}{\Gamma(2\beta-\alpha)\Gamma\alpha}\,\Pi(2\beta-\alpha,\alpha,\beta,u)\\
&=(1-u)^{\beta-\alpha}\frac{\Gamma 2\beta}{\Gamma\alpha\Gamma(2\beta-\alpha)}\,\Pi(\alpha,2\beta-\alpha,2\beta-\alpha,u)
\end{aligned}$$

$$\begin{aligned}
&=(1+\sqrt{v})^{2\alpha}\frac{\Gamma(\beta+\frac{1}{2})}{\Gamma\alpha\Gamma(\beta-\alpha+\frac{1}{2})}\,\Pi(\alpha,\beta-\alpha+\tfrac{1}{2},\alpha-\beta+\tfrac{1}{2},v)\\
&=(1+\sqrt{v})^{2\alpha}\frac{\Gamma(\beta+\frac{1}{2})1}{\Gamma(\alpha-\beta+\frac{1}{2})\Gamma(2\beta-\alpha)}\,\Pi(\alpha-\beta+\tfrac{1}{2},2\beta-\alpha,\alpha,v)\\
&=(1+\sqrt{v})^{2\beta}(1-\sqrt{v})^{2\beta-2\alpha}\frac{\Gamma(\beta+\frac{1}{2})}{\Gamma(\beta-\alpha+\frac{1}{2})\Gamma\alpha}\,\Pi(\beta-\alpha+\tfrac{1}{2},\alpha,2\beta-\alpha,v)\\
&=(1+\sqrt{v})^{2\beta}(1-\sqrt{v})^{2\beta-2\alpha}\frac{\Gamma(\beta+\frac{1}{2})}{\Gamma(2\beta-\alpha)\Gamma(\alpha-\beta+\frac{1}{2})}\,\Pi(2\beta-\alpha,\alpha-\beta+\tfrac{1}{2},\beta-\alpha+\tfrac{1}{2},v).
\end{aligned}$$

I select on the left-hand the second form, and equating it successively to the four right-hand forms, attending to the relation $\frac{\Gamma\beta\Gamma(\beta+\frac{1}{2})}{\Gamma 2\beta}=2^{1-2\beta}\Gamma\frac{1}{2}$, we find

$$\begin{aligned}
\Pi(\beta,\beta,\alpha,u)&=(1+\sqrt{v})^{2\alpha}2^{1-2\beta}\frac{\Gamma\beta\Gamma\frac{1}{2}}{\Gamma\alpha\Gamma(\beta-\alpha+\frac{1}{2})}\,\Pi(\alpha,\beta-\alpha+\tfrac{1}{2},\alpha-\beta+\tfrac{1}{2},v)\\
&=(1+\sqrt{v})^{2\alpha}2^{1-2\beta}\frac{\Gamma\beta\Gamma\frac{1}{2}}{\Gamma(\alpha-\beta+\frac{1}{2})\Gamma(2\beta-\alpha)}\,\Pi(\alpha-\beta+\tfrac{1}{2},2\beta-\alpha,\alpha,v)\\
&=(1+\sqrt{v})^{2\beta}(1-\sqrt{v})^{2\beta-2\alpha}2^{1-2\beta}\frac{\Gamma\beta\Gamma\frac{1}{2}}{\Gamma(\beta-\alpha+\frac{1}{2})\Gamma\alpha}\,\Pi(\beta-\alpha+\tfrac{1}{2},\alpha,2\beta-\alpha,v)\\
&=(1+\sqrt{v})^{2\beta}(1-\sqrt{v})^{2\beta-2\alpha}2^{1-2\beta}\frac{\Gamma\beta\Gamma\frac{1}{2}}{\Gamma(2\beta-\alpha)\Gamma(\alpha-\beta+\frac{1}{2})}\,\Pi(2\beta-\alpha,\alpha-\beta+\tfrac{1}{2},\beta-\alpha+\tfrac{1}{2},v).
\end{aligned}$$

Putting herein $\beta=\frac{1}{2}s$, $\alpha=\frac{1}{2}s+q$, the formulæ become

$$\begin{aligned}
\Pi(\tfrac{1}{2}s,\tfrac{1}{2}s,\tfrac{1}{2}s+q,u)&=(1+\sqrt{v})^{s+2q}2^{1-s}\frac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s+q)\Gamma(\frac{1}{2}-q)}\,\Pi(\tfrac{1}{2}s+q,\tfrac{1}{2}-q,\tfrac{1}{2}+q,v)\ .\ \ .\ \ .\quad \text{(I.)}\\
&=(1+\sqrt{v})^{s+2q}2^{1-s}\frac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}+q)\Gamma(\frac{1}{2}s-q)}\,\Pi(\tfrac{1}{2}+q,\tfrac{1}{2}s-q,\tfrac{1}{2}s+q,v)\ \ .\ \ .\quad \text{(II.)}\\
&=(1+\sqrt{v})^{s}(1-\sqrt{v})^{-2q}2^{1-s}\frac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}-q)\Gamma(\frac{1}{2}s+q)}\,\Pi(\tfrac{1}{2}-q,\tfrac{1}{2}s+q,\tfrac{1}{2}s-q,v).\ \ .\ \ .\text{(III.)}\\
&=(1+\sqrt{v})^{s}(1-\sqrt{v})^{-2q}2^{1-s}\frac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s-q)\Gamma(\frac{1}{2}+q)}\,\Pi(\tfrac{1}{2}s-q,\tfrac{1}{2}+q,\tfrac{1}{2}-q,v),\ \ .\ \ .\text{(IV.)}
\end{aligned}$$

where observe that on the right-hand side the Π-functions of I. and IV. only differ by the sign of q, and so also the Π-functions of II. and III. only differ by the sign of q. We hence have

$$\Pi(\tfrac{1}{2}s, \tfrac{1}{2}s, \tfrac{1}{2}s-q, u)=(1+\sqrt{v})^{s-2q}\, 2^{1-s}\,\frac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s-q)\Gamma(\frac{1}{2}+q)}\,\Pi(\tfrac{1}{2}s-q, \tfrac{1}{2}+q, \tfrac{1}{2}-q, v);$$

and comparing with (IV.),

$$\Pi(\tfrac{1}{2}s, \tfrac{1}{2}s, \tfrac{1}{2}s+q, u)=\left(\frac{1+\sqrt{v}}{1-\sqrt{v}}\right)^{2q}\Pi(\tfrac{1}{2}s, \tfrac{1}{2}s, \tfrac{1}{2}s-q, u).$$

74. The foregoing formula,

$$\text{Ring-integral}=\frac{2^{s-1}f^{s}}{(f+\kappa)^{s+2q}}\,\Pi(\tfrac{1}{2}s, \tfrac{1}{2}s, \tfrac{1}{2}s+q, u),$$

where $u=\frac{4\kappa f}{(f+\kappa)^2}$, gives, as well in the case of an exterior as an interior point, a convergent series for the integral; but this series proceeds according to the powers of $\frac{4\kappa f}{(f+\kappa)^2}$. We may obtain more convenient formulæ applying to the cases of an internal and an external point respectively.

75. Internal point $\kappa<f$, $\sqrt{1-u}=\frac{f-\kappa}{f+\kappa}$, and therefore $v=\frac{\kappa^2}{f^2}$.

$$\begin{aligned}
\Pi(\tfrac{1}{2}s, \tfrac{1}{2}s, \tfrac{1}{2}s+q, u)&=\left(\frac{f+\kappa}{f}\right)^{s+2q}2^{1-s}\frac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s+q)\Gamma(\frac{1}{2}-q)}\;\Pi\left(\tfrac{1}{2}s+q, \tfrac{1}{2}-q, \tfrac{1}{2}+q, \frac{\kappa^2}{f^2}\right)\\
&=\left(\frac{f+\kappa}{f}\right)^{s+2q}2^{1-s}\frac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}+q)\Gamma(\frac{1}{2}s-q)}\;\Pi\left(\tfrac{1}{2}+q, \tfrac{1}{2}s-q, \tfrac{1}{2}s+q, \frac{\kappa^2}{f^2}\right)\\
&=\left(\frac{f+\kappa}{f}\right)^{s}\left(\frac{f-\kappa}{f}\right)^{-2q}2^{1-s}\frac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}-q)\Gamma(\frac{1}{2}s+q)}\;\Pi\left(\tfrac{1}{2}-q, \tfrac{1}{2}s+q, \tfrac{1}{2}s-q, \frac{\kappa^2}{f^2}\right)\\
&=\left(\frac{f+\kappa}{f}\right)^{s}\left(\frac{f-\kappa}{f}\right)^{-2q}2^{1-s}\frac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s-q)\Gamma(\frac{1}{2}+q)}\;\Pi\left(\tfrac{1}{2}s-q, \tfrac{1}{2}+q, \tfrac{1}{2}-q, \frac{\kappa^2}{f^2}\right),
\end{aligned}$$

where the Π-functions on the right-hand side are respectively

$$\begin{array}{l|l}
=f^{2q+1}\displaystyle\int_0^1\frac{x^{\frac{1}{2}s+q-1}(1-x)^{-q-\frac{1}{2}}\,dx}{(f^2-\kappa^2x)^{q+\frac{1}{2}}} & =\dfrac{f^{2q+1}}{(f^2-\kappa^2)^{2q}}\displaystyle\int_0^\infty t^{\frac{1}{2}s+q-1}(t+f^2-\kappa^2)^{-\frac{1}{2}s+q}(t+f^2)^{-q-\frac{1}{2}}\,dt\\
=f^{s+2q}\displaystyle\int_0^1\frac{x^{q-\frac{1}{2}}(1-x)^{\frac{1}{2}s-q-1}\,dx}{(f^2-\kappa^2x)^{\frac{1}{2}s+q}} & =\dfrac{f^{s+2q}}{(f^2-\kappa^2)^{2q}}\displaystyle\int_0^\infty t^{q-\frac{1}{2}}(t+f^2-\kappa^2)^{q-\frac{1}{2}}(t+f^2)^{-\frac{1}{2}s-q}\,dt\\
=f^{s-2q}\displaystyle\int_0^1\frac{x^{-q-\frac{1}{2}}(1-x)^{\frac{1}{2}s+q-1}\,dx}{(f^2-\kappa^2x)^{\frac{1}{2}s-q}} & =\dfrac{f^{s-2q}}{(f^2-\kappa^2)^{-2q}}\displaystyle\int_0^\infty t^{-q-\frac{1}{2}}(t+f^2-\kappa^2)^{-q-\frac{1}{2}}(t+f^2)^{-\frac{1}{2}s+q}\,dt\\
=f^{-2q+1}\displaystyle\int_0^1\frac{x^{\frac{1}{2}s-q+1}(1-x)^{q-\frac{1}{2}}\,dx}{(f^2-\kappa^2x)^{-q+\frac{1}{2}}} & =\dfrac{f^{-2q+1}}{(f^2-\kappa^2)^{-2q}}\displaystyle\int_0^\infty t^{\frac{1}{2}s-q-1}(t+f^2-\kappa^2)^{-\frac{1}{2}s-q}(t+f^2)^{q-\frac{1}{2}}\,dt,
\end{array}$$

the t-forms being obtained by means of the transformation $x=\frac{t}{t+f^2-\kappa^2}$; viz. this gives

$$1-x=\frac{f^2-\kappa^2}{t+f^2-\kappa^2},\ f^2-\kappa^2x=\frac{(f^2-\kappa^2)(t+f^2)}{t+f^2-\kappa^2},\ dx=\frac{(f^2-\kappa^2)\,dt}{(t+f^2-\kappa^2)^2},$$

whence the results just written down.

We hence have

$$\text{Ring-integral} = \frac{f}{(f^2-x^2)^{2q}}\,\frac{\Gamma\tfrac{1}{2}s\,\Gamma\tfrac{1}{2}}{\Gamma(\tfrac{1}{2}s+q)\Gamma(\tfrac{1}{2}-q)}\int_0^\infty t^{\frac{1}{2}s+q-1}(t+f^2-x^2)^{-\frac{1}{2}s+q}(t+f^2)^{-q-\frac{1}{2}}dt$$

$$= \frac{f^s}{(f^2-x^2)^{2q}}\,\frac{\Gamma\tfrac{1}{2}s\,\Gamma\tfrac{1}{2}}{\Gamma(\tfrac{1}{2}+q)\Gamma(\tfrac{1}{2}s-q)}\int_0^\infty t^{q-\frac{1}{2}}\;(t+f^2-x^2)^{q-\frac{1}{2}}\;(t+f^2)^{-\frac{1}{2}s-q}dt$$

$$= \quad f^s \quad \frac{\Gamma\tfrac{1}{2}s\,\Gamma\tfrac{1}{2}}{\Gamma(\tfrac{1}{2}-q)\Gamma(\tfrac{1}{2}s+q)}\int_0^\infty t^{-q-\frac{1}{2}}\;(t+f^2-x^2)^{-q-\frac{1}{2}}\;(t+f^2)^{-\frac{1}{2}s+q}dt$$

$$= \quad f \quad \frac{\Gamma\tfrac{1}{2}s\,\Gamma\tfrac{1}{2}}{\Gamma(\tfrac{1}{2}s-q)\Gamma(\tfrac{1}{2}+q)}\int_0^\infty t^{\frac{1}{2}s-q-1}(t+f^2-x^2)^{-\frac{1}{2}s-q}(t+f^2)^{q-\frac{1}{2}}\;dt.$$

As a verification write $x=0$, the four integrals are

$$\int_0^\infty \frac{t^{\frac{1}{2}s+q-1}\,dt}{(t+f^2)^{\frac{1}{2}s+\frac{1}{2}}}, \;= f^{2q-1}\;\frac{\Gamma(\tfrac{1}{2}s+q)\Gamma(\tfrac{1}{2}-q)}{\Gamma(\tfrac{1}{2}s+\tfrac{1}{2})},$$

$$\int_0^\infty \frac{t^{\frac{1}{2}+q-1}\,dt}{(t+f^2)^{\frac{1}{2}s+\frac{1}{2}}}, \;= f^{2q-s}\;\frac{\Gamma(\tfrac{1}{2}+q)\Gamma(\tfrac{1}{2}s-q)}{\Gamma(\tfrac{1}{2}s+\tfrac{1}{2})},$$

$$\int_0^\infty \frac{t^{\frac{1}{2}-q-1}\,dt}{(t+f^2)^{\frac{1}{2}s+\frac{1}{2}}}, \;= f^{-2q-s}\;\frac{\Gamma(\tfrac{1}{2}-q)\Gamma(\tfrac{1}{2}s+q)}{\Gamma(\tfrac{1}{2}s+\tfrac{1}{2})},$$

$$\int_0^\infty \frac{t^{\frac{1}{2}s-q-1}\,dt}{(t+f^2)^{\frac{1}{2}s+\frac{1}{2}}}, \;= f^{-2q-1}\;\frac{\Gamma(\tfrac{1}{2}s-q)\Gamma(\tfrac{1}{2}+q)}{\Gamma(\tfrac{1}{2}s+\tfrac{1}{2})},$$

and hence from each of them

$$\text{Ring-integral} = \frac{1}{f^{2q}}\,\frac{\Gamma\tfrac{1}{2}s\,\Gamma\tfrac{1}{2}}{\Gamma(\tfrac{1}{2}s+\tfrac{1}{2})},$$

which is in fact the value obtained from

$$\text{Ring-integral} = \frac{2^{s-1}f^s}{(f+\kappa)^{s+2q}}\,\Pi\left(\tfrac{1}{2}s,\,\tfrac{1}{2}s,\,\tfrac{1}{2}s+q,\,\frac{4\kappa f}{(\kappa+f)^2}\right)$$

on putting therein $\kappa=0$; viz. the value is

$$= \frac{2^{s-1}}{f^{2q}}\int_0^1 x^{\frac{1}{2}s-1}(1-x)^{\frac{1}{2}s-1}dx, \;= \frac{1}{f^{2q}}\,\frac{2^{s-1}\Gamma\tfrac{1}{2}s\,.\,\Gamma\tfrac{1}{2}s}{\Gamma s}.$$

76. External point $\kappa>f$, $\sqrt{1-u}=\frac{\kappa-f}{\kappa+f}$, and therefore $v=\frac{f^2}{\kappa^2}$.

$$\Pi(\tfrac{1}{2}s,\,\tfrac{1}{2}s,\,\tfrac{1}{2}s+q,\,u) = \left(\frac{\kappa+f}{\kappa}\right)^{s+2q} \qquad 2^{1-s}\frac{\Gamma\tfrac{1}{2}s\,\Gamma\tfrac{1}{2}}{\Gamma(\tfrac{1}{2}s+q)\Gamma(\tfrac{1}{2}-q)}\,\Pi\left(\tfrac{1}{2}s+q,\,\tfrac{1}{2}-q,\,\tfrac{1}{2}+q,\,\frac{f^2}{\kappa^2}\right)$$

$$= \left(\frac{\kappa+f}{\kappa}\right)^{s+2q} \qquad 2^{1-s}\frac{\Gamma\tfrac{1}{2}s\,\Gamma\tfrac{1}{2}}{\Gamma(\tfrac{1}{2}+q)\Gamma(\tfrac{1}{2}s-q)}\,\Pi\left(\tfrac{1}{2}+q,\,\tfrac{1}{2}s-q,\,\tfrac{1}{2}s+q,\,\frac{f^2}{\kappa^2}\right)$$

$$= \left(\frac{\kappa+f}{\kappa}\right)^{s}\left(\frac{\kappa-f}{\kappa}\right)^{-2q}2^{1-s}\frac{\Gamma\tfrac{1}{2}s\,\Gamma\tfrac{1}{2}}{\Gamma(\tfrac{1}{2}-q)\Gamma(\tfrac{1}{2}s+q)}\,\Pi\left(\tfrac{1}{2}-q,\,\tfrac{1}{2}s+q,\,\tfrac{1}{2}s-q,\,\frac{f^2}{\kappa^2}\right)$$

$$= \left(\frac{\kappa+f}{\kappa}\right)^{s}\left(\frac{\kappa-f}{\kappa}\right)^{-2q}2^{1-s}\frac{\Gamma\tfrac{1}{2}s\,\Gamma\tfrac{1}{2}}{\Gamma(\tfrac{1}{2}s-q)\Gamma(\tfrac{1}{2}+q)}\,\Pi\left(\tfrac{1}{2}s-q,\,\tfrac{1}{2}+q,\,\tfrac{1}{2}-q,\,\frac{f^2}{\kappa^2}\right),$$

where the Π-functions on the right hand are respectively

$$\begin{array}{l|l}
=\kappa^{2q+1}\displaystyle\int_0^1 \frac{x^{\frac12 s+q-1}(1-x)^{-q-\frac12}dx}{(\kappa^2-f^2x)^{q+\frac12}} & =\dfrac{\kappa^{2q+1}}{(\kappa^2-f^2)^{2q}}\displaystyle\int_{\kappa^2-f^2}^{\infty} t^{-\frac12 s+q}(t+f^2-\kappa^2)^{\frac12 s+q-1}(t+f^2)^{-q-\frac12}\,dt, \\
=\kappa^{s+2q}\displaystyle\int_0^1 \frac{x^{q-\frac12}(1-x)^{\frac12 s-q-1}dx}{(\kappa^2-f^2x)^{\frac12 s+q}} & =\dfrac{\kappa^{s+2q}}{(\kappa^2-f^2)^{2q}}\displaystyle\int_{\kappa^2-f^2}^{\infty} t^{q-\frac12}\ (t+f^2-\kappa^2)^{q-\frac12}\ (t+f^2)^{-\frac12 s-q}dt, \\
=\kappa^{s-2q}\displaystyle\int_0^1 \frac{x^{-q-\frac12}(1-x)^{\frac12 s+q-1}dx}{(\kappa^2-f^2x)^{\frac12 s-q}} & =\dfrac{\kappa^{s-2q}}{(\kappa^2-f^2)^{-2q}}\displaystyle\int_{\kappa^2-f^2}^{\infty} t^{-q-\frac12}(t+f^2-\kappa^2)^{-q-\frac12}\ (t+f^2)^{-\frac12 s+q}dt, \\
=\kappa^{-2q+1}\displaystyle\int_0^1 \frac{x^{\frac12 s-q+1}(1-x)^{q-\frac12}dx}{(\kappa^2-f^2x)^{-q+\frac12}} & =\dfrac{\kappa^{-2q+1}}{(\kappa^2-f^2)^{-2q}}\displaystyle\int_{\kappa^2-f^2}^{\infty} t^{-\frac12 s-q}(t+f^2-\kappa^2)^{\frac12 s-q-1}(t+f^2)^{q-\frac12}\ dt\,;
\end{array}$$

we have then

$$\begin{aligned}
\text{Ring-integral}&=\frac{f^s\kappa^{1-s}}{(\kappa^2-f^2)^{2q}}\,\frac{\Gamma\frac12 s\Gamma\frac12}{\Gamma(\frac12 s+q)\Gamma(\frac12-q)}\int_{\kappa^2-f^2}^{\infty} t^{-\frac12 s+q}(t+f^2-\kappa^2)^{\frac12 s+q-1}(t+f^2)^{-q-\frac12}\,dt \\
&=\frac{f^s}{(\kappa^2-f^2)^{2q}}\,\frac{\Gamma\frac12 s\Gamma\frac12}{\Gamma(\frac12+q)\Gamma(\frac12 s-q)}\int_{\kappa^2-f^2}^{\infty} t^{q-\frac12}\ (t+f^2-\kappa^2)^{q-\frac12}\ (t+f^2)^{-\frac12 s-q}dt \\
&=\ f^s\ \frac{\Gamma\frac12 s\Gamma\frac12}{\Gamma(\frac12-q)\Gamma(\frac12 s+q)}\int_{\kappa^2-f^2}^{\infty} t^{-q-\frac12}(t+f^2-\kappa^2)^{-q-\frac12}\ (t+f^2)^{-\frac12 s+q}dt \\
&=\frac{f^s\kappa^{1-s}}{(\kappa^2-f^2)^{2q}}\,\frac{\Gamma\frac12 s\Gamma\frac12}{\Gamma(\frac12 s-q)\Gamma(\frac12+q)}\int_{\kappa^2-f^2}^{\infty} t^{-\frac12 s-q}(t+f^2-\kappa^2)^{\frac12 s-q-1}(t+f^2)^{q-\frac12}\ dt
\end{aligned}$$

Observe that in II and III. the integrals, except as to the limits, are the same as in the corresponding formulæ for the interior point.

If in the t-integrals we put $t+\kappa^2-f^2$ in place of t, and ultimately suppose κ indefinitely large in comparison with f, they severally become

$$\begin{aligned}
&\int_0^{\infty}(t+\kappa^2-f^2)^{-\frac12 s+q}t^{\frac12 s+q-1}(t+\kappa^2)^{-q-\frac12}\,dt=\int_0^{\infty}\frac{t^{\frac12 s+q-1}dt}{(t+\kappa^2)^{\frac12 s+\frac12}}=\kappa^{2q-1}\ \frac{\Gamma(\frac12 s+q)\Gamma(\frac12-q)}{\Gamma(\frac12 s+\frac12)}, \\
&\int_0^{\infty}(t+\kappa^2-f^2)^{q-\frac12}\ \ t^{q+\frac12}\ \ (t+\kappa^2)^{-\frac12 s-q}dt=\int_0^{\infty}\frac{t^{\frac12+q-1}dt}{(t+\kappa^2)^{\frac12 s+\frac12}}=\kappa^{2q-s}\ \frac{\Gamma(\frac12+q)\Gamma(\frac12 s-q)}{\Gamma(\frac12 s+\frac12)}, \\
&\int_0^{\infty}(t+\kappa^2-f^2)^{-q-\frac12}\,t^{-q-\frac12}\ (t+\kappa^2)^{-\frac12 s+q}dt=\int_0^{\infty}\frac{t^{\frac12-q-1}dt}{(t+\kappa^2)^{\frac12 s+\frac12}}=\kappa^{-2q-s}\frac{\Gamma(\frac12-q)\Gamma(\frac12 s+q)}{\Gamma(\frac12 s+\frac12)}, \\
&\int_0^{\infty}(t+\kappa^2-f^2)^{-\frac12 s-q}t^{\frac12 s-q-1}(t+\kappa^2)^{q-\frac12}\ \ dt=\int_0^{\infty}\frac{t^{\frac12 s-q-1}dt}{(t+\kappa^2)^{\frac12 s+\frac12}}=\kappa^{-2q-1}\frac{\Gamma(\frac12 s-q)\Gamma(\frac12+q)}{\Gamma(\frac12 s+\frac12)}\ ;
\end{aligned}$$

and they all four give

$$\text{Ring-integral}=\frac{f^s}{\kappa^{s+2q}}\,\frac{\Gamma\frac12 s\Gamma\frac12}{\Gamma(\frac12 s+\frac12)},$$

which agrees with the value

$$\frac{2^{s-1}f^s}{(\kappa+f)^{s+2q}}\,\Pi\left(\tfrac12 s,\ \tfrac12 s,\ \tfrac12 s+q,\ \frac{4\kappa f}{(\kappa+f)^2}\right),\ \ =\frac{2^{s-1}f^s}{\kappa^{s+2q}}\,\Pi(\tfrac12 s,\ \tfrac12 s,\ \tfrac12 s+q,\ 0)$$

when $\dfrac{\kappa}{f}$ is indefinitely large.

77. We come now to the disk-integral,

$$\int \frac{y^{s-1}\,dx\,dy}{\{(x-\varkappa)^2+y^2\}^{\frac{1}{2}s+q}},$$

over the circle $x^2+y^2=f^2$. Writing $x=\varkappa+\rho\cos\phi, y=\rho\sin\phi$, we have $dx\,dy=\rho\,d\rho\,d\phi$, and the integral therefore is

$$\int \frac{\sin^{s-1}\phi\,d\rho\,d\phi}{\rho^{2q}},$$

where the integration in regard to ρ is performed at once, viz. the integral is

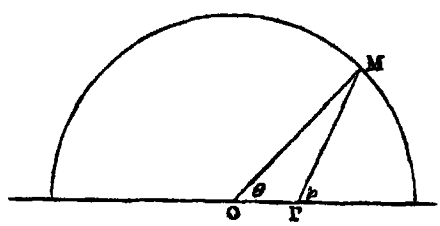
$$=\frac{1}{1-2q}\int(\rho^{1-2q})\sin^{s-1}\phi\,d\phi\,,$$

or multiplying by 2, in order that the integration may be taken only over the semicircle, $y=$positive, this is

$$=\frac{1}{\frac{1}{2}-q}\int(\rho^{1-2q})\sin^{s-1}\phi\,d\phi,$$

the term (ρ^{1-2q}) being taken between the proper limits.

78. Consider first an interior point $\varkappa<f$. As already mentioned, we exclude an indefinitely small circle radius ϵ, and the limits for ρ are from $\rho=\epsilon$ to $\rho=$its value at the



circumference; viz. if here $x=f\cos\theta$, $y=f\sin\theta$, then we have $f\cos\theta=\varkappa+\rho\cos\phi$, $f\sin\theta=\rho\sin\phi$, and consequently

$$\rho^2=\varkappa^2+f^2-2\varkappa f\cos\theta,$$

$$\sin\phi=\frac{f}{\rho}\sin\theta,=\frac{f\sin\theta}{\sqrt{\varkappa^2+f^2-2\varkappa f\cos\theta}},$$

and the integral therefore is

$$=\frac{1}{\frac{1}{2}-q}\left(\frac{f^{s-1}\sin^{s-1}\theta}{\{\varkappa^2+f^2-2\varkappa f\cos\theta\}^{\frac{1}{2}s+q-1}}-\epsilon^{1-2q}\sin^{s-1}\phi\right)d\phi.$$

As regards the second term, this is $=-\frac{\epsilon^{1-2q}}{\frac{1}{2}-q}\int\sin^{s-1}\phi\,d\phi$, $\phi=0$ to $\phi=\pi$, or, what is the same thing, we may multiply by 2 and take the integral from $\phi=0$ to $\phi=\frac{\pi}{2}$. Writing then $\sin\phi=\sqrt{x}$, and consequently $\sin^{s-1}\phi\,d\phi=\frac{1}{2}x^{\frac{1}{2}s-1}(1-x)^{-\frac{1}{2}}\,dx$, the term is $=-\frac{\epsilon^{1-2q}}{\frac{1}{2}-q}\frac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s+\frac{1}{2})}$, and the value of the disk-integral is

$$=\frac{f^{s-1}}{\frac{1}{2}-q}\int\frac{\sin^{s-1}\theta\,d\phi}{(\varkappa^2+f^2-2\varkappa f\cos\theta)^{\frac{1}{2}s+q-1}}-\frac{\epsilon^{1-2q}}{\frac{1}{2}-q}\frac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s+\frac{1}{2})}.$$

But we have

$$\sin\phi=\frac{f\sin\theta}{\rho},\quad \cos\phi=\frac{f\cos\theta-\varkappa}{\rho},$$

and thence

$$\tan\phi=\frac{f\sin\theta}{f\cos\theta-\varkappa},\quad \sec^2\phi\,d\phi=\frac{f(f-\varkappa\cos\theta)d\theta}{(f\cos\theta-\varkappa)^2};$$

that is

$$d\phi=\frac{f(f-\varkappa\cos\theta)d\theta}{\rho^2},=\frac{f(f-\varkappa\cos\theta)d\theta}{f^2+\varkappa^2-2\varkappa f\cos\theta},$$

or, what is the same thing,

$$=\frac{\frac{1}{2}\{(f^2-\varkappa^2)+(f^2+\varkappa^2-2\varkappa f\cos\theta)\}}{f^2+\varkappa^2-2\varkappa f\cos\theta};$$

and the expression for the disk-integral is therefore

$$=\frac{\frac{1}{2}f^{s-1}}{\frac{1}{2}-q}\int_0^\pi\frac{\sin^{s-1}\theta\{(f^2-\varkappa^2)+(f^2+\varkappa^2-2\varkappa f\cos\theta)\}d\theta}{\{f^2+\varkappa^2-2\varkappa f\cos\theta\}^{\frac{1}{2}s+q}}-\frac{\epsilon^{1-2q}}{\frac{1}{2}-q}\,\frac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s+\frac{1}{2})}.$$

79. Writing as before $\cos\frac{1}{2}\theta=\sqrt{x}$, $\sin\frac{1}{2}\theta=\sqrt{1+x}$, &c., and $u=\frac{4\varkappa f}{(\varkappa+f)^2}$, this is

$$=\frac{2^{s-2}f^{s-1}}{(\frac{1}{2}-q)(\varkappa+f)^{s+2q-2}}\left\{\frac{(f^2-\varkappa^2)}{(\varkappa+f)^2}\,\Pi(\tfrac{1}{2}s,\tfrac{1}{2}s,\tfrac{1}{2}s+q,u)+\Pi(\tfrac{1}{2}s,\tfrac{1}{2}s,\tfrac{1}{2}s+q-1,u)\right\}-\frac{\epsilon^{1-2q}}{\frac{1}{2}-q}\,\frac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s+\frac{1}{2})}.$$

As a verification observe that if $\varkappa=0$, each of the Π-functions becomes

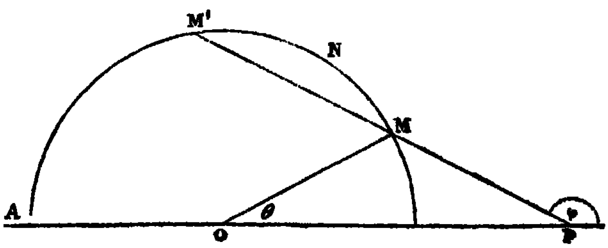
$$=\int_0^1 x^{\frac{1}{2}s-1}(1-x)^{\frac{1}{2}s-1}\,dx,=\frac{\Gamma\frac{1}{2}s\,\Gamma\frac{1}{2}s}{\Gamma s},$$

hence the whole first term is $=2\,\frac{2^{s-2}}{\frac{1}{2}-q}\,\frac{f^{1-2q}}{}\,.\,\frac{\Gamma\frac{1}{2}s\,\Gamma\frac{1}{2}s}{\Gamma s}$, viz. this is $=\frac{f^{1-2q}}{\frac{1}{2}-q}\,\frac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s+\frac{1}{2})}$, and the complete value is

$$=\frac{1}{\frac{1}{2}-q}\,\frac{\Gamma\frac{1}{2}s\,\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s+\frac{1}{2})}\{f^{1-2q}-\epsilon^{1-2q}\},$$

vanishing, as it should do, if $f=\epsilon$.

80. In the case of an exterior point $\varkappa>f$ the process is somewhat different, but the



result is of a like form. We have

$$\text{Disk-integral}=\frac{1}{\frac{1}{2}-q}\int(\rho_1{}^{1-2q}-\rho^{1-2q})\sin^{s-1}\phi\,d\phi,$$

ρ_1 referring to the point M′ and ρ to the point M. Attending first to the integral $\int\rho^{1-2q}\sin^{s-1}\phi\,d\phi$, and writing as before $f\cos\theta=\varkappa+\rho\cos\phi$, $f\sin\theta=\rho\sin\phi$, this is

$$=f^{s-1}\int\frac{\sin^{s-1}\theta\,d\phi}{\{\varkappa^2+f^2-2\varkappa f\cos\theta\}^{\frac{1}{2}s+q}}$$

$$=\tfrac{1}{2}f^{s-1}\int\frac{\sin^{s-1}\theta\{(f^2-x^2)+(f^2+x^2-2xf\cos\theta)\}d\theta}{(f^2+x^2-2fx\cos\theta)^{\frac{1}{2}s+q}},$$

the inferior and superior limits being here the values of θ which correspond to the points N, A respectively, say $\theta+\alpha$, and $\theta=0$; hence, reversing the sign and interchanging the two limits, the value of $-\int\varrho^{1-2q}\sin^{s-1}\theta\,d\phi$ is the above integral taken from 0 to α. But similarly the value of $+\int\varrho_1^{1-2q}\sin^{s-1}\theta\,d\phi$ is the same integral taken from α to π; and for the two terms together the value is the same integral from 0 to π; viz. we thus find

$$\text{Disk-integral}=\frac{\frac{1}{2}f^{s-1}}{\frac{1}{2}-q}\int_0^\pi\frac{\sin^{s-1}\theta\{-(x^2-f^2)+(f^2+x^2-2xf\cos\theta)\}d\theta}{(f^2+x^2-2fx\cos\theta)^{\frac{1}{2}s+q}};$$

viz. writing as before $\cos\frac{1}{2}\theta=\sqrt{x}$ &c., and $u=\frac{4xf}{(x+f)^2}$, this is

$$=\frac{2^{s-2}f^{s-1}}{(\frac{1}{2}-q)(x+f)^{s+2q-2}}\left\{-\frac{x^2-f^2}{(x+f)^2}\cdot\Pi(\tfrac{1}{2}s,\tfrac{1}{2}s,\tfrac{1}{2}s+q,u)+\Pi(\tfrac{1}{2}s,\tfrac{1}{2}s,\tfrac{1}{2}s+q-1)\right\}.$$

81. As a verification, suppose that x is indefinitely large. we must recur to the last preceding formula, the value is thus

$$=\frac{f^s}{(\frac{1}{2}-q)x^{s+2q-1}}\int_0^\pi\frac{\sin^{s-1}\theta\left(-\cos\theta+\frac{f}{x}\right)}{\left(1-\frac{2f}{x}\cos\theta\right)^{\frac{1}{2}s+q}};$$

viz. this is

$$=\frac{f^s}{(\frac{1}{2}-q)x^{s+2q-1}}\int_0^\pi\sin^{s-1}\theta\left\{-\cos\theta+[1-(s+2q)\cos^2\theta]\frac{f}{x}\right\}d\theta,$$

where the integral of the first term vanishes; the value is thus

$$=\frac{f^{s+1}}{(\frac{1}{2}-q)x^{s+2q}}\int_0^\pi\sin^{s-1}\theta[1-(s+2q)\cos^2\theta]\,d\theta,$$

where we may multiply by 2 and take the integral from 0 to $\frac{\pi}{2}$. Writing then $\sin\theta=\sqrt{x}$, the value is

$$=\frac{f^{s+1}}{(\frac{1}{2}-q)x^{s+2q}}\int_0^1 x^{\frac{1}{2}s-1}\{1-(s+2q)(1-x)\}(1-x)^{-\frac{1}{2}}\,dx,$$

where the integral is $=\frac{\Gamma\frac{1}{2}s\,\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s+\frac{1}{2})}\left(1-\frac{\frac{1}{2}(s+2q)}{\frac{1}{2}s+\frac{1}{2}}\right),=\frac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s+\frac{1}{2})}\cdot\frac{\frac{1}{2}-q}{\frac{1}{2}s+\frac{1}{2}}$,
and hence the value is

$$=\frac{f^{s+1}}{x^{s+2q}}\cdot\frac{\Gamma\frac{1}{2}s\,\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s+\frac{3}{2})};$$

viz. this is $=\frac{1}{x^{s+2q}}\int y^{s-1}\,dx\,dy$, over the circle $x^2+y^2=f^2$, as is easily verified.

82. Reverting to the interior point $x<f$.

Disk-integral

$$=\frac{2^{s-2}f^{s-1}}{(\tfrac{1}{2}-q)(x+f)^{s+2q-2}}\left\{\frac{f-x}{f+x}\Pi(\tfrac{1}{2}s,\tfrac{1}{2}s,\tfrac{1}{2}s+q,u)+\Pi(\tfrac{1}{2}s,\tfrac{1}{2}s,\tfrac{1}{2}s+q-1,u)\right\}-\frac{s^{1-2q}}{\tfrac{1}{2}-q}\frac{\Gamma\tfrac{1}{2}s\Gamma\tfrac{1}{2}}{\Gamma(\tfrac{1}{2}s+\tfrac{1}{2})};$$

then reducing the expression in { } by the transformations for $\Pi(\tfrac{1}{2}s, \tfrac{1}{2}s, \tfrac{1}{2}s+q, u)$ and the like transformations for $\Pi(\tfrac{1}{2}s, \tfrac{1}{2}s, \tfrac{1}{2}s+q-1, u)$, the term in { } may be expressed in the four forms:—

$$2^{1-s}\frac{\Gamma\tfrac{1}{2}s\,\Gamma\tfrac{1}{2}}{\Gamma(\tfrac{1}{2}s+q)\Gamma(\tfrac{1}{2}-q)}\frac{(f+x)^{s+2q-2}}{f^{s+2q-2}}\text{ into}$$

$$\left[\left(1-\frac{x^2}{f^2}\right)\Pi\left(\tfrac{1}{2}s+q,\tfrac{1}{2}-q,\tfrac{1}{2}+q,\frac{x^2}{f^2}\right)+\frac{\tfrac{1}{2}s+q-1}{\tfrac{1}{2}-q}\Pi\left(\tfrac{1}{2}s+q-1,\tfrac{3}{2}-q,-\tfrac{1}{2}+q,\frac{x^2}{f^2}\right)\right],$$

$$2^{1-s}\frac{\Gamma\tfrac{1}{2}s\,\Gamma\tfrac{1}{2}}{\Gamma(\tfrac{1}{2}+q)\Gamma(\tfrac{1}{2}s-q)}\frac{(f+x)^{s+2q-2}}{f^{s+2q-2}}\text{ into}$$

$$\left[\left(1-\frac{x^2}{f^2}\right)\Pi\left(\tfrac{1}{2}+q,\tfrac{1}{2}s-q,\tfrac{1}{2}s+q,\frac{x^2}{f^2}\right)+\frac{-\tfrac{1}{2}+q}{\tfrac{1}{2}s+q}\Pi\left(-\tfrac{1}{2}+q,\tfrac{1}{2}s-q+1,\tfrac{1}{2}s+q-1,\frac{x^2}{f^2}\right)\right],$$

$$2^{1-s}\frac{\Gamma\tfrac{1}{2}s\,\Gamma\tfrac{1}{2}}{\Gamma(\tfrac{1}{2}-q)\Gamma(\tfrac{1}{2}s+q)}\frac{(f+x)^{s-1}(f-x)^{1-2q}}{f^{s-2q}}\text{ into}$$

$$\left[\Pi\left(\tfrac{1}{2}-q,\tfrac{1}{2}s+q,\tfrac{1}{2}s-q,\frac{x^2}{f^2}\right)+\left(1-\frac{x^2}{f^2}\right)\frac{\tfrac{1}{2}s+q-1}{\tfrac{1}{2}-q}\Pi\left(\tfrac{3}{2}-q,\tfrac{1}{2}s+q-1,\tfrac{1}{2}s-q+1,\frac{x^2}{f^2}\right)\right],$$

$$2^{1-s}\frac{\Gamma\tfrac{1}{2}s\,\Gamma\tfrac{1}{2}}{\Gamma(\tfrac{1}{2}s-q)\Gamma(\tfrac{1}{2}+q)}\frac{(f+x)^{s-1}(f-x)^{1-2q}}{f^{s-2q}}\text{ into}$$

$$\left[\Pi\left(\tfrac{1}{2}s-q,\tfrac{1}{2}+q,\tfrac{1}{2}-q,\frac{x^2}{f^2}\right)+\left(1-\frac{x^2}{f^2}\right)\frac{-\tfrac{1}{2}+q}{\tfrac{1}{2}s-q}\Pi\left(\tfrac{1}{2}s-q+1,-\tfrac{1}{2}+q,\tfrac{3}{2}-q,\frac{x^2}{f^2}\right)\right].$$

83. The first and fourth of these are susceptible of a reduction which does not appear to be applicable to the second and third. Consider in general the function

$$(1-v)\Pi(\alpha,\beta,1-\beta,v)+\frac{\alpha-1}{\beta}\Pi(\alpha-1,\beta+1,-\beta,v);$$

the second Π-function is here

$$\int_0^1 x^{\alpha-2}(1-x\,.\,1-vx)^{\beta}\,dx;$$

viz. this is

$$=\frac{x^{\alpha-1}}{\alpha-1}(1-x\,.\,1-vx)^{\beta}-\frac{1}{\alpha-1}\int_0^1 x^{\alpha-1}\frac{d}{dx}(1-x\,.\,1-vx)^{\beta}dx,$$

or, since the first term vanishes between the limits, this is

$$=\frac{\beta}{\alpha-1}\int_0^1 x^{\alpha-1}.(1-x\,.\,1-vx)^{\beta-1}(1+v-2vx)dx,$$

$$=\frac{\beta}{\alpha-1}\left\{(1+v)\Pi(\alpha,\beta,1-\beta,v)-2v\,.\int_0^1 x^{\alpha}(1-x\,.\,1-vx)^{\beta-1}dx\right\}.$$

Hence the two Π-functions together are

$$=(1-v+1+v)\int_0^1 x^{\alpha-1}(1-x.1-vx)^{\beta-1}dx-2\int_0^1 vx.x^{\alpha-1}(1-x.1-vx)^{\beta-1}dx,$$

$$=2\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}(1-vx)^{\beta}dx,$$

that is

$$(1-v)\Pi(\alpha,\ \beta,\ 1-\beta,\ v)+\frac{\alpha-1}{\beta}\Pi(\alpha-1,\ \beta+1,\ -\beta,\ v)=2\Pi(\alpha,\ \beta,\ -\beta,\ v).$$

We have therefore

$$\left(1-\frac{x^2}{f^2}\right)\Pi\left(\tfrac{1}{2}s+q,\ \tfrac{1}{2}-q,\ \tfrac{1}{2}+q,\ \frac{x^2}{f^2}\right)+\frac{\frac{1}{2}s+q-1}{\frac{1}{2}-q}\Pi\left(\tfrac{1}{2}s+q-1,\ \tfrac{3}{2}-q,\ -\tfrac{1}{2}+q,\ \frac{x^2}{f^2}\right)$$
$$=2\Pi\left(\tfrac{1}{2}s+q,\ \tfrac{1}{2}-q,\ -\tfrac{1}{2}+q,\ \frac{x^2}{f^2}\right);$$

and from the same equation written in the form

$$\Pi(\alpha-1,\ \beta+1,\ -\beta,\ v)+\frac{\beta}{\alpha-1}(1-v)\Pi(\alpha,\ \beta,\ 1-\beta,\ v)=2\frac{\beta}{\alpha-1}\Pi(\alpha,\ \beta,\ -\beta,\ v),$$

we obtain

$$\Pi\left(\tfrac{1}{2}s-q,\ \tfrac{1}{2}+q,\ \tfrac{1}{2}-q,\ \frac{x^2}{f^2}\right)+\frac{-\frac{1}{2}+q}{\frac{1}{2}s-q}\left(1-\frac{x^2}{f^2}\right)\Pi\left(\tfrac{1}{2}s-q+1,\ -\tfrac{1}{2}+q,\ \tfrac{3}{2}-q,\ \frac{x^2}{f^2}\right)$$
$$=\frac{2(-\frac{1}{2}+q)}{\frac{1}{2}s-q}\Pi\left(\tfrac{1}{2}s-q+1,\ -\tfrac{1}{2}+q,\ \tfrac{1}{2}-q,\ \frac{x^2}{f^2}\right).$$

84. Hence the terms in [ ] are

$$=\frac{2^{2-s}\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s+q)\Gamma(\frac{1}{2}-q)}\cdot\frac{(f+x)^{s+2q-2}}{f^{s+2q-2}}\cdot\Pi\left(\tfrac{1}{2}s+q,\ \tfrac{1}{2}-q,\ -\tfrac{1}{2}+q,\ \frac{x^2}{f^2}\right),$$

$$=\frac{2^{2-s}(-\frac{1}{2}+q)}{\Gamma(\frac{1}{2}s-q+1)}\frac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}+q)}\frac{(f+x)^{s-1}(f-x)^{1-2q}}{f^{s-2q}}\Pi\left(\tfrac{1}{2}s-q+1,\ -\tfrac{1}{2}+q,\ \tfrac{1}{2}-q,\ \frac{x^2}{f^2}\right),$$

respectively, and the corresponding values of the disk-integral are

$$\frac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{3}{2}-q)\Gamma(\frac{1}{2}s+q)}f^{1-2q}\cdot\Pi\left(\tfrac{1}{2}s+q,\ \tfrac{1}{2}-q,\ -\tfrac{1}{2}+q,\ \frac{x^2}{f^2}\right)\qquad-\frac{\epsilon^{1-2q}}{\frac{1}{2}-q}\frac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s+\frac{1}{2})},$$

$$\frac{-\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s-q+1)\Gamma(\frac{1}{2}+q)}\left(\frac{f^2-x^2}{f}\right)^{1-2q}\cdot\Pi\left(\tfrac{1}{2}s-q+1,\ -\tfrac{1}{2}+q,\ \tfrac{1}{2}-q,\ \frac{x^2}{f^2}\right)-\frac{\epsilon^{1-2q}}{\frac{1}{2}-q}\frac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s+\frac{1}{2})},$$

which we may again verify by writing therein $x=0$, viz. the Π-functions thus become

$$\frac{\Gamma(\frac{1}{2}s+q)\ \Gamma(\frac{1}{2}-q)}{\Gamma(\frac{1}{2}s+\frac{1}{2})}\ \text{ and }\ \frac{\Gamma(\frac{1}{2}s-q+1)\Gamma(-\frac{1}{2}+q)}{\Gamma(\frac{1}{2}s+\frac{1}{2})},$$

and consequently the integral is

$$=\frac{1}{\frac{1}{2}-q}\cdot\frac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s+\frac{1}{2})}(f^{1-2q}-\epsilon^{1-2q}).$$

85. But the forms nevertheless belong to a system of four; from the formulæ

$$
\begin{aligned}
&\Pi(\alpha,\ \beta,\ \gamma,\ v)\\
=\ &\frac{\Gamma\alpha\Gamma\beta}{\Gamma\gamma\Gamma(\alpha+\beta-\gamma)}\,\Pi(\gamma,\ \alpha+\beta-\gamma,\ \alpha,\ v)\\
=\ &(1-v)^{\beta-\gamma}\qquad\Pi(\beta,\ \alpha,\ \alpha+\beta-\gamma,\ v)\\
=\ &(1-v)^{\beta-\gamma}\frac{\Gamma\alpha\Gamma\beta}{\Gamma(\alpha+\beta-\gamma)\Gamma\gamma}\,\Pi(\alpha+\beta-\gamma,\ \gamma,\ \beta,\ v),
\end{aligned}
$$

writing therein $\alpha=\tfrac{1}{2}s+q$, $\beta=\tfrac{1}{2}-q$, $\gamma=-\tfrac{1}{2}+q$, we deduce

$$
\begin{aligned}
&\Pi(\tfrac{1}{2}s+q,\ \tfrac{1}{2}-q,\ -\tfrac{1}{2}+q,\ v)\\
=\ &\frac{\Gamma(\frac{1}{2}s+q)\Gamma(\frac{1}{2}-q)}{\Gamma(-\frac{1}{2}+q)\Gamma(\frac{1}{2}s-q+1)}\,\Pi(-\tfrac{1}{2}+q,\ \tfrac{1}{2}s-q+1,\ \tfrac{1}{2}s+q,\ v)\\
=\ &(1-v)^{1-2q}\qquad\Pi(\tfrac{1}{2}-q,\ \tfrac{1}{2}s+q,\ \tfrac{1}{2}s-q+1,\ v)\\
=\ &(1-v)^{1-2q}\frac{\Gamma(\frac{1}{2}s+q)\Gamma(\frac{1}{2}-q)}{\Gamma(\frac{1}{2}s-q+1)\Gamma(-\frac{1}{2}+q)}\,\Pi(\tfrac{1}{2}s-q+1,\ -\tfrac{1}{2}+q,\ \tfrac{1}{2}-q,\ v),
\end{aligned}
$$

and the last-mentioned values of the disk-integral may thus be written in the four forms:

$$
\begin{aligned}
&\frac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{3}{2}-q)\Gamma(\frac{1}{2}s+q)}\quad f^{1-2q}\quad \Pi\left(\tfrac{1}{2}s+q,\ \tfrac{1}{2}-q,\ -\tfrac{1}{2}+q,\ \frac{x^2}{f^2}\right)\quad -\text{term in } s,\\
&\frac{-\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}+q)\Gamma(\frac{1}{2}s-q+1)}\quad f^{1-2q}\quad \Pi\left(-\tfrac{1}{2}+q,\ \tfrac{1}{2}s-q+1,\ \tfrac{1}{2}s+q,\ \frac{x^2}{f^2}\right)-\quad ,,\quad ,\\
&\frac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{3}{2}-q)\Gamma(\frac{1}{2}s+q)}\quad \left(f-\frac{x^2}{f}\right)^{1-2q}\Pi\left(\tfrac{1}{2}-q,\ \tfrac{1}{2}s+q,\ \tfrac{1}{2}s-q+1,\ \frac{x^2}{f^2}\right)\quad -\quad ,,\quad ,\\
&\frac{-\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}+q)\Gamma(\frac{1}{2}s-q+1)}\left(f-\frac{x^2}{f}\right)^{1-2q}\Pi\left(\tfrac{1}{2}s-q+1,\ -\tfrac{1}{2}+q,\ \tfrac{1}{2}-q,\ \frac{x^2}{f^2}\right)\quad -\quad ,,\quad ,
\end{aligned}
$$

and since the last of these is in fact the second of the original forms, it is clear that if instead of the first we had taken the second of the original forms, we should have obtained again the same system of four forms.

86. Writing as before $x=\dfrac{t}{t+f^2-x^2}$ &c., the forms are

$$
\begin{aligned}
&\frac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{3}{2}-q)\Gamma(\frac{1}{2}s+q)}\quad (f^2-x^2)^{1-2q}\quad \int_0^\infty t^{\frac{1}{2}s+q-1}(t+f^2-x^2)^{-\frac{1}{2}s+q-\frac{1}{2}}(t+f^2)^{-\frac{1}{2}s-q}\quad dt-\text{term in } s,\\
&\frac{-\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}+q)\Gamma(\frac{1}{2}s-q+1)}\,f^{s+1}(f^2-x^2)^{1-2q}\int_0^\infty t^{-\frac{3}{2}+q}\ (t+f^2-x^2)^{q-\frac{1}{2}}\quad (t+f^2)^{-\frac{1}{2}s-q}\ dt-\quad ,,\quad ,\\
&\frac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{3}{2}-q)\Gamma(\frac{1}{2}s+q)}\quad f^{s+1}\quad \int_0^\infty t^{-q-\frac{1}{2}}\ (t+f^2-x^2)^{-q+\frac{1}{2}}\ (t+f^2)^{-\frac{1}{2}s+q-1}dt-\quad ,,\quad ,\\
&\frac{-\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s-q+1)\Gamma(\frac{1}{2}+q)}\quad \int_0^\infty t^{\frac{1}{2}s-q}\ (t+f^2-x^2)^{-\frac{1}{2}s-q}\ (t+f^2)^{-\frac{1}{2}+q}\ dt-\quad ,,\quad .
\end{aligned}
$$

87. The third of these possesses a remarkable property: write mf instead of f, and at the same time change t into m^2t, the integral becomes

$$\frac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}-q)\Gamma(\frac{1}{2}s+q)}f^{s+1}\int_0^\infty t^{-q-\frac{1}{2}}\{m^2(t+f^2)-\varkappa^2\}^{-q+\frac{1}{2}}(t+f^2)^{-\frac{1}{2}s+q-1}dt-\text{term in }\epsilon\,;$$

and hence writing $mf=f+\delta f$ or $m=1+\frac{\delta f}{f}$, and therefore $m^2=1+2\frac{\delta f}{f}$, the value is

$$\frac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}-q)\Gamma(\frac{1}{2}s+q)}f^{s+1}\int_0^\infty t^{-q-\frac{1}{2}}\left\{t+f^2-\varkappa^2+\frac{2\delta f}{f}(t+f^2)\right\}^{-q+\frac{1}{2}}.(t+f^2)^{-\frac{1}{2}s+q-1}dt-\text{term in }\epsilon.$$

Hence the term in δf is

$$=2(-q+\tfrac{1}{2})\frac{\delta f}{f}\cdot\frac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}-q)\Gamma(\frac{1}{2}s+q)}f^{s+1}\int_0^\infty t^{-q-\frac{1}{2}}(t+f^2-\varkappa^2)^{-q-\frac{1}{2}}(t+f^2)^{-\frac{1}{2}s+q}dt,$$

$$=\delta f\text{ into expression }\frac{2\Gamma\frac{1}{2}s\ \Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}-q)\Gamma(\frac{1}{2}s+q)}f^{s}\int_0^\infty t^{-q-\frac{1}{2}}(t+f^2-\varkappa^2)^{-q-\frac{1}{2}}(t+f^2)^{-\frac{1}{2}s+q}dt,$$

where the factor which multiplies δf is, as it should be, the ring-integral; it in fact agrees with one of the expressions previously obtained for this integral

88. Similarly for an exterior point $\varkappa>f$, starting in like manner from, Disk-integral

$$=\frac{2^{s-2}f^{s-1}}{(\frac{1}{2}-q)(\varkappa+f)^{s+2q-2}}\left\{-\frac{\varkappa-f}{\varkappa+f}\Pi(\tfrac{1}{2}s,\ \tfrac{1}{2}s,\ \tfrac{1}{2}s+q,\ u)+\Pi(\tfrac{1}{2}s,\ \tfrac{1}{2}s,\ \tfrac{1}{2}s+q-1,\ u)\right\},$$

and reducing in like manner, the term in { } may be expressed in the four forms

$$2^{1-s}\frac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s+q)\Gamma(\frac{1}{2}-q)}\frac{(\varkappa+f)^{s+2q-2}}{\varkappa^{s+2q-2}}\text{ into}$$

$$\left[-\left(1-\frac{f^2}{\varkappa^2}\right)\Pi\left(\tfrac{1}{2}s+q,\ \tfrac{1}{2}-q,\ \tfrac{1}{2}+q,\ \frac{f^2}{\varkappa^2}\right)+\frac{\frac{1}{2}s+q-1}{\frac{1}{2}-q}\quad\Pi\left(\tfrac{1}{2}s+q-1,\ \tfrac{3}{2}-q,\ -\tfrac{1}{2}+q,\ \frac{f^2}{\varkappa^2}\right)\right],$$

$$2^{1-s}\frac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}+q)\Gamma(\frac{1}{2}s-q)}\frac{(\varkappa+f)^{s+2q-2}}{\varkappa^{s+2q-2}}\text{ into}$$

$$\left[-\left(1-\frac{f^2}{\varkappa^2}\right)\Pi\left(\tfrac{1}{2}+q,\ \tfrac{1}{2}s-q,\ \tfrac{1}{2}s+q,\ \frac{f^2}{\varkappa^2}\right)+\frac{-\frac{1}{2}+q}{\frac{1}{2}s-q}\quad\Pi\left(-\tfrac{1}{2}+q,\tfrac{1}{2}s-q+1,\tfrac{1}{2}s+q-1,\frac{f^2}{\varkappa^2}\right)\right],$$

$$2^{1-s}\frac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}-q)\Gamma(\frac{1}{2}s+q)}\left(\frac{\varkappa+f}{\varkappa}\right)^{s-1}\left(\frac{\varkappa-f}{\varkappa}\right)^{-2q+1}\text{ into}$$

$$\left[-\Pi\left(\tfrac{1}{2}-q,\ \tfrac{1}{2}s+q,\ \tfrac{1}{2}s-q,\ \frac{f^2}{\varkappa^2}\right)+\left(1-\frac{f^2}{\varkappa^2}\right)\frac{\frac{1}{2}s+q-1}{\frac{1}{2}-q}\Pi\left(\tfrac{1}{2}-q,\ \tfrac{1}{2}s+q,\ \tfrac{1}{2}s-q,\ \frac{f^2}{\varkappa^2}\right)\right],$$

$$2^{1-s}\frac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s-q)\Gamma(\frac{1}{2}+q)}\left(\frac{\varkappa+f}{\varkappa}\right)^{s-1}\left(\frac{\varkappa-f}{\varkappa}\right)^{-2q+1}\text{ into}$$

$$\left[-\Pi\left(\tfrac{1}{2}s-q,\ \tfrac{1}{2}+q,\ \tfrac{1}{2}-q,\ \frac{f^2}{\varkappa^2}\right)+\left(1-\frac{f^2}{\varkappa^2}\right)\ \frac{-\frac{1}{2}+q}{\frac{1}{2}s-q}\quad\Pi\left(\tfrac{1}{2}s-q+1,\ -\tfrac{1}{2}+q,\ \tfrac{3}{2}-q,\ \frac{f^2}{\varkappa^2}\right)\right].$$

89. For the reduction of the first and fourth of these we have to consider

$$-(1-v)\Pi(\alpha,\ \beta,\ 1-\beta,\ v)+\frac{\alpha-1}{\beta}\ \Pi(\alpha-1,\ \beta+1,\ -\beta,\ v)\,;$$

viz. this is

$$(-1+v+1+v)\int_0^1 x^{\alpha-1}(1-x\,.\,1-vx)^{\beta-1}dx-2\int_0^1 vx\,.\,x^{\alpha-1}(1-x\,.\,1-vx)^{\beta-1}dx,$$

$$=2v\,.\int_0^1 x^{\alpha-1}(1-x)(1-x\,.\,1-vx)^{\beta-1}dx,$$

$$=2v\,.\,\Pi(\alpha,\beta+1,\ -\beta+1,v)\,;$$

that is,

$$-(1-v)\,\Pi(\alpha,\beta,1-\beta,\ v)+\frac{\alpha-1}{\beta}\ \Pi(\alpha-1,\beta+1,\ -\beta,\ v)=2v\Pi(\alpha,\beta+1,\ -\beta+1,v).$$

[I repeat for comparison the foregoing equation,

$$+(1-v)\,\Pi(\alpha,\beta,1-\beta,\ v)+\frac{\alpha-1}{\beta}\ \Pi(\alpha-1,\beta+1,\ -\beta,\ v)=2\Pi(\alpha,\ \beta,\ -\beta,\ v),$$

by adding and subtracting these we obtain two new formulæ], for reduction of the fourth formula the equation may be written

$$-\Pi(\alpha-1,\beta+1,\ -\beta,\ v)+(1-v)\frac{\beta}{\alpha-1}\,.\,\Pi(\alpha,\ \beta,\ 1-\beta,\ v)=-2\,\frac{\beta}{\alpha-1}\ v\Pi(\alpha,\beta+1-\beta+1,v)$$

90. But it is sufficient to consider the first formula, the term in [ ] is

$$=\frac{2^{2-s}\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s+q)\Gamma(\frac{1}{2}-q)}\quad\left(\frac{\kappa+f}{\kappa}\right)^{s+2q-2}\frac{f^2}{\kappa^2}\quad\Pi\left(\tfrac{1}{2}s+q,\ \tfrac{3}{2}-q,\ \tfrac{1}{2}+q,\frac{f^2}{\kappa^2}\right),$$

and the corresponding value of the disk-integral is

$$=\frac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s+q)\Gamma(\frac{3}{2}-q)}\quad\frac{f^{s+1}}{\kappa^{s+2q}}\qquad\Pi\left(\tfrac{1}{2}s+q,\ \tfrac{3}{2}-q,\ \tfrac{1}{2}+q,\frac{f^2}{\kappa^2}\right),$$

which we may again verify by taking therein κ indefinitely large; viz. the value is then $=\frac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s+\frac{1}{2})}\ \frac{f^{s+1}}{\kappa^{s+2q}}$, as above. It is the first of a system of four forms, the others of which are

$$=\frac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}+q)\Gamma(\frac{1}{2}s-q+1)}\ \frac{f^{s+1}}{\kappa^{s+2q}}\qquad\Pi\left(\tfrac{1}{2}+q,\ \tfrac{1}{2}s-q+1,\ \tfrac{1}{2}s+q,\frac{f^2}{\kappa^2}\right),$$

$$=\frac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s+q)\Gamma(\frac{3}{2}-q)}\quad\frac{f^{s+1}}{\kappa^{s+2q}}\left(1-\frac{f^2}{\kappa^2}\right)^{1-2q}\Pi\left(\tfrac{3}{2}-q,\ \tfrac{1}{2}s+q,\ \tfrac{1}{2}s-q+1,\frac{f^2}{\kappa^2}\right),$$

$$=\frac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s-q+1)\Gamma(\frac{1}{2}+q)}\ \frac{f^{s+1}}{\kappa^{s+2q}}\left(1-\frac{f^2}{\kappa^2}\right)^{1-2q}\Pi\left(\tfrac{1}{2}s-q+1,\ \tfrac{1}{2}+q,\ \tfrac{3}{2}-q,\frac{f^2}{\kappa^2}\right).$$

And hence, writing as before $x=\frac{t+f^2-\kappa^2}{t}$ &c., the four values are

$$=\frac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s+q)\Gamma(\frac{3}{2}-q)}\quad\frac{f^{s+1}}{x^{s-1}}(x^2-f^2)^{1-2q}\int_{x^2-f^2}^{\infty}t^{-\frac{1}{2}s+q-1}(t+f^2-x^2)^{\frac{1}{2}s+q-1}(t+f^2)^{-\frac{1}{2}-q}dt,$$

$$=\frac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}+q)\Gamma(\frac{1}{2}s-q+1)}f^{s+1}(x^2-f^2)^{1-2q}\int_{x^2-f^2}^{\infty}t^{q-\frac{1}{2}}(t+f^2-x^2)^{q-\frac{1}{2}}(t+f^2)^{-\frac{1}{2}s-q}dt,$$

$$=\frac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s+q)\Gamma(\frac{3}{2}-q)}\quad f^{s+1}\quad\int_{x^2-f^2}^{\infty}t^{-q-\frac{1}{2}}(t+f^2-x^2)^{-q+\frac{1}{2}}(t+f^2)^{-\frac{1}{2}s+q-1}dt,$$

$$=\frac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s-q+1)\Gamma(\frac{1}{2}+q)}\quad\frac{f^{s+1}}{x^{s-1}}\quad\int_{x^2-f^2}^{\infty}t^{-\frac{1}{2}s-q}(t+f^2-x^2)^{\frac{1}{2}s-q}(t+f^2)^{q-\frac{3}{2}}dt,$$

where we may in the integrals write $t+x^2-f^2$ in place of t, making the limits ∞, 0; but the actual form is preferable.

91. In the third form for f write mf, at the same time changing t into mt, the new value of the disk-integral is

$$=\frac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s+q)\Gamma(\frac{3}{2}-q)}f^{s+1}\int_{\frac{x^2}{m^2}-f^2}^{\infty}t^{-q-\frac{1}{2}}(m^2(t+f^2)-x^2)^{-q+\frac{1}{2}}(t+f^2)^{-\frac{1}{2}s+q-1}dt$$

Writing here $mf=f+\delta f$, that is $m=1+\frac{\delta f}{f}$, $m^2=1+\frac{2\delta f}{f}$, and observing that if $-q+\frac{1}{2}$ be positive, the factor $(m^2(t+f^2)-x^2)^{-q+\frac{1}{2}}$ vanishes for the value $t=\frac{x^2}{m^2}-f^2$ at the lower limit, we see that on this supposition, $-q+\frac{1}{2}$ positive, the value is

$$=\frac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s+q)\Gamma\frac{3}{2}-q)}f^{s+1}\quad\int_{x^2-f^2}^{\infty}t^{-q-\frac{1}{2}}\left(t+f^2-x^2+\frac{2\delta f}{f}(t+f^2)\right)^{-q+\frac{1}{2}}(t+f^2)^{-\frac{1}{2}s+q-1}dt;$$

viz. the term in δf is $=\delta f$ into the expression

$$2(\tfrac{1}{2}-q)\cdot\frac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s+q)\Gamma(\frac{3}{2}-q)}f^s\int_{x^2-f^2}^{\infty}t^{-q-\frac{1}{2}}(t+f^2-x^2)^{-q-\frac{1}{2}}(t+f^2)(t+f^2)^{-\frac{1}{2}s+q-1}dt,$$

that is into

$$2\frac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s+q)\Gamma(\frac{1}{2}-q)}f^s\int_{x^2-f^2}^{\infty}t^{-q-\frac{1}{2}}(t+f^2-x^2)^{-q-\frac{1}{2}}(t+f^2)^{-\frac{1}{2}s+q}dt,$$

which is in fact $=\delta f$ into the value of the ring-integral.

92. Comparing for the cases of an interior point $x<f$ and an exterior point $x>f$, the four expressions for the disk-integral, it will be noticed that only the third expressions correspond precisely to each other; viz. these are interior point, $x<f$; the value is

$$\frac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s+q)\Gamma(\frac{3}{2}-q)}f^{s+1}\cdot\int_0^{\infty}t^{-q-\frac{1}{2}}(t+f^2-x^2)^{-q+\frac{1}{2}}(t+f^2)^{-\frac{1}{2}s+q-1}dt-\frac{\epsilon^{1-2q}}{\frac{1}{2}-q}\frac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}s}{\Gamma(\frac{1}{2}s+q)},$$

where, if $\frac{1}{2}-q$ be positive (which is in fact a necessary condition in order to the applicability of the formula), the term in ϵ vanishes, and may therefore be omitted: and

exterior point, $\kappa > f$; the value is

$$=\frac{\Gamma\tfrac{1}{2}s\,\Gamma\tfrac{1}{2}}{\Gamma(\tfrac{1}{2}s+q)\Gamma(\tfrac{1}{2}-q)}\,f^{s+1}\int_{\kappa^2-f^2}^{\infty} t^{-q-\frac{1}{2}}(t+f^2-\kappa^2)^{-q+\frac{1}{2}}(t+f^2)^{-\frac{1}{2}s+q-1}\,dt,$$

differing only from the preceding one in the inferior limit κ^2-f^2 in place of 0 of the integral. We have $\tfrac{1}{2}-q$ positive, and also $\tfrac{1}{2}s+q$ positive; viz. q may have any value diminishing from $\tfrac{1}{2}$ to $-\tfrac{1}{2}s$, the extreme values *not* admissible.

ANNEX IV. *Examples of Theorem* A.—Nos. 93 to 112.

93. It is remarked in the text that in the examples which relate to the s-coordinal sphere and ellipsoid respectively, we have a quantity θ, a function of the coordinates $(a \ldots c, e)$ of the attracted point; viz. in the case of the sphere, writing $a^2 \ldots + c^2 = \kappa^2$, we have

$$\frac{\kappa^2}{f^2+\theta}+\frac{e^2}{\theta}=1,$$

and in the case of the ellipsoid

$$\frac{a^2}{f^2+\theta}\cdots+\frac{c^2}{h^2+\theta}+\frac{e^2}{\theta}=1,$$

the equation having in each case a positive root which is called θ. The properties of the equation are the same in each case, but for the sphere, the equation being a quadric one, can be solved The equation in fact is

$$\theta^2-\theta(e^2+\kappa^2-f^2)-e^2f^2=0,$$

and the positive root is therefore

$$\theta=\tfrac{1}{2}\{e^2+\kappa^2-f^2+\sqrt{(e^2+\kappa^2-f^2)^2+4e^2f^2}\}$$

Suppose e to gradually diminish and become $=0$, for an exterior point, $\kappa > f$, the value of the radical is $=\kappa^2-f^2$, and we have $\theta=\kappa^2-f^2$, for an interior point, $\kappa < f$, the value of the radical, supposing e only indefinitely small, is $=f^2-\kappa^2+\frac{f^2+\kappa^2}{f^2-\kappa^2}e^2$, and we have $\theta=\tfrac{1}{2}e^2\left(1+\frac{f^2+\kappa^2}{f^2-\kappa^2}\right), =\frac{e^2f^2}{f^2-\kappa^2}$, or, what is the same thing, $\frac{e^2}{\theta}=\left(1-\frac{\kappa^2}{f^2}\right)$; viz. the positive root of the equation continually diminishes with e, and becomes ultimately $=0$

If κ or e be indefinitely large, then the radical may be taken $=e^2+\kappa^2$, and we have θ indefinitely large, $=e^2+\kappa^2$

94. Every thing is the same with the general equation

$$\frac{a^2}{f^2+\theta}\cdots+\frac{c^2}{h^2+\theta}+\frac{e^2}{\theta}=1;$$

the left-hand side is $=0$ for $\theta=\infty$, and (as θ decreases) continually increases, becoming infinite for $\theta=0$; there is consequently a single positive value of θ for which the value is $=1$; viz. the equation has a single positive root, and θ is taken to denote this root

In the last-mentioned equation, let e gradually diminish and become $=0$; then for an exterior point, viz. if

$$\frac{a^2}{f^2}\cdots+\frac{c^2}{h^2}>1, \text{ the equation } \frac{a^2}{f^2+\theta}\cdots+\frac{c^2}{h^2+\theta}=1$$

has (as is at once seen) a single positive root, and θ becomes equal to the positive root of this equation; but for an interior point, or $\frac{a^2}{f^2}\cdots+\frac{c^2}{h^2}<1$, the equation just written down has no positive root, and θ becomes $=0$, that is the positive root of the original equation continually diminishes with e, and for $e=0$ becomes ultimately $=0$; its value for e small is in fact given by $\frac{e^2}{\theta}=\left(1-\frac{a^2}{f^2}\cdots-\frac{c^2}{h^2}\right)$. Also $a\ldots c, e$ or any of them indefinitely large, θ is indefinitely large, $=a^2 \ldots +c^2+e^2$.

95. We have an interesting geometrical illustration in the case $s+1=2$; θ is here determined by the equation

$$\frac{a^2}{f^2+\theta}+\frac{b^2}{g^2+\theta}+\frac{e^2}{\theta}=1;$$

viz. θ is the squared z-semiaxis of the ellipsoid, confocal with the conic $\frac{x^2}{f^2}+\frac{y^2}{g^2}=1$, which passes through the point (a, b, e). Taking $e=0$, the point in question, if $\frac{a^2}{f^2}+\frac{b^2}{g^2}>1$, is a point in the plane of xy, outside the ellipse, and we have through the point a proper confocal ellipsoid, whose squared z-semiaxis does not vanish; but if $\frac{a^2}{f^2}+\frac{b^2}{g^2}<1$, then the point is within the ellipse, and the only confocal ellipsoid through the point is the indefinitely thin ellipsoid, squared semiaxes $(f^2, g^2, 0)$, which in fact coincides with the ellipse.

96. The positive root θ of the equation

$$\mathrm{J}, = 1-\frac{a^2}{f^2+\theta}\cdots-\frac{c^2}{h^2+\theta}-\frac{e^2}{\theta}, =0$$

has certain properties which connect themselves with the function

$$\Theta, =\theta^{-q-1}(\theta+f^2\ldots\theta+h^2)^{-\frac{1}{2}}.$$

We have (the accents denoting differentiations in regard to θ)

$$\mathrm{J}'\frac{d\theta}{da}-\frac{2a}{\theta+f^2}=0, \text{ or } \frac{d\theta}{da}=\frac{1}{\mathrm{J}'}\frac{2a}{\theta+f^2},$$

where

$$\mathrm{J}'=\frac{a^2}{(f^2+\theta)^2}\cdots+\frac{c^2}{(h^2+\theta)^2}+\frac{e^2}{\theta^2},$$

and we have the like formulæ for $\ldots\frac{d\theta}{dc}, \frac{d\theta}{de}$.

We deduce

$$\frac{a}{\theta+f^2}\frac{d\theta}{da}\cdots+\frac{c}{\theta+h^2}\frac{d\theta}{dc}+\frac{e}{\theta}\frac{d\theta}{de}=\frac{2}{\mathrm{J}'}\left\{\frac{a^2}{(\theta+f^2)^2}\cdots+\frac{c^2}{(\theta+h^2)^2}+\frac{e^2}{\theta}\right\}, =2;$$

and to this we may join, η being arbitrary,

$$\frac{a}{\theta+\eta+f^2}\frac{d\theta}{da}\ldots+\frac{c}{\theta+\eta+h^2}\frac{d\theta}{dc}+\frac{e}{\theta+\eta}\frac{d\theta}{de}=\frac{2}{\mathrm{J}'}\left\{\frac{a^2}{\theta+f^2.\theta+\eta+f^2}\ldots+\frac{c^2}{\theta+h^2.\theta+\eta+h^2}+\frac{e^2}{\theta.\theta+\eta}\right\}.$$

Again, defining $\nabla_1\theta$, $\square\theta$ as immediately appears, we have

$$\nabla_1\theta,=\left(\frac{d\theta}{da}\right)^2\ldots+\left(\frac{d\theta}{dc}\right)^2,=\frac{1}{\mathrm{J}'^2}.4\mathrm{J}',=\frac{4}{\mathrm{J}'};$$

and passing to the second differential coefficients, we have

$$\frac{d^2\theta}{da^2}=\frac{2}{\mathrm{J}'(\theta+f^2)}-\frac{8a^2}{\mathrm{J}'^2(\theta+f^2)^3}-\frac{4a^2\mathrm{J}''}{\mathrm{J}'^3(\theta+f^2)^2},$$

where

$$\mathrm{J}''=-2\left\{\frac{a^2}{(\theta+f^2)^3}\ldots+\frac{c^2}{(\theta+h^2)^3}+\frac{e^2}{\theta^3}\right\},$$

with the like formulæ for $\ldots\frac{d^2\theta}{dc^2},\frac{d^2\theta}{de^2}$. Joining to these $\frac{2q+1}{e}\frac{d\theta}{de}=\frac{4q+2}{\mathrm{J}'\theta}$, we obtain

$$\begin{aligned}\square\theta,&=\left(\frac{d^2\theta}{da^2}\ldots+\frac{d^2\theta}{dc^2}+\frac{d^2\theta}{de^2}+\frac{2q+1}{e}\frac{d\theta}{de}\right),\\&=\frac{2}{\mathrm{J}'}\left\{\frac{1}{\theta+f^2}\ldots+\frac{1}{\theta+h^2}+\frac{1+(2q+1)}{\theta}\right\}\\&-\frac{8}{\mathrm{J}'^2}(-\tfrac{1}{2}\mathrm{J}'')-\frac{4\mathrm{J}''}{\mathrm{J}'^3}(\mathrm{J}'),\end{aligned}$$

where the last two terms destroy each other; and observing that we have

$$\frac{\Theta'}{\Theta}=-\tfrac{1}{2}\left(\frac{1}{\theta+f^2}\ldots+\frac{1}{\theta+h^2}+\frac{2q+1}{\theta}\right),$$

the result is

$$\square\theta=\frac{2}{\mathrm{J}'}\left(-\frac{2\Theta'}{\Theta}\right),=-\frac{4\Theta'}{\mathrm{J}'\Theta}.$$

97. First example. $x^2=a^2\ldots+c^2$, and θ is the positive root of $\frac{x^2}{f^2+\theta}+\frac{e^2}{\theta}=1$.

V is assumed $=\int_\theta^\infty t^{-q-1}(t+f^2)^{-\frac{1}{2}s}dt$, where $q+1$ is positive.

I do not work the example out; it corresponds step by step with, and is hardly more simple than, the next example, which relates to the ellipsoid. The result is

$$\rho=0,\text{ if }x^2\ldots+z^2>f^2,$$

$$\rho=\frac{\Gamma(\frac{1}{2}s+q)}{(\Gamma\frac{1}{2})^s\Gamma(q+1)}f^{-s}\left(1-\frac{x^2\ldots+z^2}{f^2}\right)^q,\text{ if }x^2\ldots+z^2<f^2;$$

hence the integral

$$\int\frac{\left(1-\frac{x^2\ldots+z^2}{f^2}\right)^q dx\ldots dz}{\{(a-x)^2\ldots+(c-z)^2+e^2\}^{\frac{1}{2}s+q}},$$

taken over the sphere $x^2 \ldots + z^2 = f^2$,

$$= \frac{(\Gamma\frac{1}{2})^s \Gamma(q+1)}{\Gamma(\frac{1}{2}s+q)} \int_0^\infty t^{-q-1}(t+f^2)^{-\frac{1}{2}s}\, dt.$$

98. Second example. θ the positive root of $\frac{a^2}{f^2+\theta} \ldots + \frac{c^2}{h^2+\theta} + \frac{e^2}{\theta} = 1$; $q+1$ positive.

Consider here the function

$$\mathrm{V} = \int_\theta^\infty t^{-q-1}(t+f^2 \ldots t+h^2)^{-\frac{1}{2}}\, dt;$$

this satisfies the prepotential equation We have in fact

$$\frac{d\mathrm{V}}{da} = -\Theta \frac{d\theta}{da}\,;\ \frac{d^2\mathrm{V}}{da^2} = -\Theta \frac{d^2\theta}{da^2} - \Theta' \left(\frac{d\theta}{da}\right)^2,$$

with the like expressions for $\frac{d^2\mathrm{V}}{dc^2}, \frac{d^2\mathrm{V}}{de^2}$; also

$$\frac{2q+1}{e} \frac{d\mathrm{V}}{de} = -\Theta \frac{2q+1}{e} \frac{d\theta}{de}.$$

Hence

$$\square \mathrm{V} = -\Theta \square \theta - \Theta' \nabla_1 \theta,$$

or, substituting for $\square \theta$ and $\nabla_1 \theta$ their values, this is

$$= -\Theta \left(-\frac{4\Theta'}{J'\Theta}\right) - \Theta' . 4\mathrm{J}, = 0.$$

Moreover V does not become infinite for any values of $(a \ldots c, e)$, e not $=0$; and it vanishes for points at ∞, and not only so, but for indefinitely large values of any of the coordinates $(a \ldots e, e)$ it reduces itself to a numerical multiple of $(a^2 .. + c^2 + e^2)^{-\frac{1}{2}s+q}$; in fact in this case θ is indefinitely large, $= a^2 \ldots + c^2 + e^2$: consequently throughout the integral t is indefinitely large, and we may therefore write

$$\mathrm{V} = \int_\theta^\infty t^{-q-1}\ t^{-\frac{1}{2}s}\, dt,\ = -\frac{1}{\frac{1}{2}s+q}\left(t^{-\frac{1}{2}s-q}\right)_\theta^\infty,\ = \frac{1}{\frac{1}{2}s+q}\theta^{-\frac{1}{2}s-q},$$

that is

$$\mathrm{V} = \frac{1}{\frac{1}{2}s+q}(a^2 \ldots + c^2 + e^2)^{-\frac{1}{2}s-q}$$

The conditions of the theorem are thus satisfied, and we have for ρ either of the formulæ.

$$\rho = \frac{\Gamma(\frac{1}{2}s+q)}{(\Gamma\frac{1}{2})^s \Gamma q} (e^{2q}\mathrm{W})_0,\quad \rho = \frac{-\Gamma(\frac{1}{2}s+q)}{2(\Gamma\frac{1}{2})^s \Gamma(q+1)} \left(e^{2q+1}\frac{d\mathrm{W}}{de}\right)_0$$

(in the former of them q must be positive, in the latter it is sufficient if $q+1$ be positive)

99. We have W the same function of $(x \ldots z, e)$ that V is of $(a \ldots c, e)$; viz. writing λ for the positive root of

$$\frac{x^2}{f^2+\lambda} \cdots + \frac{z^2}{h^2+\lambda} + \frac{e^2}{\lambda} = 1,$$

the value of W is

$$= \int_\lambda^\infty t^{-q-1}(t+f^2 \ldots t+h^2)^{-\frac{1}{2}} dt.$$

Considering the formula which involves e^{2q}W,—first, if $\frac{x^2}{f^2} \ldots + \frac{z^2}{h^2} > 1$, then when e is $=0$ the value of λ is not $=0$, the integral W is therefore finite (not indefinitely large), and we have $e^{2q}W=0$, consequently $\varrho=0$.

But if $\frac{x^2}{f^2} \ldots + \frac{z^2}{h^2} < 1$, then when e is indefinitely small, λ is also indefinitely small, viz. we then have $\frac{e^2}{\lambda} = 1 - \frac{x^2}{f^2} \ldots - \frac{z^2}{h^2}$, the value of W is

$$W = (f \ldots h)^{-1} \int_\lambda^\infty t^{-q-1} dt, \quad = (f \ldots h)^{-1} \frac{1}{q} \lambda^{-q},$$

and hence

$$\varrho = \frac{\Gamma(\frac{1}{2}s+q)}{(\Gamma\frac{1}{2})^s \Gamma q} \cdot \frac{1}{q} \left(\frac{e^2}{\lambda}\right)^q (f \ldots h)^{-1}, \quad = \frac{\Gamma(\frac{1}{2}s+q)}{(\Gamma\frac{1}{2})^s \Gamma(q+1)} (f \ldots h)^{-1} \left(1 - \frac{x^2}{f^2} \ldots - \frac{z^2}{h^2}\right)^q.$$

100. Again, using the formula which involves $\left(e^{2q+1}\frac{dW}{de}\right)$; we have here $\frac{dV}{de} = -\Theta \frac{d\theta}{de}$, or substituting for Θ and $\frac{d\theta}{de}$ their values and multiplying by e^{2q+1}, we find

$$e^{2q+1}\frac{dV}{de} = 2e^{2q+2}\theta^{-1} J'^{-1} \Theta,$$

$$= 2e^{2q+2}\theta^{-q-2}\left[\frac{a^2}{(f^2+\theta)^2} \cdots + \frac{c^2}{(h^2+\theta)^2} + \frac{e^2}{\theta^2}\right]^{-1} (\theta+f^2 \ldots \theta+h^2)^{-\frac{1}{2}},$$

and therefore

$$e^{2q+1}\frac{dW}{de} = 2e^{2q+2}\lambda^{-q-2}\left[\frac{x^2}{(f^2+\lambda)^2} \cdots + \frac{c^2}{(h^2+\lambda)^2} + \frac{e^2}{\lambda^2}\right]^{-1} (\lambda+f^2 \ldots \lambda+h^2)^{-\frac{1}{2}}.$$

Hence, writing $e=0$, first for an exterior point or $\frac{x^2}{f^2} \ldots + \frac{z^2}{h^2} > 1$, λ is not $=0$, and the expression vanishes in virtue of the factor e^{2q+2}, whence also $\varrho=0$, next for an interior point or $\frac{x^2}{f^2} \ldots + \frac{z^2}{h^2} < 1$, λ is $=0$, hence also $\frac{e^2}{\lambda^2} = \frac{1}{\lambda}\left(1 - \frac{x^2}{f^2} \ldots - \frac{z^2}{h^2}\right)$ is infinite, and neglecting in comparison with it the terms $\frac{x^2}{(f^2+\lambda)^2}$ &c., the value is

$$2\left(\frac{e^2}{\lambda}\right)^q (f \ldots h)^{-1}, \quad = 2\left(1 - \frac{x^2}{f^2} \ldots - \frac{z^2}{h^2}\right)^q (f \ldots h)^{-1},$$

and we have as before,

$$\rho=\frac{\Gamma(\tfrac{1}{2}s+q)}{(\Gamma\tfrac{1}{2})^s\Gamma(q+1)}(f\ldots h)^{-1}\left(1-\frac{x^2}{f^2}\ldots-\frac{z^2}{h^2}\right)^q.$$

101 Hence in the formula

$$\mathrm{V}=\int\frac{\rho\,dx\ldots dz}{\{(a-x)^2\ldots+(c-z)^2+e^2\}^{\frac{1}{2}s+q}}$$

$$=\int_0^\infty t^{-q-1}(t+f^2\ldots t+h^2)^{-\frac{1}{2}}dt,$$

ρ has the value just found, or, what is the same thing, we have

$$\int\frac{\left(1-\frac{x^2}{f^2}\ldots-\frac{z^2}{h^2}\right)^q dx\ldots dz}{\{(a-x)^2\ldots+(c-z)^2+e^2\}^{\frac{1}{2}s+q}}$$

over ellipsoid $\frac{x^2}{f^2}\ldots+\frac{z^2}{h^2}=1$,

$$=\frac{(\Gamma\tfrac{1}{2})^s\,\Gamma(q+1)}{\Gamma(\tfrac{1}{2}s+q)}(f\ldots h)\int_0^\infty t^{-q-1}(t+f^2\ldots t+h^2)^{-\frac{1}{2}}dt.$$

102. We may in this result write $e=0$. There are two cases, according as the attracted point is exterior or interior: if it is exterior, $\frac{a^2}{f^2}\ldots+\frac{c^2}{h^2}>1$, θ will denote the positive root of the equation $\frac{a^2}{f^2+\theta}\ldots+\frac{c^2}{h^2+\theta}=1$; if it be interior, $\frac{a^2}{f^2}\ldots+\frac{c^2}{h^2}<1$, θ will be $=0$; and we thus have

$$\int\frac{\left(1-\frac{x^2}{f^2}\ldots-\frac{z^2}{h^2}\right)^q dx\ldots dz}{\{(a-x)^2\ldots+(c-z)^2\}^{\frac{1}{2}s+q}}$$

$$=\frac{(\Gamma\tfrac{1}{2})^s\Gamma(q+1)}{\Gamma(\tfrac{1}{2}s+q)}(f\ldots h)\int_\theta^\infty t^{-q-1}(t+f^2\ldots t+h^2)^{-\frac{1}{2}}dt,\ \text{for exterior point}\ \frac{a^2}{f^2}\ldots+\frac{c^2}{h^2}>1,$$

$$=\frac{(\Gamma\tfrac{1}{2})^s\Gamma(q+1)}{\Gamma(\tfrac{1}{2}s+q)}(f\ldots h)\int_0^\infty t^{-q-1}(t+f^2\ldots t+h^2)^{-\frac{1}{2}}dt,\ \text{for interior point}\ \frac{a^2}{f^2}\ldots+\frac{c^2}{h^2}<1;$$

but as regards the value for an interior point it is to be observed that unless q be negative (between 0 and -1, since $1+q$ is positive by hypothesis) the two sides of the equation will be each of them infinite.

103. Third example. We assume here

$$\mathrm{V}=\int_\theta^\infty dt\,\mathrm{I}^m\,\mathrm{T},$$

where

$$\mathrm{I}=1-\frac{a^2}{f^2+t}\ldots-\frac{c^2}{h^2+t}-\frac{e^2}{t},$$

$$\mathrm{T}=t^{-q-1}(t+f^2\ldots t+h^2)^{-\frac{1}{2}},$$

and, as before, θ is the positive root of the equation

$$J=1-\frac{a^2}{f^2+\theta}\cdots-\frac{c^2}{h^2+\theta}-\frac{e^2}{\theta},\ =0.$$

$\frac{1}{2}s+q$ is positive in order that the integral may be finite; also m is positive.

104. In order to show that V satisfies the prepotential equation $\square V=0$, I shall, in the first place, consider the more general expression,

$$V=\int_{\theta+\eta}^{\infty} dt\, I^m T,$$

where η is a constant positive quantity which will be ultimately put $=0$. The functions previously called J and Θ will be written J_0 and Θ_0, and J, Θ will now denote

$$J,\ =1-\frac{a^2}{\theta+\eta+f^2}\cdots-\frac{c^2}{\theta+\eta+h^2}-\frac{e^2}{\theta+\eta},$$

$$\Theta,\ =(\theta+\eta)^{-q-1}(\overline{\theta+\eta+f^2}\ .\ \ \overline{\theta+\eta+h^2})^{-\frac{1}{2}},$$

whence also, subtracting from J the evanescent function J_0, we have

$$J=\eta\left(\frac{a^2}{\theta+f^2\,.\,\theta+\eta+f^2}\cdots+\frac{c^2}{\theta+h^2\,.\,\theta+\eta+h^2}+\frac{e^2}{\theta\,.\,\theta+\eta}\right),$$

say this is

$$J=\eta P,$$

and we have thence, by former equations and in the present notation,

$$\frac{a}{\theta+\eta+f^2}\frac{d\theta}{da}\cdot\ \cdot+\frac{c}{\theta+\eta+h^2}\frac{d\theta}{dc}+\frac{e}{\theta+\eta}\frac{d\theta}{de}=\frac{2}{J_0}\ P,$$

$$\nabla_1\theta=\frac{4}{J_0},$$

$$\square\theta=\frac{-4\Theta_0'}{J_0'\Theta_0}.$$

In virtue of the equation which determines θ, we have

$$\frac{dV}{da}=\int_{\theta+\eta}^{\infty} dt\, mI^{m-1}\frac{-2a}{t+f^2}T$$
$$-J^m\Theta\frac{d\theta}{da};$$

and thence

$$\frac{d^2V}{da^2}=\int_{\theta+\eta}^{\infty} dt\left\{mI^{m-1}\frac{-2}{t+f^2}+m(m-1)I^{m-2}\frac{4a^2}{(t+f^2)^2}\right\}T$$
$$\left.\begin{aligned}&-mJ^{m-1}\left(-\frac{2a}{\theta+\eta+f^2}\right)\Theta\frac{d\theta}{da}\\&-mJ^{m-1}\left(\frac{-2a}{\theta+\eta+f^2}\right)\Theta\frac{d\theta}{da}\end{aligned}\right\}$$
$$-\frac{d}{d\theta}(J^m\Theta)\left(\frac{d\theta}{da}\right)^2$$
$$-J^m\Theta\frac{d^2\theta}{da^2},$$

with like expressions for $\ldots\frac{d^2V}{dc^2},\frac{d^2V}{de^2}$.

Also

$$\frac{2q+1}{e}\frac{dV}{de}=\int_{\theta+\eta}^{\infty} dt\, m\mathrm{I}^{m-1}\frac{-4q-2}{t}\mathrm{T}$$
$$-\frac{2q+1}{e}\mathrm{J}^m\Theta\frac{d\theta}{de};$$

and hence

$$\square V=\int_{\theta+\eta}^{\infty} dt\Big[-2m\mathrm{I}^{m-1}\Big\{\frac{1}{t+f^2}\ldots-\frac{1}{t+h^2}+\frac{1+(2q+1)}{t}\Big\}\mathrm{T}$$
$$+m(m-1)\,\mathrm{I}^{m-2}.\,4\Big\{\frac{a^2}{(t+f^2)^2}\ldots+\frac{c^2}{(t+h^2)^2}+\frac{e^2}{t^2}\Big\}\mathrm{T}\Big]$$
$$+4m\mathrm{J}^{m-1}\Theta\left(\frac{a}{\theta+\eta+f^2}\frac{d\theta}{da}\ \ldots+\frac{c}{\theta+\eta+h^2}\frac{d\theta}{dc}+\frac{e}{\theta+\eta}\frac{d\theta}{de}\right)$$
$$-\frac{d}{d\theta}(\mathrm{J}^m\Theta)\left(\left(\frac{d\theta}{da}\right)^2\ldots+\left(\frac{d\theta}{dc}\right)^2+\left(\frac{d\theta}{de}\right)^2\right)$$
$$-\mathrm{J}^m\Theta\left(\frac{d^2\theta}{da^2}\ldots+\frac{d^2\theta}{dc^2}+\frac{d^2\theta}{de^2}+\frac{2q+1}{e}\frac{d\theta}{de}\right).$$

105. Writing I′, T′ for the first derived coefficients of I, T in regard to t, we have

$$\mathrm{I}'=\frac{a^2}{(t+f^2)^2}\ldots+\frac{c^2}{(t+h^2)^2}+\frac{e^2}{t^2},\qquad \frac{\mathrm{T}'}{\mathrm{T}}=-\tfrac{1}{2}\left(\frac{1}{t+f^2}\ldots+\frac{1}{t+h^2}+\frac{2q+2}{t}\right),$$

and the integral is therefore

$$\int_{\theta+\eta}^{\infty} dt\left(2m\mathrm{I}^{m-1}\frac{2\mathrm{T}'}{\mathrm{T}}\mathrm{T}+m(m-1)\mathrm{I}^{m-2}.\,4\mathrm{I}'\mathrm{T}\right),$$
$$=\int_{\theta+\eta}^{\infty} dt(4m\mathrm{I}^{m-1}\mathrm{T}'+4m(m-1)\mathrm{I}^{m-2}\mathrm{I}'\mathrm{T}),$$
$$=\int_{\theta+\eta}^{\infty} dt\,4m\frac{d}{dt}(\mathrm{I}^{m-1}\mathrm{T}),$$

viz., $\mathrm{I}^{m-1}\mathrm{T}$ vanishing for $t=\infty$, this is

$$=-4m\,\mathrm{J}^{m-1}\Theta$$

Hence, writing $(\mathrm{J}^m\Theta)'$ instead of $\frac{d}{d\theta}(\mathrm{J}^m\Theta)$, we have

$$\square V=-4m\,\mathrm{J}^{m-1}\Theta$$
$$+4m\,\mathrm{J}^{m-1}\Theta\left(\frac{a}{\theta+\eta+f^2}\frac{d\theta}{da}\ldots+\frac{c}{\theta+\eta+f^2}\frac{d\theta}{dc}+\frac{e}{\theta+\eta}\frac{d\theta}{de}\right)$$
$$-(\mathrm{J}^m\Theta)'\,\nabla_1\theta$$
$$-\mathrm{J}^m\Theta\ \square\theta;$$

viz this is

$$\square V=-4m\mathrm{J}^{m-1}\Theta$$
$$+8m\mathrm{J}^{m-1}\Theta\ \frac{\mathrm{P}}{\mathrm{J}_0'}$$
$$-4(\mathrm{J}^m\Theta)'\frac{1}{\mathrm{J}_0'}$$
$$+4\,\mathrm{J}^m\Theta\,\frac{\Theta_0'}{\mathrm{J}_0'\Theta_0};$$

or, instead of $(J^m\Theta)'$, writing $mJ^{m-1}J'\Theta+J^m\Theta'$, this is

$$\square V=-\frac{4mJ^{m-1}\Theta}{J_0'}(J'-2P+J)-\frac{4J^m}{J_0'\Theta_0}(\Theta'\Theta_0-\Theta\Theta'_0).$$

We have here

$$J'-2P+J=a^2\left\{\frac{1}{(\theta+\eta+f^2)^2}-\frac{2}{(\theta+\eta+f^2)(\theta+f^2)}+\frac{1}{(\theta+f^2)^2}\right\}\dots+e^2\left\{\frac{1}{(\theta+\eta)^2}-\frac{2}{(\theta+\eta)\theta}+\frac{1}{\theta^2}\right\}$$

$$=\eta^2\left\{\frac{a^2}{(\theta+f^2)^2(\theta+\eta+f^2)^2}\dots+\frac{c^2}{(\theta+h^2)^2(\theta+\eta+h^2)^2}+\frac{e^2}{\theta^2(\theta+\eta)^2}\right\}$$

$$=\eta^2\,.\,Q,\text{ suppose.}$$

Also $\Theta'\Theta_0-\Theta\Theta_0'$ contains the factor η, is $=\eta M$ suppose.

106. Substituting for J, $J'-2P+J$, and $\Theta'\Theta_0-\Theta\Theta_0'$ their values ηP, ηQ, and ηM, the whole result contains the factor η^{m+1}, viz. we have

$$\square V=-\frac{4\eta^{m+1}P^{m-1}}{J_0'}\left(Q\Theta+\frac{PM}{\Theta_0}\right),$$

and if here, except in the term η^{m+1}, we write $\eta=0$, we have

$$P=\frac{a^2}{(\theta+f^2)^2}\dots+\frac{c^2}{(\theta+h^2)^2}+\frac{e^2}{\theta^2},=J_0,$$

$$Q=\frac{a^2}{(\theta+f^2)^4}\dots+\frac{c^2}{(\theta+h^2)^4}+\frac{e^2}{\theta^4},=\tfrac{1}{6}J_0''',$$

$$M=\Theta_0\Theta_0''-\Theta_0'^2,$$

and the formula becomes

$$\square V=-4\eta^{m+1}J_0'^{m-2}\left\{\tfrac{1}{6}J_0'''\Theta_0+J_0'\left(\Theta_0''-\frac{\Theta_0'^2}{\Theta_0}\right)\right\},$$

or (instead of J_0, Θ_0) using now J, Θ in their original significations,

$$J=1-\frac{a^2}{\theta+f^2}\dots-\frac{c^2}{\theta+h^2}-\frac{e^2}{\theta},\text{ and }\Theta=\theta^{-q-1}(\theta+f^2\dots\theta+h^2)^{-\frac{1}{2}},$$

this is

$$\square V=-4\eta^{m+1}J'^{m-2}\left\{\tfrac{1}{6}J'''\Theta+J'\left(\Theta''-\frac{\Theta'^2}{\Theta}\right)\right\},$$

or, what is the same thing,

$$=-4\eta^{m+1}J'^{m-2}\Theta\left\{\tfrac{1}{6}J'''+J'\left(\frac{\Theta'}{\Theta}\right)'\right\},$$

viz. the expression in { } is

$$=\left[\frac{a^2}{(\theta+f^2)^4}\dots+\frac{c^2}{(\theta+h^2)^4}+\frac{e^2}{\theta^4}\right]+\tfrac{1}{2}\left[\frac{a^2}{(\theta+f^2)^2}\dots+\frac{c^2}{(\theta+h^2)^2}+\frac{e^2}{\theta^2}\right]\left[\frac{1}{(\theta+f^2)^2}\dots+\frac{1}{(\theta+h^2)^2}+\frac{e^2}{\theta^2}\right]$$

We thus see that η being infinitesimal $\square V$ is infinitesimal of the order η^{m+1}, and hence η being $=0$, we have

$$\square V=0;$$

viz. the prepotential equation is satisfied by the value

$$V=\int_e^\infty dt\, I^m T,$$

where $m+1$ is positive.

107. We have consequently a value of ρ corresponding to the foregoing value of V; and this value is

$$\rho=-\frac{\Gamma(\tfrac{1}{2}s+q)}{2\pi^{\frac{1}{2}s}\Gamma(q+1)}\left(e^{2q+1}\frac{dW}{de}\right)_{e=0},$$

where, writing λ for the positive root of

$$1-\frac{x^2}{\lambda+f^2}\cdots-\frac{z^2}{\lambda+h^2}-\frac{e^2}{\lambda}=0,$$

we have

$$W=\int_\lambda^\infty dt\left(1-\frac{x^2}{t+f^2}\cdots-\frac{z^2}{t+h^2}-\frac{e^2}{t}\right)^m t^{-q-1}(t+f^2\ldots+t+h^2)^{-\frac{1}{2}},$$

we thence obtain

$$\frac{dW}{de}=\int_\lambda^\infty dt\,.-\frac{2me}{t}\left(1-\frac{x^2}{t+f^2}\cdots-\frac{z^2}{t+h^2}-\frac{e^2}{t}\right)^{m-1}t^{-q-1}(t+f^2\ldots t+h^2)^{-\frac{1}{2}}$$
$$-\left(1-\frac{x^2}{\lambda+f^2}\cdots-\frac{z^2}{\lambda+h^2}-\frac{e^2}{\lambda}\right)^m\lambda^{-q-1}(\lambda+f^2\,..\,\lambda+h^2)^{-\frac{1}{2}}.\frac{d\lambda}{de};$$

or multiplying by e^{2q+1}, and substituting for $\frac{d\lambda}{de}$ its value

$$=\frac{\frac{2e}{\lambda}}{\left\{\frac{x^2}{(\lambda+f^2)^2}\cdots+\frac{z^2}{(\lambda+h^2)^2}+\frac{e^2}{\lambda^2}\right\}},$$

we have

$$e^{2q+1}\frac{dW}{de}=\int_\lambda^\infty dt\,.-\frac{2me^{2q+2}}{t^{q+2}}\left(1-\frac{x^2}{t+f^2}\cdots-\frac{z^2}{t+h^2}-\frac{e^2}{t}\right)^{m+1}(t+f^2\ldots t+h^2)^{-\frac{1}{2}}$$
$$-\frac{\frac{2e^{2q+2}}{\lambda^{q+2}}}{\left\{\frac{x^2}{(\lambda+f^2)^2}\cdots+\frac{z^2}{(\lambda+h^2)^2}+\frac{e^2}{\lambda^2}\right\}}\left(1-\frac{x^2}{\lambda+f^2}\cdots-\frac{z^2}{\lambda+h^2}-\frac{e^2}{\lambda}\right)^m.(\lambda+f^2\ldots\lambda+h^2)^{-\frac{1}{2}},$$

where the second term, although containing the evanescent factor

$$\left(1-\frac{x^2}{\lambda+f^2}\cdots-\frac{z^2}{\lambda+h^2}-\frac{e^2}{\lambda}\right)^m,$$

is for the present retained.

108. I attend to the second term.

1°. Suppose $\frac{x^2}{f^2}\cdots+\frac{z^2}{h^2}>1$, then as e diminishes and becomes $=0$, λ does not become zero, but it becomes the positive root of the equation

$$1-\frac{x^2}{\lambda+f^2}\cdots-\frac{z^2}{\lambda+h^2}=0;$$

hence the term, containing as well the evanescent factor e^{2q+2} as the other evanescent factor $\left(1-\frac{x^2}{\lambda+f^2}\dots-\frac{z^2}{\lambda+h^2}-\frac{e^2}{\lambda}\right)^m$, is $=0$.

2°. Suppose $\frac{x^2}{f^2}\dots+\frac{z^2}{h^2}<1$, then as e diminishes to zero, λ tends to become $=0$, but $\frac{e^2}{\lambda}$ is finite and $=1-\frac{x^2}{f^2}\dots-\frac{z^2}{h^2}$, whence $\frac{e^2}{\lambda^2}$ is indefinitely large, and since $\frac{x^2}{(\lambda+f^2)^2}\dots+\frac{z^2}{(\lambda+h^2)^2}$ becomes $=\frac{x^2}{f^4}\dots+\frac{z^2}{h^4}$, which is finite, the denominator may be reduced to $\frac{e^2}{\lambda^2}$, and the term therefore is

$$=-2\left(\frac{e^2}{\lambda}\right)^q\left(1-\frac{x^2}{\lambda+f^2}\dots-\frac{z^2}{\lambda+h^2}-\frac{e^2}{\lambda}\right)^m.(\lambda+f^2.\ \ .\lambda+h^2)^{-\frac{1}{2}},$$

$$=-2\left(1-\frac{x^2}{f^2}\dots-\frac{z^2}{h^2}\right)^q\left(1-\frac{x^2}{\lambda+f^2}\dots-\frac{z^2}{\lambda+h^2}-\frac{e^2}{\lambda}\right)^m(f\dots h)^{-1},$$

which, the other factor being finite, vanishes in virtue of the evanescent factor

$$\left(1-\frac{x^2}{\lambda+f^2}\dots-\frac{z^2}{\lambda+h^2}-\frac{e^2}{\lambda}\right)^m$$

Hence the second term always vanishes, and we have (e being $=0$)

$$e^{2q+1}\frac{dW}{de}=\int_\lambda^\infty dt\ -\frac{2me^{2q+2}}{t^{q+2}}\left(1-\frac{x^2}{t+f^2}\dots-\frac{z^2}{t+h^2}-\frac{e^2}{t}\right)^m(t+f^2\dots t+h^2)^{-\frac{1}{2}}$$

109. Considering first the case $\frac{x^2}{f^2}\dots+\frac{z^2}{h^2}>1$, then as e diminishes to zero, λ does not become $=0$; the integral contains no infinite element, and it consequently vanishes in virtue of the factor e^{2q+2}.

But if $\frac{x^2}{f^2}\dots+\frac{z^2}{h^2}<1$, then introducing instead of t the new variable ξ, $=\frac{e^2}{t}$, that is $t=\frac{e^2}{\xi}$, $dt=\frac{-e^2}{\xi^2}$, and writing for shortness,

$$R=1-\frac{x^2}{f^2+\frac{e^2}{\xi}}\dots-\frac{z^2}{h^2+\frac{e^2}{\xi}},$$

the term becomes

$$=\int d\xi.\,2m(R-\xi)^{m-1}\xi^q\left(f^2+\frac{e^2}{\xi}\ \dots h^2+\frac{e^2}{\xi}\right)^{-\frac{1}{2}},$$

where, as regards the limits corresponding to $t=\infty$, we have $\xi=0$, and corresponding to $t=\lambda$ we have ξ the positive root of $R-\xi=0$. But e is indefinitely small, except for indefinitely small values of ξ, we have

$$R=1-\frac{x^2}{f^2}\dots-\frac{z^2}{h^2},\ \text{and}\ \left(f^2+\frac{e^2}{\xi}\dots h^2+\frac{e^2}{\xi}\right)^{-\frac{1}{2}}=(f\dots h)^{-1};$$

and if ξ be indefinitely small, then whether we take the accurate or the reduced

expressions, the elements are finite, and the corresponding portion of the integral is indefinitely small. We may consequently reduce as above; viz. writing now

$$R=1-\frac{x^2}{f^2}\ldots-\frac{z^2}{h^2},$$

the formula is

$$e^{2q+1}\frac{dW}{de}=\int_R^0 d\xi\,.\,2m(R-\xi)^{m-1}\xi^q(f\ldots h)^{-1},$$

$$=-2m(f\ldots h)^{-1}\,.\int_0^R d\xi\,.\,\xi^q(R-\xi)^{m-1},$$

or writing $\xi=Ru$, the integral becomes $=R^{q+m}\int_0^1 du\, u^q(1-u)^{m-1}$, which is

$$=\frac{\Gamma(1+q)\Gamma(m)}{\Gamma(1+q+m)}R^{q+m};$$

that is, we have

$$e^{2q+1}\frac{dW}{de}=-2(f\,.\,h)^{-1}\,.\,\frac{\Gamma(1+q)\Gamma(1+m)}{\Gamma(1+q+m)},$$

and consequently

$$\varrho=\frac{\Gamma(\tfrac{1}{2}s+q)}{2(\Gamma\tfrac{1}{2})^s\Gamma(1+q)}\,.\,2(f\,..h)^{-1}\,.\,\frac{\Gamma(1+q)\Gamma(1+m)}{\Gamma(1+q+m)}R^{q+m},$$

that is

$$\varrho=(f\ldots h)^{-1}\frac{\Gamma(\tfrac{1}{2}s+q)\Gamma(1+m)}{(\Gamma\tfrac{1}{2})^s\Gamma(1+q+m)}R^{q+m},$$

viz. ϱ has this value for values of $(x \quad z)$ such that $\frac{x^2}{f^2}\ldots+\frac{z^2}{h^2}<1$, but is $=0$ if $\frac{x^2}{f^2}\ldots+\frac{z^2}{h^2}>1$.

110. Multiplying by a constant factor so as to reduce ϱ to the value R^{q+m}, the final result is

$$V=\int\frac{\left(1-\frac{x^2}{f^2}\ .-\frac{z^2}{h^2}\right)^{q+m}dx\,..\,dz}{[(a-x)^2.\ +(c-z)^2\ ..+e^2]^{\frac{1}{2}s+q}},$$

the limits being given by the equation

$$\frac{x^2}{f^2}\ldots+\frac{z^2}{h^2}=1$$

is

$$=\frac{\Gamma(\tfrac{1}{2})^s\Gamma(1+q+m)}{\Gamma(\tfrac{1}{2}s+q)\Gamma(1+m)}(f\ldots h)\int_\theta^\infty dt\, t^{-q-1}\left(1-\frac{a^2}{t+f^2}\cdots-\frac{c^2}{t+h^2}\right)^m(t+f^2\ldots t+h^2)^{-\frac{1}{2}},$$

where θ is the positive root of

$$1-\frac{a^2}{\theta+f^2}\ ..-\frac{c^2}{\theta+h^2}-\frac{e^2}{\theta}=0.$$

In particular if $e=0$, or

$$V=\int\frac{\left(1-\frac{x^2}{f^2}\cdot\ -\frac{z^2}{h^2}\right)^{q+m}dx\,.\ dz}{\{(a-x)^2\ldots+(c-z)^2\}^{\frac{1}{2}s+q}},$$

there are two cases,

exterior, $\frac{a^2}{f^2}\ldots+\frac{c^2}{h^2}>1$, θ is positive root of $1-\frac{a^2}{f^2}\ldots-\frac{c^2}{h^2}=0$,

interior, $\frac{a^2}{f^2}\ldots+\frac{c^2}{h^2}<1$, θ vanishes, viz. the limits in the integral are ∞, 0;

q must be *negative*, $1+q$ positive as before, in order that the t-integral may not be infinite in regard to the element $t=0$.

It is assumed in the proof that m and $1+q$ are each of them positive, but, as appears by the second example, the theorem is true for the extreme value $m=0$; it does not, however, appear that the proof can be extended to include the extreme value $q=-1$. The formula seems, however, to hold good for values of m, q beyond the foregoing limits; and it would seem that the only necessary conditions are $\frac{1}{2}s+q$, $1+m$, and $1+q+m$, each of them positive. The theorem is in fact a particular case of the following one, proved, Annex X. No. 162, viz

$$V=\int\frac{\phi\left(1-\frac{x^2}{f^2}\ldots-\frac{z^2}{h^2}\right)dx\,..dz}{\{(a-x)^2\ldots+(c-z)^2+e^2\}^{\frac{1}{2}s+q}}$$

over the ellipsoid $\frac{x^2}{f^2}\,..+\frac{z^2}{h^2}=1$,

$$=\frac{(\Gamma\frac{1}{2})^s(f..h)}{\Gamma(-q)\Gamma(\frac{1}{2}s+q)}\int_0^\infty dt\,t^{-q-1}(t+f^2\,.\,t+h^2)^{-\frac{1}{2}}(1-\sigma)^{-q}\int_0^1 x^{-q-1}\phi(\sigma+(1-\sigma)x)dx,$$

where σ denotes $\frac{a^2}{f^2+t}\ldots+\frac{c^2}{h^2+t}+\frac{e^2}{t}$. assuming $\phi u=(1-u)^{q+m}$, we have

$$\phi(\sigma+(1-\sigma)x)=(1-\sigma)^{q+m}(1-x)^{q+m},$$

and the theorem is thus proved.

111. Particular cases

$$m=0;\quad\int\frac{\left(1-\frac{x^2}{f^2}\ldots-\frac{z^2}{h^2}\right)^q dx..\,dz}{[(a-x)^2\ldots+(c-z)^2+e^2]^{\frac{1}{2}s+q}}=\frac{(\Gamma\frac{1}{2})^s\Gamma(1+q)}{\Gamma(\frac{1}{2}s+q)}(f\ .h)\int_0^\infty dt\,t^{-q-1}(t+f^2\ldots t+h^2)^{-\frac{1}{2}}.$$

Cor. In a somewhat similar manner it may be shown that

$$\int\frac{\left(1-\frac{x^2}{f^2}\ldots-\frac{z^2}{h^2}\right)^q x\,dx\,.\,dz}{\{(a-x)^2\,..+(c-z)^2+e^2\}^{\frac{1}{2}s+q}}=\frac{(\Gamma\frac{1}{2})^s\Gamma(1+q)}{\Gamma(\frac{1}{2}s+q)}.(f\ldots h)\int_0^\infty dt\,\frac{af^2}{t+f^2}\,t^{-q-1}(t+f^2\ldots t+h^2)^{-\frac{1}{2}}.$$

Multiply the first by a and subtract the second, we have

$$\int\frac{\left(1-\frac{x^2}{f^2}\ldots-\frac{z^2}{h^2}\right)^q(a-x)\,dx\ldots dz}{\{(a-x)^2\ldots+(c-z)^2+e^2\}^{\frac{1}{2}s+q}}=\frac{(\Gamma\frac{1}{2})^s\Gamma(1+q)}{\Gamma(\frac{1}{2}s+q)}(f..\,h)\int_0^\infty dt\,.\,\frac{a}{t+f^2}\,t^{-q}(t+f^2\ldots t+h^2)^{-\frac{1}{2}};$$

or writing $q+1$ for q, this is

$$\int\frac{\left(1-\frac{x^2}{f^2}\,..-\frac{z^2}{h^2}\right)^{q+1}(a-x)\,dx\,..dz}{\{(a-x)^2\ldots+(c-z)^2+e^2\}^{\frac{1}{2}s+q+1}}=\frac{(\Gamma\frac{1}{2})^s\Gamma(2+q)}{\Gamma(\frac{1}{2}s+q+1)}(f\,..h)\int_0^\infty dt\,.\,\frac{a}{t+f^2}\,t^{-q-1}(t+f^2\ldots t+h^2)^{-\frac{1}{2}};$$

and we have similar formulæ with (instead of $(a-x)$) $\ldots c-z$, e in the numerator.

112. If $m=1$, we have

$$\int\frac{\left(1-\frac{x^2}{f^2}\cdot\,.-\frac{z^2}{h^2}\right)^{q+1}dx\,.\,dz}{\{(a-x)^2\ldots+(c-z)^2+e^2\}^{\frac{1}{2}s+q}}=\frac{(\Gamma\tfrac{1}{2})^s\Gamma(2+q)}{\Gamma(\tfrac{1}{2}s+q)}\,(f\ldots h)\int_\theta^\infty\left\{1-\frac{a^2}{t+f^2}\ldots-\frac{c^2}{t+h^2}-\frac{e^2}{t}\right\}t^{-q-1}(t+f^2\ldots t+h^2)^{-\frac{1}{2}},$$

which, differentiated in respect to a, gives the $(a-x)$formula, hence conversely, assuming the $a-x$, $\ldots c-z$, e formulæ, we obtain by integration the last preceding formula to a constant *près*, viz. we thereby obtain the multiple integral $=\mathrm{C}+$ right-hand function, where C is independent of $(a\ldots c, e)$; and by taking these all infinite, observing that then $\theta=\infty$, the two integrals each vanish, and we obtain $\mathrm{C}=0$.

In particular $s=3$, $q=-1$, then

$$\int\frac{dx\,dy\,dz}{\{(a-x)^2+(b-y)^2+(c-z)^2+e^2\}^{\frac{1}{2}}}=\pi fgh\int_\theta^\infty\left\{1-\frac{a^2}{t+f^2}-\frac{b^2}{t+g^2}-\frac{c^2}{t+h^2}-\frac{e^2}{t}\right\}(t+f^2.\,t+g^2.\,t+h^2)^{-\frac{1}{2}},$$

which, putting therein $e=0$, gives the potential of an ellipsoid for the cases of an exterior point and an interior point respectively.

ANNEX V. GREEN'S *Integration of the Prepotential Equation* $\left(\frac{d^2}{da^2}\ldots+\frac{d^2}{dc^2}+\frac{d^2}{de^2}+\frac{2q+1}{e}\frac{d}{de}\right)\mathrm{V}=0$.—Nos. 113 to 128.

113. In the present Annex I in part reproduce GREEN'S process for the integration of this equation by means of a series of functions analogous to LAPLACE'S Functions, and which may be termed "Greenians" (see his Memoir on the Attraction of Ellipsoids, referred to above), each such function gives rise to a Prepotential Integral.

GREEN shows, by a complicated and difficult piece of general reasoning, that there exist solutions of the form $\mathrm{V}=\Theta\phi$ (see *post*, No. 116), where ϕ is a function of the s new variables $\alpha, \beta\ \ldots\ \gamma$ without θ, such that $\nabla\phi=\varkappa\phi$, $\varkappa$ being a function of θ only; these functions ϕ of the variables $\alpha, \beta\ldots\gamma$ are in fact the Greenian Functions in question. The function of the order 0 is $\phi=1$; those of the order 1 are $\phi=\alpha$, $\phi=\beta\ldots\phi=\gamma$; those of the order 2 are $\phi=\alpha\beta$, &c., and s-functions each of the form

$$\tfrac{1}{2}\{\mathrm{A}\alpha^2+\mathrm{B}\beta^2\ .\ .+\mathrm{C}\gamma^2\}+\mathrm{D}.$$

The existence of the functions just referred to other than the s-functions involving the squares of the variables is obvious enough, the difficulty first arises in regard to these s-functions; and the actual development of them appears to me important by reason of the light which is thereby thrown upon the general theory. This I accomplish in the present Annex, and I determine by GREEN'S process the corresponding prepotential integrals. I do not go into the question of the Greenian Functions of orders superior to the second.

114 I write for greater clearness $(a, b\ldots c, e)$ instead of $(a\ldots c, e)$ to denote the series of $(s+1)$ variables; viz. $(a, b\ldots c)$ will denote a series of s variables; corresponding to these we have the semiaxes $(f, g\ldots h)$, and the new variables $(\alpha, \beta\ldots\gamma)$,

these last, with the before-mentioned function θ, are the $s+1$ new variables of the problem; and for convenience there is introduced also a quantity ϵ, viz. we have

$$\begin{aligned} a &= \sqrt{f^2+\theta}\ \alpha, \\ b &= \sqrt{g^2+\theta}\ \beta, \\ &\vdots \\ c &= \sqrt{h^2+\theta}\ \gamma, \\ e &= \sqrt{\theta}\ \ \ \epsilon, \end{aligned}$$

where $1=\alpha^2+\beta^2\ldots+\gamma^2+\epsilon^2$.

That is, we have θ a function of $a, b \ldots c, e$ determined by

$$\frac{a^2}{f^2+\theta}+\frac{b^2}{g^2+\theta}\ldots+\frac{c^2}{h^2+\theta}+\frac{e^2}{\theta}=1;$$

and then $\alpha, \beta \ldots \gamma$ are given as functions of the same quantities $a, b \ldots c, e$ by the equations

$$\alpha^2=\frac{a^2}{f^2+\theta}, \quad \beta^2=\frac{b^2}{g^2+\theta} \ldots \quad \gamma^2=\frac{c^2}{h^2+\theta},$$

also ϵ, considered as a function of the same quantities, is

$$=\sqrt{1-\frac{a^2}{(f^2+\theta)}-\frac{b^2}{(g^2+\theta)}\ldots-\frac{c^2}{(h^2+\theta)}}.$$

115. Introducing instead of $a, b \ldots c, e$ the new variables $\alpha, \beta \ldots \gamma, \theta$, the transformed differential equation is

$$4\theta\frac{d^2V}{d\theta^2}+2\frac{dV}{d\theta}\left(s+2q+2-\frac{f^2}{f^2+\theta}\ldots-\frac{h^2}{h^2+\theta}\right)+\nabla V=0,$$

where for shortness

$$\begin{aligned} \nabla V = & \frac{1}{f^2+\theta}\left\{-\alpha^2-\frac{g^2}{g^2+\theta}\beta^2\ldots-\frac{h^2}{h^2+\theta}\gamma^2+1\right\}\frac{d^2V}{d\alpha^2} \\ & +\frac{1}{g^2+\theta}\left\{-\frac{f^2}{f^2+\theta}\alpha^2-\beta^2\ldots-\frac{h^2}{h^2+\theta}\gamma^2+1\right\}\frac{d^2V}{d\beta^2} \\ & \ \vdots \\ & +\frac{1}{h^2+\theta}\left\{-\frac{f^2}{f^2+\theta}\alpha^2-\frac{g^2}{g^2+\theta}\beta^2\ldots-\gamma^2+1\right\}\frac{d^2V}{d\gamma^2} \\ & -\frac{2\theta}{f^2+\theta\,.\,g^2+\theta}\frac{d^2V}{d\alpha d\beta}-\&c \\ & +\frac{1}{f^2+\theta}\left\{-2q-2-\theta\left(\frac{1}{g^2+\theta}\ldots+\frac{1}{h^2+\theta}\right)\right\}\alpha\frac{dV}{d\alpha} \\ & +\frac{1}{g^2+\theta}\left\{-2q-2-\theta\left(\frac{1}{f^2+\theta}\ldots+\frac{1}{h^2+\theta}\right)\right\}\beta\frac{dV}{d\beta} \\ & \ \vdots \\ & +\frac{1}{h^2+\theta}\left\{-2q-2-\theta\left(\frac{1}{f^2+\theta}+\frac{1}{g^2+\theta}\ldots\right)\right\}\gamma\frac{dV}{d\gamma} \end{aligned}$$

Also

$$e^{2q+1}\frac{d\mathrm{V}}{de}=-\theta^{q+1}e^{2q+2}\left\{1-\frac{f^2\alpha^2}{f^2+\theta}\cdots-\frac{h^2\gamma^2}{\gamma^2+\theta}\right\}^{-1}\left(\frac{1}{f^2+\theta}\,\alpha\frac{d\mathrm{V}}{d\alpha}\cdot\ \cdot+\frac{1}{h^2+\theta}\,\gamma\frac{d\mathrm{V}}{d\gamma}-2\frac{d\mathrm{V}}{d\theta}\right).$$

116 To integrate the equation for V we assume

$$\mathrm{V}=\Theta\phi,$$

where Θ is a function of θ only, and ϕ a function of $\alpha, \beta\ldots\gamma$ (without θ), such that

$$\nabla\phi=\varkappa\phi,$$

$\varkappa$ being a function of θ only. Assuming that this is possible, the remaining equation to be satisfied is obviously

$$4\theta\frac{d^2\Theta}{d\theta^2}+2\frac{d\Theta}{d\theta}\left\{2q+2+\theta\left(\frac{1}{f^2+\theta}\cdot\ \cdot+\frac{1}{h^2+\theta}\right)\right\}+\varkappa\Theta=0.$$

Solutions of the form in question are

$$\begin{aligned}
&\phi=1\ ,\quad \varkappa=0,\\
&\phi=\alpha\ ,\quad \varkappa=\frac{1}{f^2+\theta}\left\{-2q-2-\frac{\theta}{g^2+\theta}\cdots-\frac{\theta}{h^2+\theta}\right\},\\
&\phi=\beta\ ,\quad \varkappa=\quad ,, \qquad\qquad ,,\\
&\phi=\alpha\beta,\quad \varkappa=\frac{-2\theta}{f^2+\theta\ g^2+\theta}+\frac{1}{f^2+\theta}\left\{-2q-2-\frac{\theta}{g^2+\theta}\cdots-\frac{\theta}{h^2+\theta}\right\},\\
&\qquad\qquad\qquad +\frac{1}{g^2+\theta}\cdot\left\{-2q-2-\frac{\theta}{f^2+\theta}\cdots-\frac{\theta}{g^2+\theta}\right\};
\end{aligned}$$

and it can be shown next that there is a solution of the form

$$\phi=\tfrac{1}{2}(\mathrm{A}\alpha^2+\mathrm{B}\beta^2\ldots+\mathrm{C}\gamma^2)+\mathrm{D}.$$

117. In fact, assuming that this satisfies $\nabla\phi-\varkappa\phi=0$, we must have identically

$$\begin{aligned}
&\frac{\mathrm{A}}{f^2+\theta}\left\{-\alpha^2-\frac{g^2}{g^2+\theta}\beta^2\ldots-\frac{h^2}{h^2+\theta}\gamma^2+1\right\}\\
&+\frac{\mathrm{B}}{g^2+\theta}\left\{-\frac{f^2}{f^2+\theta}\alpha^2-\beta^2\ldots-\frac{h^2}{h^2+\theta}\gamma^2+1\right\}\\
&+\frac{\mathrm{C}}{h^2+\theta}\left\{-\frac{f^2}{f^2+\theta}\alpha^2-\frac{g^2}{g^2+\theta}\beta^2\ldots-\gamma^2+1\right\}\\
&+\frac{\mathrm{A}}{f^2+\theta}\left\{-s-2q-1\ +\frac{g^2}{g^2+\theta}\cdots+\frac{h^2}{h^2+\theta}\right\}\\
&+\frac{\mathrm{B}}{g^2+\theta}\left\{-s-2q-1\ +\frac{f^2}{f^2+\theta}\cdots+\frac{h^2}{h^2+\theta}\right\}\\
&\quad\vdots\\
&+\frac{\mathrm{C}}{h^2+\theta}\left\{-s-2q-1\ +\frac{f^2}{f^2+\theta}+\frac{g^2}{g^2+\theta}\cdots\right\}\\
&+\varkappa\left\{\tfrac{1}{2}(\mathrm{A}\alpha^2+\mathrm{B}\beta^2\ldots+\mathrm{C}\gamma^2)\ +\ \mathrm{D}\ \right\};
\end{aligned}$$

so that from the term in α^2 we have

$$\frac{A}{f^2+\theta}\left\{-s-2q-2+\frac{g^2}{g^2+\theta}\ldots+\frac{h^2}{h^2+\theta}\right\}-\tfrac{1}{2}\varkappa A-\frac{Bf^2}{f^2+\theta\,.\,g^2+\theta}\ \ldots-\frac{Cf^2}{f^2+\theta\,.\,h^2+\theta}=0\,,$$

or, what is the same thing,

$$A\left\{-2q-3-\frac{\theta}{g^2+\theta}\ldots+\frac{\theta}{h^2+\theta}-\tfrac{1}{2}\varkappa(f^2+\theta)\right\}-B\frac{f^2}{g^2+\theta}\ldots-\frac{Cf^2}{h^2+\theta}=0,$$

with the like equations from $\beta^2 \ldots \gamma^2$, and from the constant term we have

$$A\frac{1}{f^2+\theta}+B\frac{1}{g^2+\theta}\ldots+\frac{C}{h^2+\theta}-\varkappa D=0.$$

118 Multiplying this last by f^2, and adding it to the first, we obtain

$$A\left\{-2q-2-\frac{\theta}{f^2+\theta}-\frac{\theta}{g^2+\theta}\ldots-\frac{\theta}{h^2+\theta}-\tfrac{1}{2}\varkappa(f^2+\theta)\right\}-\varkappa f^2 D=0\,,$$

viz. putting for shortness $\Omega=\theta\left(\frac{1}{f^2+\theta}+\frac{1}{g^2+\theta}\ldots+\frac{1}{h^2+\theta}\right)$, this is

$$A\{2q+2+\Omega+\tfrac{1}{2}\varkappa(f^2+\theta)\}+\varkappa f^2D=0,$$

and similarly

$$B\{2q+2+\Omega+\tfrac{1}{2}\varkappa(g^2+\theta)\}+\varkappa g^2D=0,$$

$$\vdots$$

$$C\{2q+2+\Omega+\tfrac{1}{2}\varkappa(h^2+\theta)\}+\varkappa h^2D=0,$$

and to these we join the foregoing equation

$$\frac{A}{f^2+\theta}+\frac{B}{g^2+\theta}\ldots+\frac{C}{h^2+\theta}-\varkappa D=0$$

Eliminating A, B ... C, D we have an equation which determines $\varkappa$ as a function of θ, and the equations then determine the ratios of A, B ... C, D, so that these quantities will be given as determinate multiples of an arbitrary quantity M The equation for $\varkappa$ is in fact

$$\frac{f^2}{(f^2+\theta)\{2q+2+\Omega+\frac{1}{2}\varkappa(f^2+\theta)\}}+\frac{g^2}{(g^2+\theta)\{2q+2+\Omega+\frac{1}{2}\varkappa(g^2+\theta)\}}\ldots$$
$$+\frac{h^2}{(h^2+\theta)\{2q+2+\Omega+\frac{1}{2}\varkappa(h^2+\theta)\}}+1=0\,,$$

and the values of A, B ... C, D are then

$$\frac{Mf^2}{2q+2+\Omega+\frac{1}{2}\varkappa(f^2+\theta)},\ \frac{Mg^2}{2q+2+\Omega+\frac{1}{2}\varkappa(g^2+\theta)},\ \ldots\frac{Mh^2}{2q+2+\Omega+\frac{1}{2}\varkappa(h^2+\theta)},\ -\frac{M}{\varkappa},$$

values which seem to be dependent on θ: if they were so, it would be fatal to the success of the process, but they are really independent of θ.

119. That they are independent of θ depends on the theorem that we have

$$\varkappa=\frac{(2q+2+\Omega)\varkappa_0}{2q+2-\frac{1}{2}\varkappa_0\theta},$$

where $\varkappa_0$ is a quantity independent of θ determined by the equation

$$\frac{1}{2q+2+\frac{1}{2}\varkappa_0 f^2}+\frac{1}{2q+2+\frac{1}{2}\varkappa_0 g^2}\cdots+\frac{1}{2q+2+\frac{1}{2}\varkappa_0 h^2}+1=0,$$

($\varkappa_0$ is in fact the value of $\varkappa$ on writing $\theta=0$), and that, omitting the arbitrary multiplier, the values of A, B ... C, D then are

$$\frac{f^2}{2q+2+\frac{1}{2}\varkappa_0 f^2},\ \frac{g^2}{2q+2+\frac{1}{2}\varkappa_0 g^2},\ \cdots \frac{h^2}{2q+2+\frac{1}{2}\varkappa_0 h^2},\ -\frac{1}{\varkappa_0};$$

or, what is the same thing, the value of ϕ is

$$=\frac{\frac{1}{2}f^2\alpha^2}{2q+2+\frac{1}{2}\varkappa_0 f^2}+\frac{\frac{1}{2}g^2\beta^2}{2q+2+\frac{1}{2}\varkappa_0 g^2}\cdots+\frac{\frac{1}{2}h^2\gamma^2}{2q+2+\frac{1}{2}\varkappa_0 h^2}-\frac{1}{\varkappa_0}.$$

120. [To explain the ground of the assumption

$$\varkappa=\frac{(2q+2+\Omega)\varkappa_0}{2q+2-\frac{1}{2}\varkappa_0\theta},$$

observe that, assuming

$$\frac{2q+2+\Omega+\frac{1}{2}\varkappa(f^2+\theta)}{2q+2+\frac{1}{2}\varkappa_0 f^2}=\frac{2q+2+\Omega+\frac{1}{2}\varkappa(g^2+\theta)}{2q+2+\frac{1}{2}\varkappa_0 g^2},$$

then multiplying out and reducing, we obtain

$$\tfrac{1}{2}\varkappa_0(2q+2+\Omega)(g^2-f^2)+(2q+2)\,.\,\tfrac{1}{2}\varkappa(f^2-g^2)+\tfrac{1}{4}\varkappa_0\varkappa(g^2-f^2)\theta=0;$$

viz. the equation divides out by the factor g^2-f^2, thereby becoming

$$\varkappa_0(2q+2+\Omega)-(2q+2)\varkappa+\tfrac{1}{2}\varkappa\varkappa_0\theta=0,$$

that is, it gives for $\varkappa$ the foregoing value. hence clearly, $\varkappa$ having this value, we obtain by symmetry

$$2q+2+\Omega+\tfrac{1}{2}\varkappa(f^2+\theta),\ 2q+2+\Omega+\tfrac{1}{2}\varkappa(g^2+\theta),\ldots 2q+2+\Omega+\tfrac{1}{2}\varkappa(h^2+\theta),$$

proportional to

$$2q+2+\tfrac{1}{2}\varkappa_0 f^2,\qquad 2q+2+\tfrac{1}{2}\varkappa_0 g^2,\ldots\qquad 2q+2+\tfrac{1}{2}\varkappa_0 h^2,$$

viz. the ratios, not only of A : B, but of A : B ... : C will be independent of θ.]

121. To complete the transformation, starting with the foregoing value of $\varkappa$, we have

$$2q+2+\Omega+\tfrac{1}{2}\varkappa(f^2+\theta)=(2q+2+\Omega)\,.\,\frac{2q+2+\frac{1}{2}\varkappa_0 f^2}{2q+2-\frac{1}{2}\varkappa_0\theta},\ \&c.;$$

so that we have

$$\begin{aligned}&\mathrm{A}\{2q+2+\tfrac{1}{2}\varkappa_0 f^2\}+\varkappa_0 f^2\mathrm{D}=0,\\ &\mathrm{B}\{2q+2+\tfrac{1}{2}\varkappa_0 g^2\}+\varkappa_0 g^2\mathrm{D}=0,\\ &\vdots\\ &\mathrm{C}\{2q+2+\tfrac{1}{2}\varkappa_0 h^2\}+\varkappa_0 h^2\mathrm{D}=0,\end{aligned}$$

and

$$\frac{\mathrm{A}}{f^2+\theta}+\frac{\mathrm{B}}{g^2+\theta}\cdots+\frac{\mathrm{C}}{h^2+\theta}-\frac{(2q+2+\Omega)\varkappa_0\mathrm{D}}{2q+2-\frac{1}{2}\varkappa_0\theta}=0.$$

Substituting for A, B, ... C their values, this last becomes

$$-\frac{\varkappa_0 D}{2q+2+\frac{1}{2}k_0\theta}\left\{\frac{2q+2}{2q+2+\frac{1}{2}\varkappa_0 f^2}-\frac{\theta}{f^2+\theta}\right\}\cdots-\frac{\varkappa_0 D}{2q+2-\frac{1}{2}\varkappa_0\theta}\left\{\frac{2q+2}{2q+2+\frac{1}{2}\varkappa_0 h^2}-\frac{\theta}{h^2+\theta}\right\}$$

$$-\frac{\varkappa_0 D}{2q+2-\frac{1}{2}\varkappa_0\theta}\{2q+2+\Omega\}=0,$$

viz. this is

$$\left\{\frac{2q+2}{2q+2+\frac{1}{2}\varkappa_0 f^2}-\frac{\theta}{f^2+\theta}\right\}\ \cdots+\left\{\frac{2q+2}{2q+2+\frac{1}{2}\varkappa_0 h^2}-\frac{\theta}{h^2+\theta}\right\}+2q+2+\Omega=0,$$

or substituting for Ω its value, and dividing out by $2q+2$, we have

$$\frac{1}{2q+2+\frac{1}{2}\varkappa_0 f^2}+\frac{1}{2q+2+\frac{1}{2}\varkappa_0 g^2}\cdots+\frac{1}{2q+2+\frac{1}{2}\varkappa_0 h^2}+1=0,$$

the equation for the determination of $\varkappa_0$.

122 The equation for $\varkappa_0$ is of the order s, there are consequently s functions of the form in question, and each of the terms $\alpha^2, \beta^2, \ldots \gamma^2$ can be expressed as a linear function of these It thus appears that any quadric function of $\alpha, \beta, \ldots \gamma$ can be expressed as a sum of Greenian functions; viz. the form is

$$\begin{aligned}
&\mathrm{A}\\
&+\mathrm{B}\alpha+\&c.\\
&+\mathrm{C}\alpha\beta+\&c.\\
&+\mathrm{D}'\left(\frac{\frac{1}{2}f^2\alpha^2}{2q+2+\frac{1}{2}\varkappa_0' f^2}+\frac{\frac{1}{2}g^2\beta^2}{2q+2+\frac{1}{2}\varkappa_0' g^2}+\cdots\frac{\frac{1}{2}h^2\gamma^2}{2q+2+\frac{1}{2}\varkappa_0' h^2}-\frac{1}{\varkappa_0'}\right)\\
&+\mathrm{D}''(\qquad ,, \qquad\qquad ,, \qquad\qquad ,, \qquad)\\
&\quad (s \text{ lines}),
\end{aligned}$$

viz the terms multiplied by D′, D″, &c respectively are those answering to the roots $\varkappa_0', \varkappa_0'', \ldots$ of the equation in $\varkappa_0$.

The general conclusion is that any rational and integral function of $\alpha, \beta, \ldots \gamma$ can be expressed as a sum of Greenian functions.

123. We have next to integrate the equation

$$4\theta\frac{d^2\Theta}{d\theta^2}+2\frac{d\Theta}{d\theta}\left(2q+2+\frac{\theta}{f^2+\theta}+\frac{\theta}{g^2+\theta}\cdots+\frac{\theta}{h^2+\theta}\right)-\varkappa\Theta=0.$$

Suppose $\varkappa=0$, a particular solution is $\Theta=1$;

$$\varkappa=\frac{1}{f^2+\theta}\left(-2q-2-\frac{\theta}{g^2+\theta}\cdots-\frac{\theta}{h^2+\theta}\right), \text{ a particular solution is } \frac{\sqrt{f^2+\theta}}{\sqrt{f^2+g^2\ldots+h^2}};$$

in fact, omitting the constant denominator, or writing $\Theta=\sqrt{f^2+\theta}$, and therefore

$$\frac{d\Theta}{d\theta}=\frac{1}{2\sqrt{f^2+\theta}},\quad \frac{d^2\Theta}{d\theta^2}=-\frac{1}{4(f^2+\theta)^{\frac{3}{2}}},$$

the equation to be verified is

$$-\frac{\theta}{(f^2+\theta)^{\frac{3}{2}}}+\frac{1}{\sqrt{f^2+\theta}}\left\{\quad 2q+2+\frac{\theta}{f^2+\theta}+\frac{\theta}{g^2+\theta}\cdots+\frac{\theta}{h^2+\theta}\right\}$$

$$+\frac{1}{\sqrt{f^2+\theta}}\left\{-2q-2\qquad -\frac{\theta}{g^2+\theta}\cdots-\frac{\theta}{h^2+\theta}\right\}=0, \text{ which is right.}$$

Again, suppose $x=\frac{-2\theta}{f^2+\theta\,.\,g^2+\theta}+$&c. (value belonging to $\phi=\alpha\beta$, see No. 116), a particular solution is $\frac{\sqrt{f^2+\theta}\,\sqrt{g^2+\theta}}{f^2+g^2\ldots+h^2}$; in fact omitting the constant factor, or writing

$$\Theta=\sqrt{f^2+\theta}\,\sqrt{g^2+\theta},$$

and therefore

$$\frac{d\Theta}{d\theta}=\tfrac{1}{2}\left\{\frac{\sqrt{g^2+\theta}}{\sqrt{f^2+\theta}}+\frac{\sqrt{f^2+\theta}}{\sqrt{g^2+\theta}}\right\},$$

$$\frac{d^2\Theta}{d\theta^2}=\tfrac{1}{4}\left\{-\frac{\sqrt{g^2+\theta}}{(f^2+\theta)^{\frac{3}{2}}}+\frac{2}{\sqrt{f^2+\theta}\,\sqrt{g^2+\theta}}-\frac{\sqrt{f^2+\theta}}{(g^2+\theta)^{\frac{3}{2}}}\right\},$$

the equation to be verified is

$$\theta\left\{-\frac{\sqrt{g^2+\theta}}{(f^2+\theta)^{\frac{3}{2}}}+\frac{2}{\sqrt{f^2+\theta}\,\sqrt{g^2+\theta}}-\frac{\sqrt{f^2+\theta}}{(g^2+\theta)^{\frac{3}{2}}}\right\}$$

$$+\left(\frac{\sqrt{g^2+\theta}}{\sqrt{f^2+\theta}}+\frac{\sqrt{f^2+\theta}}{\sqrt{g^2+\theta}}\right)\left\{2q+2+\frac{\theta}{f^2+\theta}+\frac{\theta}{g^2+\theta}\cdots+\frac{\theta}{h^2+\theta}\right\}$$

$$+\sqrt{f^2+\theta}\,\sqrt{g^2+\theta}\quad\left\{\frac{-2\theta}{f^2+\theta\,g^2+\theta}+\frac{1}{f^2+\theta}\left(-2q-2-\frac{\theta}{g^2+\theta}\cdots-\frac{\theta}{h^2+\theta}\right)\right.$$

$$\left.+\frac{1}{g^2+\theta}\left(-2q-2-\frac{\theta}{f^2+\theta}\cdots+\frac{\theta}{h^2+\theta}\right)\right\}=0,$$

or putting for shortness $\Omega=\frac{\theta}{f^2+\theta}+\frac{\theta}{g^2+\theta}\;\ldots+\frac{\theta}{h^2+\theta}$, this is

$$-\frac{\theta\sqrt{g^2+\theta}}{(f^2+\theta)^{\frac{3}{2}}}+\frac{2\theta}{\sqrt{f^2+\theta}\,\sqrt{g^2+\theta}}-\frac{\theta\sqrt{f^2+\theta}}{(g^2+\theta)^{\frac{3}{2}}}\qquad+\left(\frac{\sqrt{g^2+\theta}}{\sqrt{f^2+\theta}}+\frac{\sqrt{f^2+\theta}}{\sqrt{g^2+\theta}}\right)(2q+2+\Omega)$$

$$-\frac{2\theta}{\sqrt{f^2+\theta}\,\sqrt{g^2+\theta}}+\frac{\sqrt{g^2+\theta}}{\sqrt{f^2+\theta}}\left(-2q-2+\frac{\theta}{f^2+\theta}-\Omega\right)+\frac{\sqrt{f^2+\theta}}{\sqrt{g^2+\theta}}\left(-2q-2+\frac{\theta}{g^2+\theta}-\Omega\right)=0,$$

which is true

And generally the particular solution is deduced from the value of ϕ by writing therein

$$\frac{\sqrt{f^2+\theta}}{\sqrt{f^2+g^2\ldots+h^2}},\ \frac{\sqrt{g^2+\theta}}{\sqrt{f^2+g^2\ldots+h^2}},\qquad\frac{\sqrt{h^2+\theta}}{\sqrt{f^2+g^2\ldots+h^2}}$$

in place of $\alpha, \beta, \ldots \gamma$ respectively say the value thus obtained is $\Theta=\mathrm{H}$, where H is what ϕ becomes by the above substitution.

124. Represent for a moment the equation in Θ by

$$4\theta\frac{d^2\Theta}{d\theta^2}+2\frac{d\Theta}{d\theta}\mathrm{P}+x\Theta=0,$$

and assume that this is satisfied by $\Theta=\mathrm{H}\int z d\theta$, then we have

$$
\begin{aligned}
&4\theta\left(\frac{d^2\mathrm{H}}{d\theta^2}\int z d\theta+2\frac{d\mathrm{H}}{d\theta}z+\mathrm{H}\frac{dz}{d\theta}\right)\\
&+2\mathrm{P}\left(\frac{d\mathrm{H}}{d\theta}\int z d\theta+\quad \mathrm{H}z\right)\\
&+\varkappa\,.\ \mathrm{H}\int z d\theta \qquad\qquad =0,
\end{aligned}
$$

and therefore

$$\left(8\theta\frac{d\mathrm{H}}{d\theta}+2\mathrm{PH}\right)z+4\theta\mathrm{H}\frac{dz}{d\theta}=0;$$

viz., multiplying by $\frac{\mathrm{H}}{4\theta}$, this is

$$\frac{d}{d\theta}(\mathrm{H}^2z)+\frac{1}{2\theta}\mathrm{PH}^2z=0,$$

or

$$\frac{1}{\mathrm{H}^2z}\frac{d}{d\theta}(\mathrm{H}^2z)+\frac{1}{2\theta}\mathrm{P}=0,$$

viz. substituting for P its value, this is

$$\frac{1}{\mathrm{H}^2z}\frac{d}{d\theta}(\mathrm{H}^2z)+\frac{1}{2\theta}\left(2q+2+\frac{\theta}{f^2+\theta}+\frac{\theta}{g^2+\theta}\cdots+\frac{\theta}{h^2+\theta}\right)=0$$

Hence, integrating,

$$\mathrm{H}^2z=\frac{\mathrm{C}\theta^{-q-1}}{\sqrt{f^2+\theta\,.\,g^2+\theta\ldots h^2+\theta}},\qquad \text{C an arbitrary constant,}$$

and

$$\Theta=\mathrm{CH}\int_\chi\frac{\theta^{-q-1}d\theta}{\mathrm{H}^2\sqrt{f^2+\theta\,.\,g^2+\theta\ldots h^2+\theta}},\qquad \chi \text{ arbitrary,}$$

where the constants of integration are C, λ; or, what is the same thing, taking T the same function of t that H is of θ (viz. T is what ϕ becomes on writing therein

$$\frac{\sqrt{f^2+t}}{\sqrt{f^2+g^2\ldots+h^2}}\cdots\frac{\sqrt{g^2+t}}{\sqrt{f^2+g^2\ldots+h^2}}\quad.\frac{\sqrt{h^2+t}}{\sqrt{f^2+g^2\ldots+h^2}},$$

in place of α, β, γ respectively), then

$$\Theta=-\mathrm{CH}\int_0^\chi\frac{t^{-q-1}dt}{\mathrm{T}^2\sqrt{f^2+t\,.\,g^2+t\ldots h^2+t}},$$

where χ may be taken $=\infty$ we thus have

$$\mathrm{V}=\Theta\phi=-\mathrm{CH}\phi\int_0^\infty\frac{t^{-q-1}dt}{\mathrm{T}^2\sqrt{f^2+t\,.\,g^2+t\ldots h^2+t}}.$$

Recollecting that

$$1=\frac{a^2}{f^2+\theta}+\frac{b^2}{g^2+\theta}\cdots+\frac{c^2}{h^2+\theta}+\frac{e^2}{\theta},$$

so that for $\theta=\infty$ we have $a^2+b^2\ldots+c^2+e^2=\theta$, the assumption $\chi=\infty$ comes to making V vanish for infinite values of $(a, b, \ldots c, e)$.

125. We have to find the value of ρ corresponding to the foregoing value of V; viz. W being the value of V, on writing therein $(x, y, \ldots z)$ in place of $(a, b, \ldots c)$, then (theorem A)

$$\rho = -\frac{\Gamma(\tfrac{1}{2}s+q)}{2(\Gamma\tfrac{1}{2})^s\Gamma(q+1)}\left(e^{2q+1}\frac{dW}{de}\right)_0.$$

Take λ the same function of $(x, y, \ldots z, e)$ that θ is of $(a, b, \ldots c, e)$, viz. λ the positive root of

$$\frac{x^2}{f^2+\lambda}+\frac{y^2}{g^2+\lambda}\cdots+\frac{z^2}{h^2+\lambda}-\frac{e^2}{\lambda}=1,$$

and $(\xi, \eta, \ldots \zeta, \tau)$ corresponding to $(\alpha, \beta, \ldots \gamma, \epsilon)$, viz.

$$\xi=\frac{x}{\sqrt{f^2+\lambda}},\ \eta=\frac{y}{\sqrt{g^2+\lambda}}\cdots\zeta=\frac{z}{\sqrt{h^2+\lambda}},\ \tau=\sqrt{1-\frac{x^2}{f^2+\lambda}-\frac{y^2}{g^2+\lambda}\cdots-\frac{z^2}{h^2+\lambda}},$$

so that W is the same function of $(\xi, \eta, \ldots \lambda)$ that V is of $(\alpha, \beta, \ldots \theta)$: say this is

$$W=-C\Lambda\psi\int_\lambda^\infty \frac{t^{-q-1}\,dt}{T^2\sqrt{f^2+t\,.\,g^2+t\ldots h^2+t}},$$

then we have for ρ the value

$$\rho=\frac{\Gamma(\tfrac{1}{2}s+q)}{2(\Gamma\tfrac{1}{2})^s\Gamma(q+1)}\lambda^{q+1}\tau^{2q+2}\left(1-\frac{f^2\xi^2}{f^2+\lambda}\cdots-\frac{h^2\zeta^2}{h^2+\lambda}\right)^{-1}\cdot\left(\frac{1}{f^2+\lambda}\xi\frac{dW}{d\xi}\cdots+\frac{1}{h^2+\lambda}\zeta\frac{dW}{d\zeta}-2\frac{dW}{d\lambda}\right),$$

where e is to be put $=0$.

126. Suppose e is $=0$, then if $\frac{x^2}{f^2}+\frac{y^2}{g^2}\cdots+\frac{z^2}{h^2}>1$, λ is not $=0$, but is the positive root of $\frac{x^2}{f^2+\lambda}+\frac{y^2}{g^2+\lambda}\cdots+\frac{z^2}{h^2+\lambda}=1$, τ, $=\sqrt{1-\frac{x^2}{f^2+\lambda}-\frac{y^2}{g^2+\lambda}\cdots-\frac{z^2}{h^2+\lambda}}$, is $=0$, and we have $\rho=0$, viz. ρ is $=0$ for all points outside the ellipsoid $\frac{x^2}{f^2}+\frac{y^2}{g^2}\cdots+\frac{z^2}{h^2}=1$.

But if $\frac{x^2}{f^2}+\frac{y^2}{g^2}\cdots+\frac{z^2}{h^2}<1$, then on writing $e=0$, we have $\lambda=0$, $\tau^2=\frac{e^2}{\lambda}$,

$$\begin{aligned}\rho&=\frac{\Gamma(\tfrac{1}{2}s+q)}{2\pi^{\frac{1}{2}s}\Gamma(q+1)}\cdot\lambda^{q+1}\frac{e^{2q+2}}{\lambda^{q+1}}\frac{\lambda}{e^2}\left(\frac{1}{f^2}\xi\frac{dW}{d\xi}+\frac{1}{g^2}\eta\frac{dW}{d\eta}+\ldots+\frac{1}{h^2}\zeta\frac{dW}{d\zeta}-2\frac{dW}{d\lambda}\right)_{\lambda=0}\\&=\frac{\Gamma(\tfrac{1}{2}s+q)}{2\pi^{\frac{1}{2}s}\Gamma(q+1)}\cdot\quad e^{2q}\lambda\ \cdot\ \left(\frac{1}{f^2}\xi\frac{dW}{d\xi}+\frac{1}{g^2}\eta\frac{dW}{d\eta}\cdots+\frac{1}{h^2}\zeta\frac{dW}{d\zeta}-2\frac{dW}{d\lambda}\right)_{\lambda=0},\end{aligned}$$

where term in ( ) is

$$\begin{aligned}&=-C\Lambda_0\psi_0\,.+\frac{2\lambda^{-q-1}}{\Lambda_0^2 fg\ldots h}\\&=-2C\frac{\psi_0}{\Lambda_0 fg\ldots h}\cdot\frac{1}{\lambda^{q+1}}.\end{aligned}$$

Hence

$$\rho=\frac{\Gamma(\frac{1}{2}s+q)}{2\pi^{\frac{1}{2}s}\Gamma(q+1)}\cdot\frac{2C\psi_0}{\Lambda_0 fg..h}\left(\frac{e^2}{\lambda}\right)^q$$

$$=\frac{-\Gamma(\frac{1}{2}s+q)}{2\pi^{\frac{1}{2}s}\Gamma(q+1)}\frac{2C\psi_0}{\Lambda_0 fg\,.\,h}\left(1-\frac{x^2}{f^2}-\frac{y^2}{g^2}\cdots-\frac{z^2}{h^2}\right)^q,$$

where ψ_0, Λ_0 are what ψ, Λ become on writing therein $\lambda=0$ It will be remembered that Λ is what H becomes on changing therein θ into λ, hence Λ_0 is what H becomes on writing therein $\theta=0$.

Moreover ψ is what ϕ becomes on changing therein $\alpha, \beta\,.\,\gamma$ into $\xi, \eta\,..\,\zeta$: writing $\lambda=0$, we have $\xi=\frac{x}{f}$, $\eta=\frac{y}{g}\,.\,\zeta=\frac{z}{h}$, hence ψ_0 is what ϕ becomes on changing therein $\alpha, \beta\,.\,.\gamma$ into $\frac{x}{f}, \frac{y}{g}\ \frac{z}{h}$. And it is proper in ϕ to restore the original variables by writing $\frac{a}{\sqrt{f^2+\theta}}, \frac{b}{\sqrt{g^2+\theta}}\,\cdot\,\frac{c}{\sqrt{h^2+\theta}}$ in place of $\alpha, \beta\ldots\gamma$

127 Recapitulating,

$$V=\int\frac{\rho\, da\ \ dz}{[(a-x)^2\ \ +(c-z)^2+e^2]^{\frac{1}{2}s+q}},$$

where, since for the value of V about to be mentioned ρ vanishes for points outside the ellipsoid, the integral is to be taken over the ellipsoid

$$\frac{x^2}{f^2}\cdots+\frac{z^2}{h^2}=1,$$

and then (transferring a constant factor) if

$$V=\frac{(\Gamma\frac{1}{2})^s\Gamma(q+1)}{\Gamma(\frac{1}{2}s+q)}\Lambda_0(f\ ..h)\ \ H\phi\int_\theta^\infty\frac{t^{-q-1}dt}{T^2\sqrt{t+f^2\ldots t+h^2}},$$

the corresponding value of ρ is

$$\rho=\psi_0\left(1-\frac{x^2}{f^2}\cdots-\frac{z^2}{h^2}\right)^q,$$

where Λ_0 is what H becomes on writing therein $\theta=0$, and ψ_0 is what ψ becomes on writing

$$\frac{x}{f}\cdot\cdot\ \frac{z}{h}\text{ in place of }\alpha\,..\,\gamma.$$

128. Thus putting for shortness $\Omega=t^{-q-1}(t+f^2.\ .t+h^2)^{-\frac{1}{2}}$, we have in the three several cases $\phi=1$, $\phi=\frac{a}{\sqrt{f^2+\theta}}$, $\phi=\frac{ab}{\sqrt{f^2+\theta}\,.\,\sqrt{g^2+\theta}}$ respectively,

$$H=1,\qquad \rho=\ \left(1-\frac{x^2}{f^2}\cdots-\frac{z^2}{h^2}\right)^q,\ V=\frac{(\Gamma\frac{1}{2})^s\Gamma(1+q)}{\Gamma(\frac{1}{2}s+q)}(f\ ..h)\int_\theta^\infty\qquad\Omega dt,$$

$$H=\frac{\sqrt{f^2+\theta}}{\sqrt{f^2\ldots+h^2}},\qquad \rho=x\ (\ \ ,,\ \ \ ,,\ \ )^q,\ V=\ \ ,,\ \ \ ,,\ \ a\int_\theta^\infty\frac{f^2}{f^2+t}\qquad\Omega dt,$$

$$H=\frac{\sqrt{f^2+\theta}\,.\,\sqrt{g^2+\theta}}{f^2\ldots+h^2},\ \rho=xy(\ \ ,,\ \ \ ,,\ \ )^q,\ V=\ \ ,,\ \ \ ,,\ \ ab\int_\theta^\infty\frac{f^2g^2}{f^2+t\,..g^2+t}\Omega dt,$$

and for the case last considered

$$\phi = \frac{\tfrac{1}{2}\dfrac{f^2a^2}{f^2+\theta}}{2q+2+\tfrac{1}{2}\varkappa_0 f^2}\cdots+\frac{\tfrac{1}{2}\dfrac{h^2c^2}{h^2+\theta}}{2q+2+\tfrac{1}{2}\varkappa_0 f^2}-\frac{1}{\varkappa_0},$$

$$\mathrm{H} = \frac{\tfrac{1}{2}f^2(f^2+\theta)}{2q+2+\tfrac{1}{2}\varkappa_0 f^2}\cdots+\frac{\tfrac{1}{2}h^2(h^2+\theta)}{2q+2+\tfrac{1}{2}\varkappa_0 h^2}-\frac{1}{\varkappa_0},\ \text{T same function with } t \text{ for } \theta,$$

$$\psi_0 = \frac{\tfrac{1}{2}x^2}{2q+2+\tfrac{1}{2}\varkappa_0 f^2}\ \ +\frac{\tfrac{1}{2}z^2}{2q+2+\tfrac{1}{2}\varkappa_0 h^2}-\frac{1}{\varkappa_0},$$

$$\Lambda_0 = \frac{\tfrac{1}{2}f^4}{2q+2+\tfrac{1}{2}\varkappa_0 f^2}\cdot+\frac{\tfrac{1}{2}h^4}{2q+2+\tfrac{1}{2}\varkappa_0 h^2}-\frac{1}{\varkappa_0},$$

where $\varkappa_0$ is the root of the equation $\dfrac{1}{2q+2+\tfrac{1}{2}\varkappa_0 f^2}\cdot+\dfrac{1}{2q+2+\tfrac{1}{2}\varkappa_0 h^2}+1=0,$

$$\varrho=\left(1-\frac{x^2}{f^2}\ \cdot-\frac{z^2}{h^2}\right)^q\psi_0,\ \ \mathrm{V}=\frac{(\Gamma\tfrac{1}{2})^s\Gamma(1+q)}{\Gamma(\tfrac{1}{2}s+q)}(f.\ .h)\Lambda_0\mathrm{H}\phi\int_\theta^\infty \mathrm{T}^{-2}t^{-q-1}(t+f^2\ldots t+h^2)^{-\frac{1}{2}}dt.$$

Annex VI. *Examples of Theorem* C.—Nos 129 to 132

129 First example relating to the $(s+1)$ coordinal sphere $x^2\ldots+z^2+w^2=f'^2$. Assume

$$\mathrm{V}'=\frac{\mathrm{M}}{(a^2\ .\ +c^2+e^2)^{\frac{1}{2}(s-1)}},\ \ \mathrm{V}''=\frac{\mathrm{M}}{f^{s-1}},\ \text{(a constant)},$$

these values each satisfy the potential equation.

V′ is not infinite for any point outside the surfaces, and for indefinitely large distances it is of the proper form

V″ is not infinite for any point inside the surface, and at the surface V′=V″.

The conditions of the theorem are therefore satisfied, and writing

$$\mathrm{V}=\int\frac{\varrho\, d\mathrm{S}}{\{(a-x^2)\ \ .+(c-z)^2+(e-w)^2\}^{\frac{1}{2}s-\frac{1}{2}}},$$

we have

$$\varrho=-\frac{\Gamma(\tfrac{1}{2}s-\tfrac{1}{2})}{4(\Gamma\tfrac{1}{2})^{s+1}}\left(\frac{d\mathrm{W}'}{df'}+\frac{d\mathrm{W}''}{df''}\right),$$

where

$$\mathrm{W}'=\frac{\mathrm{M}}{(x^2\ldots+z^2+w^2)^{\frac{1}{2}s-\frac{1}{2}}},\ \ \mathrm{W}''=\frac{\mathrm{M}}{f^{s-1}};\ \text{hence}\ \frac{d\mathrm{W}''}{df''}=0,$$

$$\frac{d\mathrm{W}'}{df'}=\left(\frac{x}{f}\frac{d}{dx}\ \ .+\frac{z}{f}\frac{d}{dz}+\frac{w}{f}\frac{d}{dw}\right)\frac{\mathrm{M}}{(x^2\ .\ z^2+w^2)^{\frac{1}{2}s-\frac{1}{2}}}$$

$$=-\frac{(s-1)\dfrac{1}{f}(x^2\ .\ .+z^2+w^2)\,\mathrm{M}}{(x^2\ \ldots+z^2+w^2)^{\frac{1}{2}s+\frac{1}{2}}},$$

which at the surface is $=\frac{-(s-1)M}{f^s}$.

Hence

$$\rho=\frac{(s-1)\Gamma(\frac{1}{2}s-\frac{1}{2})}{4(\Gamma\frac{1}{2})^{s+1}}\frac{M}{f^s}, \quad =\frac{\Gamma(\frac{1}{2}s+\frac{1}{2})\,.\,M}{2(\Gamma\frac{1}{2})^{s+1}f^s} \text{ (viz. } \rho \text{ is constant)}$$

130. Writing for convenience $M=\frac{2(\Gamma\frac{1}{2})^{s+1}f^s}{\Gamma(\frac{1}{2}s+1)}\delta f$ (δf a constant which may be put $=1$), also $a^2\ldots+c^2+e^2=\kappa^2$, we have $\rho=\delta f$, and consequently

$$\int\frac{\delta f\, dS}{\{(a-x)^2\ldots+(c-z)^2+(e-w)^2\}^{\frac{1}{2}s-\frac{1}{2}}}$$

$$=\frac{2(\Gamma\frac{1}{2})^{s+1}f^s\delta f}{\Gamma(\frac{1}{2}s+\frac{1}{2})}\frac{1}{\kappa^{s-1}} \text{ for exterior point } \kappa>f,$$

$$=\frac{2(\Gamma\frac{1}{2})^{s+1}f^s\delta f}{\Gamma(\frac{1}{2}s+\frac{1}{2})}\frac{1}{f^{s-1}} \text{ for interior point } \kappa<f.$$

By making $a\ .\ c, e$ all indefinitely large we find

$$\int\delta f\, dS=\frac{2(\Gamma\frac{1}{2})^{s+1}f^s\delta f}{\Gamma(\frac{1}{2}s+\frac{1}{2})},$$

viz. the expression on the right-hand side is here the mass of the shell thickness δf

Taking $s=3$ we have the ordinary formulæ for the Potential of a uniform spherical shell

131. Suppose $s=3$, but let the surface be the infinite cylinder $x^2+y^2=f^2$. Take here

$$V'=M\log\sqrt{a^2+b^2},\quad V''=M\log f,$$

each satisfying the potential equation $\frac{d^2V}{da^2}+\frac{d^2V}{db^2}=0$, but V', instead of vanishing, is infinite at infinity, and the conditions of the theorem are not satisfied, the Potential of the cylinder is in fact infinite But the failure is a mere consequence of the special value of s, viz. this is such that $s-2$, instead of being positive, is $=0$. Reverting to the general case of $(s+1)$dimensional space, let the surface be the infinite cylinder $x^2\ldots+z^2=f^2$, and assume

$$V'=\frac{M}{(a^2\ldots+c^2)^{\frac{1}{2}(s-2)}},\quad V''=\frac{M}{f^{s-2}} \text{ (a constant)},$$

these satisfy the potential equation, viz. as regards V', we have

$$\left(\frac{d^2}{da^2}\ .\ +\frac{d^2}{dc^2}+\frac{d^2}{de^2}\right)V'=0, \text{ that is } \left(\frac{d^2}{da^2}\ldots+\frac{d^2}{dc^2}\right)V'=0.$$

V' is not infinite at any point outside the cylinder, and it vanishes at infinity, except indeed when only the coordinate e is infinite, and its form at infinity is not

$$=M\div(a^2\ldots+c^2+e^2)^{\frac{1}{2}(s-1)}.$$

V'' is not infinite for any point within the cylinder, and at the surface we have $V'=V''$.

We have

$$\rho=-\frac{\Gamma(\frac{1}{2}s-\frac{1}{2})}{4(\Gamma\frac{1}{2})^{s+1}}\left(\frac{dW'}{dv'}+\frac{dW''}{dv''}\right),$$

where

$$\frac{dW'}{dv'}=\frac{-(s-2)\frac{1}{f}(x^2\ldots+z^2)M}{(x^2+\ldots z^2)^{\frac{1}{2}s}},=\frac{-(s-2)M}{f^{s-1}}\text{ at the surface},\ \frac{dW''}{dv''}=0,$$

and therefore

$$\rho=\frac{(s-2)\Gamma(\frac{1}{2}s-\frac{1}{2})M}{4(\Gamma\frac{1}{2})^{s+1}f^{s-1}}\ (\text{viz } \rho \text{ is constant});$$

or, what is the same thing, writing $M=\frac{4(\Gamma\frac{1}{2})^{s+1}f^{s-1}\delta f}{(s-2)\Gamma\frac{1}{2}(s-1)}$, whence $\rho=\delta f$, and writing also $a^2\ldots+c^2=\kappa^2$, we have

$$\int\frac{\delta f\,dS}{\{(a-x)^2\ldots+(c-z)^2+(e-w)^2\}^{\frac{1}{2}s-\frac{1}{2}}}$$

$$=\frac{4(\Gamma\frac{1}{2})^{s+1}f^{s-1}\delta f}{(s-2)\Gamma(\frac{1}{2}s-\frac{1}{2})}\frac{1}{\kappa^{s-2}}\text{ for an exterior point } \kappa>f,$$

$$=\frac{4(\Gamma\frac{1}{2})^{s+1}f^{s-1}\delta f}{(s-2)\Gamma(\frac{1}{2}s-\frac{1}{2})}\frac{1}{f^{s-2}}\text{ for interior point } \kappa<f$$

132. This is right, but we can without difficulty bring it to coincide with the result obtained for the $(s+1)$dimensional sphere with only $s-1$ in place of s, we may in fact, by a single integration, pass from the cylinder $x^2\ldots+z^2=f^2$ to the s-dimensional sphere or circle $x^2\ldots+z^2f^2$, which is the base of this cylinder. Writing first $dS=d\Sigma dw$, where $d\Sigma$ refers to the s variables $(x\ldots z)$ and the sphere $x^2\ldots+z^2=f^2$, or using now dS in this sense, then in place of the original dS we have $dSdw$ and the limits of w being ∞, $-\infty$, then in place of $e-w$ we may write simply w. This being so, and putting for shortness $(a-x)^2\ldots+(c-z)^2=A^2$, the integral is

$$\int_{-\infty}^{\infty}dw\int\frac{\delta f\,dS\,dw}{(A^2+w^2)^{\frac{1}{2}(s-1)}},$$

and we have without difficulty

$$\int_{-\infty}^{\infty}\frac{dw}{(A^2+w^2)^{\frac{1}{2}(s-1)}}=\frac{1}{A^{s-2}}\frac{\Gamma\frac{1}{2}\Gamma\frac{1}{2}(s-2)}{\Gamma\frac{1}{2}(s-1)}.$$

[To prove it write $w=A\tan\theta$, then the integral is in the first place converted into $\frac{2}{A^{s-2}}\int_0^{\frac{\pi}{2}}\cos^{s-3}\theta\,d\theta$, which, putting $\cos\theta=\sqrt{x}$ and therefore $\sin\theta=\sqrt{1-x}$, becomes

$$=\frac{1}{A^{s-2}}\int_0^1 x^{\frac{1}{2}-1}(1-x)^{\frac{1}{2}(s-2)-1}dx,$$

which has the value in question.]

Hence replacing A by its value we have

$$\frac{\Gamma\frac{1}{2}\Gamma\frac{1}{2}(s-2)}{\Gamma\frac{1}{2}(s-1)}\int\frac{\delta f\,dS}{\{(a-x)^2\ldots+(c-z)^2\}^{\frac{1}{2}(s-2)}}=\frac{4\pi^{\frac{1}{2}s}\Gamma(\frac{1}{2})f^{s-1}\delta f}{(s-2)\Gamma\frac{1}{2}(s-1)}\left\{\frac{1}{(a^2\ldots+c^2)^{\frac{1}{2}(s-2)}}\text{ or }\frac{1}{f^{s-2}}\right\};$$

that is

$$\int\frac{\delta f\,dS}{\{(a-x)^2\ldots+(c-z)^2\}^{\frac{1}{2}(s-2)}}=\frac{4\pi^{\frac{1}{2}s}f^{s-1}\,\delta f}{(s-2)\Gamma\frac{1}{2}(s-2)}\left\{\frac{1}{(a^2\ldots+c^2)^{\frac{1}{2}(s-2)}}\text{ or }\frac{1}{f^{s-2}}\right\}$$

$$=\frac{2\pi^{\frac{1}{2}s}f^{s-1}\delta f}{\Gamma\frac{1}{2}s}\left\{\frac{1}{(a^2\ldots+c^2)^{\frac{1}{2}(s-2)}}\text{ or }\frac{1}{f^{s-2}}\right\},$$

viz. this is the formula for the sphere with $s-1$ instead of s

ANNEX VII *Example of Theorem* D—Nos. 133 & 134

133. The example relates to the $(s+1)$dimensional sphere $x^2\ldots+z^2+w^2=f^2$ Instead of at once assuming for V a form satisfying the proper conditions as to continuity, we assume a form with indeterminate coefficients, and make it satisfy the conditions in question Write

$$V=\frac{M}{(a^2\ldots+c^2+e^2)^{\frac{1}{2}s-\frac{1}{2}}}\qquad\text{for } a^2\ldots+c^2+e^2>f^2,$$

$$=A(a^2\ldots+c^2+e^2)+B\text{ for } a^2\ldots+c^2+e^2<f^2.$$

In order that the two values may be equal at the surface, we must have

$$\frac{M}{f^{s-1}}=Af^2+B,$$

and in order that the derived functions $\frac{dV}{da}$ &c. may be equal, we must have

$$\frac{-(s-1)aM}{f^{s+1}}=2Aa,\ \&c.,$$

viz. these are all satisfied if only $\frac{-(s-1)M}{f^{s+1}}=2A$

We have thus the values of A and B, or the exterior potential being as above

$$=\frac{M}{(a^2\ldots+c^2+e^2)^{\frac{1}{2}s-\frac{1}{2}}},$$

the value of the interior potential must be

$$=\frac{M}{f^{s-1}}\left\{(\tfrac{1}{2}s+\tfrac{1}{2})-(\tfrac{1}{2}s-\tfrac{1}{2})\cdot\frac{a^2\ldots+c^2+e^2}{f^2}\right\}.$$

The corresponding values of W are of course

$$\frac{M}{(x^2\ldots+z^2+w^2)^{\frac{1}{2}s-\frac{1}{2}}}\text{ and }\frac{M}{f^{s-1}}\left\{(\tfrac{1}{2}s+\tfrac{1}{2})-(\tfrac{1}{2}s-\tfrac{1}{2})\frac{x^2\ldots+z^2+w^2}{f^2}\right\},$$

and we thence find

$$\rho=0\qquad\qquad\text{if } x^2\ldots+z^2+w^2>f^2,$$

$$\rho=-\frac{\Gamma(\frac{1}{2}s-\frac{1}{2})}{4(\Gamma\frac{1}{2})^{s+1}}\{-4(\tfrac{1}{2}s-\tfrac{1}{2})(\tfrac{1}{2}s+\tfrac{1}{2})\}\frac{M}{f^{s+1}},\ =\frac{\Gamma(\frac{1}{2}s+\frac{3}{2})}{(\Gamma\frac{1}{2})^{s+1}}\frac{M}{f^{s+1}}$$

$$\text{if } x^2\ldots+z^2+w^2<f^2.$$

Assuming for M the value $\frac{(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s+\frac{1}{2})}f^{s+1}$, the last value becomes $\varrho=1$; and writing for shortness $a^2\ldots+c^2+e^2=\kappa^2$, we have

$$V=\int\frac{dx\ \ldots dz\,dw}{\{(a-x)^2\ldots+(c-z)^2+(e-w)^2\}^{\frac{1}{2}s+\frac{1}{2}}}\text{ over }(s+1)\text{dimensional sphere }x^2\ldots+z^2+w^2=f^2,$$

$$=\frac{(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s+\frac{1}{2})}\frac{f^{s+1}}{\kappa^{s-1}}\qquad\text{, for exterior point }\kappa>f,$$

$$=\frac{(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s+\frac{1}{2})}\{(\tfrac{1}{2}s+\tfrac{1}{2})f^2-(\tfrac{1}{2}s-\tfrac{1}{2})\kappa^2\},\text{ for interior point }\kappa<f.$$

134 The case of the ellipsoid $\frac{x^2}{f^2}\ldots+\frac{z^2}{h^2}=1$ for $s+1$-dimensional space may be worked out by the theorem, this is in fact what is done in tridimensional space by LEJEUNE DIRICHLET in his Memoir of 1846 above referred to.

ANNEX VIII. *Prepotentials of the Homaloids.*—Nos. 135 to 137

135 We have in tridimensional space the series of figures—the plane, the line, the point, and there is in like manner in $(s+1)$dimensional space a corresponding series of $(s+1)$ terms, the $(s+1)$coordinal plane—the line, the point say these are the homaloids or homaloidal figures. And (taking the density as uniform, or, what is the same thing, $=1$) we may consider the prepotentials of these several figures in regard to an attracted point, which, for greater simplicity, is taken not to be on the figure.

136. The integral may be written

$$V=\int\frac{dw\ \ dt}{\{(a-x)^2\ \ +(c-z)^2+(d-w)^2\ .\ +(e-t)^2+u^2\}^{\frac{1}{2}s+q}},$$

which still relates to a $(s+1)$dimensional space the $(s+1)$ coordinates of the attracted point instead of being $(a\ \ c,e)$ are $(a\ldots c,d\ \ e,u)$, viz we have the s' coordinates $(a\ .\ \ c)$, the $s-s'$ coordinates $(d\ \ e)$, and the $(s+1)$th coordinate u: and the integration is extended over the $(s-s')$dimensional figure $w=-\infty$ to $+\infty,\ldots t=-\infty$ to $+\infty$. And it is also assumed that q is positive

It is at once clear that we may reduce the integral to

$$V=\int\frac{dw\ldots dt}{\{(a-x)^2\ldots+(c-z)^2+u^2+w^2\ldots+t^2\}^{\frac{1}{2}s+q}},$$

say for shortness

$$=\int\frac{dw\ .\ dt}{(A^2+w^2\ .\ +t^2)^{\frac{1}{2}s+q}},$$

where $A^2,=(a-x)^2\ \ +(c-z)^2+u^2$, is a constant as regards the integration, and where the limits in regard to each of the $s-s'$ variables are $-\infty$, $+\infty$.

We may for these variables write $r\xi\ldots r\zeta$, where $\xi^2\ldots+\zeta^2=1$; and we then have

$w^2 \ldots + t^2 = r^2$, $dw \ldots dt = r^{s-s'-1} dr\, dS$, where dS is the element of surface of the $(s-s')$-coordinal unit-sphere $\xi^2 \ldots + \zeta^2 = 1$. We thus obtain

$$V = \int \frac{r^{s-s'-1}\, dr}{\{A^2 + r^2\}^{\frac{1}{2}s+q}} \int dS,$$

where the integral in regard to r is taken from 0 to ∞, and the integral $\int dS$ over the surface of the unit-sphere, hence by Annex I the value of this last factor is $= \frac{2(\Gamma\frac{1}{2})^{s-s'}}{\Gamma\frac{1}{2}(s-s')}$. The integral represented by the first factor will be finite, provided only $\frac{1}{2}s' + q$ be positive, which is the case for any value whatever of s' if only q be positive

The first factor is an integral such as is considered in Annex II., to find its value we have only to write $r = A\sqrt{x}$, and we thus find it to be

$$= \frac{1}{(A^2)^{\frac{1}{2}s'+q}} \frac{1}{2} \int_0^\infty \frac{x^{\frac{1}{2}s-\frac{1}{2}s'-1}\, dx}{(1+x)^{\frac{1}{2}s+q}}, \text{ viz.} = \frac{1}{A^{s'+2q}} \cdot \frac{\frac{1}{2}\Gamma\frac{1}{2}(s-s')\Gamma(\frac{1}{2}s'+q)}{\Gamma(\frac{1}{2}s+q)},$$

and we thus have

$$V = \frac{1}{A^{s'+2q}} \frac{(\Gamma\frac{1}{2})^{s-s'}\Gamma(\frac{1}{2}s'+q)}{\Gamma(\frac{1}{2}s+q)},$$

$$= \frac{(\Gamma\frac{1}{2})^{s-s'}\Gamma(\frac{1}{2}s'+q)}{\Gamma(\frac{1}{2}s+q)} \frac{1}{\{(a-x)^2 \ldots + (c-z)^2 + u^2\}^{\frac{1}{2}s'+q}}.$$

137. As a verification observe that the prepotential equation $\square V = 0$, that is

$$\left(\frac{d^2}{da^2} \cdots + \frac{d^2}{dc^2} + \frac{d^2}{dd^2} \cdots + \frac{d^2}{de^2} + \frac{d^2}{du^2} + \frac{2q+1}{u}\frac{d}{du}\right) V = 0,$$

for a function V which contains only the $s'+1$ variables $(a \ldots c, u)$ becomes

$$\left(\frac{d^2}{da^2} \cdots + \frac{d^2}{dc^2} + \frac{d^2}{du^2} + \frac{2q+1}{u}\frac{d}{du}\right) V = 0,$$

which is satisfied by V a constant multiple of $\{(a-x)^2 \ldots + (c-z)^2 + u^2\}^{\frac{1}{2}-s'-q}$.

ANNEX IX *The* GAUSS-JACOBI *Theory of Epispheric Integrals*—No. 138.

138. The formula obtained (Annex IV No. 110) is proved only for positive values of m; but writing therein $q = 0$, $m = -\frac{1}{2}$, it becomes

$$\int \frac{dx \ldots dz}{\sqrt{1 - \frac{x^2}{f^2} \cdots - \frac{z^2}{h^2}}\, \{(a-x)^2 \ldots + (c-z)^2 + e^2\}^{\frac{1}{2}s}}$$

$$= \frac{(\Gamma\frac{1}{2})^s}{\Gamma\frac{1}{2}s} f \ldots h \int^\infty dt \,.\, t^{-1}\left(1 - \frac{a^2}{t+f^2} \cdots - \frac{c^2}{t+h^2} - \frac{e}{t}\right)^{-\frac{1}{2}} (t+f^2 \ldots t+h^2)^{-\frac{1}{2}},$$

a formula which is obtainable as a particular case of a more general one

$$\int \frac{dS}{\{(*\between x \ldots z, w)^2\}^{\frac{1}{2}s}} = \frac{2(\Gamma\frac{1}{2})^s}{\Gamma(\frac{1}{2}s)} \int_{-\Lambda}^\infty dt \frac{1}{\sqrt{-\text{Disct.}\{(*\between X \ldots Z, W, T)^2 + t(X^2 \ldots + Z^2 + W^2 + T^2)\}}}$$

(notation to be presently explained), being a result obtained by JACOBI by a process which is in fact the extension to any number of variables of that made use of by GAUSS in his Memoir 'Determinatio attractionis quam ..... exerceret planeta, &c.' (1818) I proceed to develop this theory

139. JACOBI's process has reference to a class of s-tuple integrals (including some of those here previously considered) which may be termed "epispheric": viz. considering the $(s+1)$ variables $(x \ldots z, w)$ connected by the equation $x^2 \ldots +z^2+w^2=1$, or say they are the coordinates of a point on a $(s+1)$tuple unit-sphere, then the form is $\int \mathrm{U}d\mathrm{S}$, where $d\mathrm{S}$ is the element of the surface of the unit-sphere, and U is any function of the $s+1$ coordinates; the integral is taken to be of the form $\int \frac{d\mathrm{S}}{\{(*\between x \ldots z, w, 1)^2\}^{\frac{1}{2}s}}$ and we then obtain the general result above referred to

Before going further it is convenient to remark that taking as independent variables the s coordinates $x \ldots z$, we have $d\mathrm{S}=\frac{dx \ldots dz}{dw}$ where w stands for $\pm\sqrt{1-x^2 \ldots -z^2}$, we must in obtaining the integral take account of the two values of w, and finally extend the integral to the values of $x \ldots z$ which satisfy $x^2 \ldots +z^2<1$.

If, as is ultimately done, in place of $x \ldots z$ we write $\frac{x}{f} \ldots \frac{z}{h}$ respectively, then the value of $d\mathrm{S}$ is $=\frac{1}{f \ldots h} \frac{dx \ldots dz}{w}$, where w now stands for $\pm\sqrt{1-\frac{x^2}{f^2} \ldots -\frac{z^2}{h^2}}$, we must in finding the value of the integral take account of the two values of w, and finally extend the integral to the values of $x \ldots z$ which satisfy $\frac{x^2}{f^2} \ldots +\frac{z^2}{h^2}<1$.

140 The determination of the integral depends upon formulæ for the transformation of the spherical element $d\mathrm{S}$, and of the quadric function $(x, y \ldots z, w, 1)^2$.

First, as regards the spherical element $d\mathrm{S}$, let the $s+1$ variables $x, y \ldots z, w$ which satisfy $x^2+y^2 \ldots + z^2+w^2=1$ be regarded as functions of the s independent variables $\theta, \phi, \ldots \psi$, then we have

$$d\mathrm{S}=\begin{vmatrix} x, & y & \ldots & z, & w \\ \frac{dx}{d\theta}, & \frac{dy}{d\theta} & \ldots & \frac{dz}{d\theta}, & \frac{dw}{d\theta} \\ \frac{dx}{d\phi}, & \frac{dy}{d\phi} & \ldots & \frac{dz}{d\phi}, & \frac{d}{d\phi} \\ \frac{dx}{d\psi}, & \frac{dy}{d\psi} & \ldots & \frac{dz}{d\psi}, & \frac{dw}{d\psi} \end{vmatrix} d\theta d\phi \ldots d\psi, =\frac{\partial(x, y \ldots z, w)}{\partial(\theta, \phi \ldots \psi, *)} d\theta\, d\phi \ldots d\psi, \text{ for shortness.}$$

Suppose we effect on the $s+1$ variables $(x, y \ldots z, w)$ a transformation

$$x, y \ldots z, w=\frac{\mathrm{X}}{\mathrm{T}}, \frac{\mathrm{Y}}{\mathrm{T}}, \ldots \frac{\mathrm{Z}}{\mathrm{T}}, \frac{\mathrm{W}}{\mathrm{T}},$$

thus introducing for the moment $s+2$ variables $X, Y, \ldots Z, W, T$, which satisfy identically $X^2+Y^2\ldots+Z^2+W^2-T^2=0$, then considering these as functions of the foregoing s independent variables $\theta, \phi, \ldots \psi$, we have

$$dS=\frac{1}{T^{s+1}}\cdot\left|\begin{matrix} X, & Y\ldots & Z, & W \\ \frac{dX}{d\theta}, & \frac{dY}{d\theta}\cdot\cdot & \frac{dZ}{d\theta}, & \frac{dW}{d\theta} \\ \frac{dX}{d\phi}, & \frac{dY}{d\phi}\cdot\cdot & \frac{dZ}{d\phi}, & \frac{dW}{d\phi} \\ \cdot & & & \\ \frac{dX}{d\psi}, & \frac{dY}{d\psi}\cdots & \frac{dZ}{d\psi}, & \frac{dW}{d\psi} \end{matrix}\right| d\theta\, d\phi\ldots d\psi=\frac{1}{T^{s+1}}\frac{\partial(X, Y\ldots Z, W)}{\partial(\theta, \phi\ldots \psi, *)}\, d\theta\, d\phi\ldots d\psi.$$

141. Considering next the $s+2$ variables $X, Y, \ldots Z, W, T$ as linear functions (with constant terms) of the $s+1$ new variables $\xi, \eta\ldots \zeta, \omega$, or say as linear functions of the $s+2$ quantities $\xi, \eta\ldots \zeta, \omega, 1$, which implies between them a linear relation

$$aX+bY\ldots+cZ+dW+eT=1,$$

and assuming that we have *identically*

$$X^2+Y^2\ldots+Z^2+W^2-T^2=\xi^2+\eta^2\ldots+\zeta^2+\omega^2-1,$$

so that in consequence of the left-hand side being $=0$, the right-hand side is also $=0$, viz. $\xi, \eta\ldots \zeta, \omega$ are connected by

$$\xi^2+\eta^2\ldots+\zeta^2+\omega^2=1$$

let $d\Sigma$ represent the spherical element belonging to the coordinates $\xi, \eta, \ldots \zeta, \omega$. Considering these as functions of the foregoing s independent variables $\theta, \phi, \ldots \psi$, we have

$$d\Sigma=\left|\begin{matrix} \xi, & \eta & \ldots & \zeta, & \omega \\ \frac{d\xi}{d\theta}, & \frac{d\eta}{d\theta} & \ldots & \frac{d\zeta}{d\omega}, & \frac{d\omega}{d\theta} \\ \frac{d\xi}{d\phi}, & \frac{d\eta}{d\phi} & \ldots & \frac{d\zeta}{d\phi}, & \frac{d\omega}{d\phi} \\ \frac{d\xi}{d\psi}, & \frac{d\eta}{d\psi} & \ldots & \frac{d\zeta}{d\psi}, & \frac{d\omega}{d\psi} \end{matrix}\right| d\theta\, d\phi\ldots d\psi=\frac{\partial(\xi, \eta\ldots \zeta, \omega)}{\partial(\theta, \phi\ldots \psi, *)}\, d\theta\, d\phi\ldots d\psi.$$

142. We have in this expression $\xi, \eta\ldots \zeta, \omega$, each of them a linear function of the $s+2$ quantities $X, Y, \ldots Z, W, T$; the determinant is consequently a linear function of $s+2$ like determinants obtained by substituting for the variables any $s+1$ out of the $s+2$ variables $X, Y\ldots Z, W, T$; but in virtue of the equation $X^2+Y^2\ldots+Z^2+W^2-T^2=0$,

these $s+2$ determinants are proportional to the quantities $X, Y \dots Z, W, T$ respectively, and the determinant thus assumes the form

$$\frac{aX+bY \dots + cZ + dW + eT}{T} \Delta,$$

where Δ is the like determinant with $(X, Y, \dots Z, W)$, and where the coefficients $a, b, \dots c, d, e$ are precisely those of the linear relation $aX+bY \dots +cZ+dW+eT=1$; the last-mentioned expression is thus $=\frac{1}{T}\Delta$, or, substituting for Δ its value, we have

$$d\Sigma = \frac{1}{T} \frac{\partial(X, Y \dots Z, W)}{\partial(\theta, \phi \dots \psi, *)} d\theta \, d\phi \dots d\psi;$$

viz. comparing with the foregoing expression for dS we have

$$dS = \frac{1}{T^s} d\Sigma,$$

which is the requisite formula for the transformation of dS.

143. Consider the integral

$$\int \frac{dS}{\{(*\between x, y \dots z, w, 1)^2\}^{\frac{1}{2}s}},$$

which, from its containing a single quadric function, may be called "one-quadric." Then effecting the foregoing transformation,

$$x, y \dots z, w = \frac{X}{T}, \frac{Y}{T}, \dots \frac{Z}{T}, \frac{W}{T},$$

and observing that

$$(*\between x, y \dots z, w, 1)^2 = \frac{1}{T^2} (*\between X, Y \dots Z, W, T)^2,$$

the integral becomes

$$= \int \frac{d\Sigma}{\{(*\between X, Y \dots Z, W, T)^2\}^{\frac{1}{2}s}},$$

where $X, Y \dots Z, W, T$ denote given linear functions (with constant terms) of the $s+1$ variables $\xi, \eta \dots \zeta, \omega$, or, what is the same thing, given linear functions of the $s+2$ quantities $\xi, \eta \dots \zeta, \omega, 1$, such that identically $X^2+Y^2 \dots +Z^2+W^2-T^2=\xi^2+\eta^2 \dots +\zeta^2+\omega^2-1$. We have then $\xi^2+\eta^2 \dots +\zeta^2+\omega^2-1=0$, and $d\Sigma$ as the corresponding spherical element.

144. We may have $X, Y \dots Z, W, T$ such linear functions of $\xi, \eta \dots \zeta, \omega, 1$ that not only

$$X^2+Y^2 \dots +Z^2+W^2-T^2=\xi^2+\eta^2 \dots +\zeta^2+\omega^2-1$$

as above, but also

$$(*\between X, Y, \dots Z, W, T)^2 = A\xi^2+B\eta^2 \dots +C\zeta^2+EW^2-L,$$

and this being so, the integral becomes

$$\int \frac{d\Sigma}{\{A\xi^2+B\eta^2 \ldots +C\zeta^2+EW^2-L\}^{\frac{1}{2}s'}}$$

where the $s+2$ coefficients A, B ... C, E, L are given by means of the identity

$$-(\theta+A)(\theta+B)\ldots(\theta+C)(\theta+E)(\theta+L)$$
$$=\text{Disct.}\{(*\mathcal{X}X, Y \ldots Z, W, T)^2+\theta(X^2+Y^2\ldots+Z^2+W^2-T^2)\},$$

viz. equating the discriminant to zero, we have an equation in θ, the roots whereof are $-A, -B \ldots -C, -E, -L$.

The integral is

$$\int \frac{d\Sigma}{\{(A-L)\xi^2+(B-L)\eta^2 \ldots +(C-L)\zeta^2+(E-L)\omega^2\}^{\frac{1}{2}s'}}$$

which is of the form

$$\int \frac{d\Sigma}{\{a\xi^2+b\eta^2 \ldots +c\zeta^2+e\omega^2\}^{\frac{1}{2}s'}}$$

where I provisionally assume that $a, b \ldots c, e$ are all positive.

145. To transform this, in place of the $s+1$ variables $\xi, \eta \ldots \zeta, \omega$ connected by $\xi^2+\eta^2 \ldots +\zeta^2+\omega^2=1$, we introduce the $s+1$ variables $x, y \ldots z, w$ such that

$$x=\frac{\xi\sqrt{a}}{\varrho},\quad y=\frac{\eta\sqrt{b}}{\varrho},\quad \ldots\quad z=\frac{\zeta\sqrt{c}}{\varrho},\quad w=\frac{\omega\sqrt{d}}{\varrho},$$

where

$$\varrho^2=a\xi^2+b\eta^2\ldots+c\zeta^2+ew^2,$$

and consequently

$$x^2+y^2 \ldots +z^2+w^2=1.$$

Hence writing dS to denote the spherical element corresponding to the point $(x, y \ldots z, w)$, we have by a former formula

$$dS=\frac{1}{\varrho^{s+1}}\frac{\partial(\xi\sqrt{a},\ \eta\sqrt{b} \ldots \zeta\sqrt{c},\ \omega\sqrt{e})}{\partial(\theta, \phi \ldots \psi, *)}\,d\theta\, d\phi \ldots d\psi$$
$$=\frac{(ab \ldots ce)^{\frac{1}{2}}}{\varrho^{s+1}}\,d\Sigma;$$

or, what is the same thing,

$$\frac{d\Sigma}{\{a\xi^2+b\eta^2\ldots+c\zeta^2+e\omega^2\}^{\frac{1}{2}(s+1)}}=\frac{1}{(ab\ldots ce)^{\frac{1}{2}}}\,dS.$$

Hence integrating each side, and observing that $\int dS$, taken over the whole spherical surface $x^2+y^2\ldots+z^2+w^2=1$, is $=2(\Gamma\frac{1}{2})^{s+1}\div\Gamma(\frac{1}{2}s+\frac{1}{2})$, we have

$$\int\frac{d\Sigma}{\{a\xi^2+b\eta^2\ldots c\zeta^2+e\omega^2\}^{\frac{1}{2}(s+1)}}=\frac{2(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s+\frac{1}{2})}\cdot\frac{1}{(ab\ldots ce)^{\frac{1}{2}}}.$$

146. For $a, b \ldots c, e$ write herein $a+\theta, b+\theta \ldots c+\theta, e+\theta$ respectively, and multiplying each side by θ^{q-1}, where q is any positive integer or fractional number less than $\frac{1}{2}s$, integrate from $\theta=0$ to $\theta=\infty$. On the left-hand side, attending to the relation $\xi^2+\eta^2 \ldots +\zeta^2+\omega^2=1$, the integral in regard to θ is

$$\int_0^\infty \frac{\theta^{q-1}\,d\theta}{\{\varrho^2+\theta\}^{\frac{1}{2}(s+1)}},$$

where $\varrho^2, =a\xi^2+b\eta^2 \ldots +c\zeta^2+e\omega^2$, as before is independent of θ; the value of the definite integral is

$$=\frac{\Gamma(\frac{1}{2}(s+1)-q)\Gamma(q)}{\Gamma\frac{1}{2}(s+1)}\,\frac{1}{\varrho^{s+1-2q}},$$

which, replacing ϱ by its value and multiplying by $d\Sigma$, and prefixing the integral sign, gives the left-hand side, hence forming the equation and dividing by a numerical factor, we have

$$\int\frac{d\Sigma}{(a\xi^2 \ldots +c\zeta^2+e\omega^2)^{\frac{1}{2}(s+1)-q}}=\frac{2(\Gamma\frac{1}{2})^{s+1}}{\Gamma q\Gamma\frac{1}{2}(s+1)-q}\int_0^\infty dt\ \ t^{-q-1}(t+a \ldots t+c\,.\,t+e)^{-\frac{1}{2}},$$

and in particular if $q=-\frac{1}{2}$, then

$$\int\frac{d\Sigma}{(a\xi^2 \ldots +c\zeta^2+e^2\omega^2)^{\frac{1}{2}s}}=\frac{2(\Gamma\frac{1}{2})^s}{\Gamma\frac{1}{2}s}\int_0^\infty dt\,.\,t^{-\frac{1}{2}}(t+a \ldots t+c\,.\,t+e)^{-\frac{1}{2}},$$

or, if for $a \ldots c, e$ we restore the values $A-L \ldots C-L$, $E-L$, then

$$\int\frac{d\Sigma}{(A\xi^2 \ldots +C\zeta^2+E\omega^2-L)^{\frac{1}{2}s}}=\frac{2(\Gamma\frac{1}{2})^s}{\Gamma\frac{1}{2}s}\int_0^\infty dt\,.\,t^{-\frac{1}{2}}(t+A-L \ldots t+C-L\,.\,t+E-L)^{-\frac{1}{2}},$$

$$=\frac{2(\Gamma\frac{1}{2})^s}{\Gamma\frac{1}{2}s}\int_{-L}^\infty dt\,.\,(t+A \ldots t+C\,.\,t+E\,.\,t+L)^{-\frac{1}{2}},$$

viz. we thus have

$$\int\frac{dS}{\{(*\between x \ldots z, w, 1)^2\}^{\frac{1}{2}s}}=\frac{2(\Gamma\frac{1}{2})^s}{\Gamma\frac{1}{2}s}\int_{-L}^\infty dt(t+A \ldots t+C\,.\,t+E\,.\,t+L)^{-\frac{1}{2}},$$

where $t+A \ldots t+C\,.\,t+E\,.\,t+L$ is in fact a given rational and integral function of t; viz. it is

$$=-\text{Disct}\,\{(*\between X \ldots Z, W, T)^2+t(X^2 \ldots +Z^2+W^2-T^2)\}.$$

147. Consider in particular the integral

$$\int\frac{dS}{\{(a-fx)^2 \ldots +(c-hz)^2+(e-kw)^2+l^2\}^{\frac{1}{2}s}};$$

here

$$\begin{aligned}(*\between X \ldots Z, W, T)^2+t(X^2 \ldots +Z^2+W^2-T^2)\\ =(aT-fX)^2 \ldots +(cT-hZ)^2+(eT-kW)^2+l^2T^2\\ +t(X^2 \ldots +Z^2+W^2-T^2)\\ =(f^2+t)X^2 \ldots +(h^2+t)Z^2+(k^2+t)W^2+(a^2 \ldots +c^2+e^2+l^2-t)T^2\\ -2afXT \ldots 2chZT-2ekWT;\end{aligned}$$

viz. the discriminant taken negatively is

$$\begin{vmatrix} t+f^2\ldots & & , -af \\ \vdots & & \\ & t+h^2, & -ch \\ -af\ .\ .\ -ch & & -(a^2\ldots+c^2+e^2+l^2)+t \end{vmatrix}$$

which is

$$=t+f^2\ldots t+h^2\,.\,t+k^2\left(t-a^2\ldots-c^2-e^2-l^2+\frac{a^2f^2}{t+f^2}\ldots+\frac{c^2h^2}{t+h^2}+\frac{e^2k^2}{t+k^2}\right),$$

$$=t\,.\,(t+f^2\ldots t+h^2\,.\,t+k^2)\left(1-\frac{a^2}{t+f^2}\ldots-\frac{c^2}{t+h^2}-\frac{e^2}{t+k^2}-\frac{l^2}{t}\right)$$

$$=t+\mathrm{A}\ldots\ t+\mathrm{C}\,.\,t+\mathrm{E}\,.\,t+\mathrm{L},$$

and consequently $-\mathrm{A}\ \ldots -\mathrm{C}, -\mathrm{E}, -\mathrm{L}$ are the roots of the equation

$$1-\frac{a^2}{t+f^2}\ldots-\frac{c^2}{t+h^2}-\frac{e^2}{t+k^2}-\frac{l^2}{t}=0$$

148. The roots are all real, moreover there is one and only one positive root. Hence taking $-\mathrm{L}$ to be the positive root, we have $\mathrm{A}\ldots\ \mathrm{C}, \mathrm{E}, -\mathrm{L}$ all positive, and therefore *à fortiori* $\mathrm{A}-\mathrm{L}, \ldots\ \mathrm{C}-\mathrm{L}, \mathrm{E}-\mathrm{L}$ all positive, which agrees with a foregoing provisional assumption. Or, writing for greater convenience θ to denote the positive quantity $-\mathrm{L}$, that is taking θ to be the positive root of the equation

$$1-\frac{a^2}{\theta+f^2}\ldots-\frac{c^2}{\theta+h^2}-\frac{e^2}{\theta+k^2}-\frac{l^2}{\theta}=0,$$

we have

$$\int\frac{d\mathrm{S}}{\{(a-fx)^2\ldots+(c-hz)^2+(e-kw)^2+l^2\}^{\frac{1}{2}s}}$$

$$=\frac{2(\Gamma\frac{1}{2})^s}{\Gamma\frac{1}{2}s}\int_\theta^\infty dt\,\frac{1}{\sqrt{t\,.\,t+f^2\ldots\ t+h^2\,.\,t+k^2}\left(1-\frac{a^2}{t+f^2}\ldots-\frac{c^2}{t+h^2}-\frac{e^2}{t+k^2}-\frac{l^2}{t}\right)};$$

or, what is the same thing, we have

$$\frac{1}{f\ldots h}\int\frac{dx\ldots dz}{\pm w\{(a-x)^2\ldots+(c-z)^2+(e\mp kw)^2+l^2\}^{\frac{1}{2}s}}$$

$$=\frac{\Gamma\frac{1}{2}s}{2(\Gamma\frac{1}{2})^s}\int_\theta^\infty dt\left(1-\frac{a^2}{t+f^2}\ldots-\frac{c^2}{t+h^2}-\frac{e^2}{t+k^2}-\frac{l^2}{t}\right)^{-\frac{1}{2}}(t\,.\,t+f^2\ldots t+h^2\,.\,t+k^2)^{-\frac{1}{2}},$$

where on the left-hand side w now denotes $\sqrt{1-\frac{x^2}{f^2}\ldots-\frac{z^2}{h^2}}$, and the limiting equation is $\frac{x^2}{f^2}\ldots+\frac{z^2}{h^2}=1$.

149. Suppose $l=0$, then if

$$\frac{a^2}{f^2}\ldots+\frac{c^2}{h^2}+\frac{e^2}{k^2}>1,$$

the equation

$$1-\frac{a^2}{\theta+f^2}\cdots-\frac{c^2}{\theta+h^2}-\frac{e^2}{\theta+k^2}=0$$

has a positive root differing from zero, which may be represented by the same letter θ; but if

$$\frac{a^2}{f^2}\cdots+\frac{c^2}{h^2}+\frac{e^2}{k^2}<1,$$

then the positive root of the original equation becomes $=0$; viz. as l gradually diminishes to zero the positive root θ also diminishes, and becomes ultimately zero.

Hence writing $l=0$, we have

$$\int\frac{dS}{\{(a-fx)^2\ldots+(c-hz)^2+(e-kw)^2\}^{\frac{1}{2}s}},$$

or, what is the same thing,

$$\frac{1}{f\ldots h}\int\frac{dx\ldots dz}{\pm w\{(a-x)^2\ldots+(c-z)^2+(e\mp kw)^2\}^{\frac{1}{2}s}}$$

$$=\frac{2(\Gamma\frac{1}{2})^s}{\Gamma\frac{1}{2}s}\int_\theta^\infty dt\left(1-\frac{a^2}{t+f^2}\cdots-\frac{c^2}{t+h^2}-\frac{e^2}{t+k^2}\right)^{-\frac{1}{2}}(t\,.\,t+f^2\ldots t+h^2\,.\,t+k^2)^{-\frac{1}{2}},$$

θ now denoting the positive root of the equation

$$1-\frac{a^2}{\theta+f^2}\cdots-\frac{c^2}{\theta+h^2}-\frac{e^2}{\theta+k^2}=0,$$

or else denoting 0, according as

$$\frac{a^2}{f^2}\ldots+\frac{c^2}{h^2}+\frac{e^2}{k^2}>1 \text{ or } <1.$$

In the case $\frac{a^2}{f^2}\ldots+\frac{e^2}{k^2}<1$, the inferior limit being then 0, this is in fact JACOBI'S theorem (Crelle, t. XII. p. 69, 1834), but JACOBI does not consider the general case where l is not $=0$, nor does he give explicitly the formula in the other case

$$l=0,\ \frac{a^2}{f^2}\ldots+\frac{c^2}{h^2}+\frac{e^2}{k^2}>1.$$

150. Suppose $k=0$, e being in the first instance not $=0$, then the former alternative holds good, and observing, in regard to the form which contains $\pm w$ in the denominator, that we can now take account of the two values by simply multiplying by 2, we have

$$\int\frac{dS}{\{(a-fx)^2\ldots+(c-hz)^2+e^2\}^{\frac{1}{2}s}},\quad=\frac{2}{f\ldots h}\int\frac{dx\ldots dz}{w\{(a-x)^2\ldots+(c-z)^2+e^2\}^{\frac{1}{2}s}},$$

(w on the right-hand side denoting $\sqrt{1-\frac{x^2}{f^2}\ldots-\frac{z^2}{h^2}}$, and the limiting equation being $\frac{x^2}{f^2}\ldots+\frac{z^2}{h^2}=1$), each

$$=\frac{2(\Gamma\frac{1}{2})^s}{\Gamma\frac{1}{2}s}\int_\theta^\infty dt\left(1-\frac{a^2}{t+f^2}\cdots-\frac{c^2}{t+h^2}-\frac{e^2}{t}\right)^{-\frac{1}{2}}t^{-1}(t+f^2\ldots t+h^2)^{-\frac{1}{2}},$$

where θ is here the positive root of the equation $1-\frac{a^2}{\theta+f^2}\cdots-\frac{c^2}{\theta+h^2}-\frac{e^2}{\theta}=0$, which is the formula referred to at the beginning of the present Annex. We may in the formula write $e=0$, thus obtaining the theorem under two different forms for the cases $\frac{a^2}{f^2}\cdots+\frac{c^2}{h^2}>1$ and <1 respectively

ANNEX X. *Methods of* LEJEUNE-DIRICHLET *and* BOOLE.—Nos. 151 to 162.

151 The notion that the density ρ is a discontinuous function vanishing for points outside the attracting mass has been made use of in a different manner by LEJEUNE-DIRICHLET (1839) and BOOLE (1857). viz. supposing that ρ has a given value $f(x\ldots z)$ within a given closed surface S and is $=0$ outside the surface, these geometers in the expression of a potential or prepotential integral replace ρ by a definite integral which possesses the discontinuity in question, viz. it is $=f(x\ldots z)$ for points inside the surface and $=0$ for points outside the surface; and then in the potential or prepotential integral they extend the integration over the whole of infinite space, thus getting rid of the equation of the surface as a limiting equation for the multiple integral.

152. LEJEUNE-DIRICHLET's paper "Sur une nouvelle méthode pour la détermination des intégrales multiples" is published in 'Comptes Rendus,' t VIII. pp 155–160 (1839), and Liouv. t. IV pp. 164–168 (same year). The process is applied to the form

$$-\frac{1}{p-1}\frac{d}{da}\int\frac{dx\,dy\,dz}{\{(a-x)^2+(b-y)^2+(c-z)^2\}^{\frac{1}{2}(p-1)}}$$

over the ellipsoid $\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}+\frac{z^2}{\gamma^2}=1$, but it would be equally applicable to the triple integral itself, or say to the s-tuple integral

$$\int\frac{dx\ldots dz}{\{(a-x)^2\ldots+(c-z)^2\}^{\frac{1}{2}s+q}},$$

or, indeed, to

$$\int\frac{dx\ldots dz}{\{(a-x)^2\ldots+(c-z)^2+e^2\}^{\frac{1}{2}s+q}}$$

over the ellipsoid $\frac{x^2}{f^2}\cdots+\frac{z^2}{h^2}=1$; but it may be as well to attend to the first form, as more resembling that considered by the author

153. Since $\frac{2}{\pi}\int_0^\infty\frac{\sin\phi}{\phi}\cos\lambda\phi\,d\phi$ is $=1$ or 0, according as λ is <1 or >1, it follows that the integral is equal to the real part of the following expression,

$$\frac{2}{\pi}\int_0^\infty d\phi\frac{\sin\phi}{\phi}\int e^{i\left(\frac{x^2}{f^2}\cdots+\frac{z^2}{h^2}\right)}\frac{dx\ldots dz}{\{(a-x)^2\ldots+(c-z)^2\}^{\frac{1}{2}s+q}},$$

where the integrations in regard to $x \dots z$ are now to be extended from $-\infty$ to $+\infty$ for each variable. A further transformation is necessary: since

$$\frac{1}{\sigma^r}=\frac{1}{\Gamma r}e^{-r\pi i}\int_0^\infty d\psi\,.\,\psi^{r-1}e^{i\sigma\psi}, \qquad \sigma \text{ positive and } r \text{ positive and } <1,$$

writing herein $(a-x)^2 \dots +(c-z)^2$ for σ, and $\frac{1}{2}s+q$ for r, we have

$$\frac{1}{\{(a-x)^2 \dots +(c-z)^2\}^{\frac{1}{2}s+q}}=\frac{1}{\Gamma(\frac{1}{2}s+q)}\,e^{-(\frac{1}{2}s+q)\pi i}\int_0^\infty d\psi\,.\,\psi^{\frac{1}{2}s+q-1}\,e^{i\psi\{(a-x)^2 \dots +(c-z)^2\}},$$

and the value is thus

$$=\frac{2}{\pi\Gamma(\frac{1}{2}s+q)}\,e^{-(\frac{1}{2}s+q)\frac{\pi i}{2}}\int_0^\infty d\phi\,\frac{\sin\phi}{\phi}\int_0^\infty d\psi\,.\,\psi^{\frac{1}{2}s+q-1}\int e^{i\left(\frac{x^2}{f^2} \dots +\frac{z^2}{h^2}\right)\phi}\,e^{-i\psi\{(a-x)^2 \dots +(c-z)^2\}}\,dx \dots dz,$$

where the integral in regard to the variables $(x \dots z)$ is

$$=e^{i\psi(a^2 \dots +c^2)}\int dx\,e^{i\left\{\left(\psi+\frac{\phi}{f^2}\right)x^2+2a\psi x\right\}} \dots \int dz\,e^{i\left\{\left(\psi+\frac{\phi}{h^2}\right)z^2-2c\psi z\right\}},$$

and here the x-integral is

$$=e^{\frac{1}{4}i\pi}\sqrt{\frac{f^2\pi}{f^2\psi+\phi}}\,e^{-\frac{a^2f^2\psi^2 i}{f^2\psi+\phi}},$$

and the like for the other integrals up to the z-integral. The resulting value is thus

$$=\frac{2}{\pi\Gamma(\frac{1}{2}s+q)}\,e^{-\frac{1}{2}q\pi i}\int_0^\infty\frac{\sin\phi}{\phi}\,.\int_0^\infty d\psi\;\psi^{\frac{1}{2}s+q-1}\,e^{\psi\phi i\left(\frac{a^2}{\phi+f^2\psi} \dots +\frac{c^2}{\phi+h^2\psi}\right)}\,\frac{\pi^{\frac{1}{2}s}f \dots h}{\sqrt{\phi+f^2\psi \dots \phi+h^2\psi}},$$

which, putting therein $\psi=\frac{\phi}{t}$, $d\psi=-\frac{\phi}{t^2}\,dt$, is

$$=\frac{2\pi^{\frac{1}{2}s-1}}{\Gamma(\frac{1}{2}s+q)}\,(f \dots h)\,e^{-\frac{1}{2}q\pi i}\int_0^\infty dt\,\frac{t^{-q-1}}{\sqrt{f^2+t \dots h^2+t}}\int_0^\infty e^{i\phi\left(\frac{a^2}{f^2+t} \dots +\frac{c^2}{h^2+t}\right)}\sin\phi\,.\,\phi^{q-1}\,d\phi.$$

154. But we have to consider only the real part of this expression, viz. writing for shortness $\sigma=\frac{a^2}{f^2+t} \dots +\frac{c^2}{h^2+t}$, we require the real part of

$$e^{-\frac{1}{2}q\pi i}\int_0^\infty e^{i\sigma\phi}\,.\,\phi^{q-1}\sin\phi\,d\phi.$$

Writing here for $\sin\phi$ its exponential value $\frac{1}{2i}(e^{i\phi}-e^{-i\phi})$, and using the formula

$$\frac{1}{\sigma^q}=\frac{1}{\Gamma q}\,e^{-q\pi i}\int_0^\infty d\phi\,.\,\phi^{q-1}\,.\,e^{i\sigma\phi} \qquad (\sigma \text{ positive}),$$

and the like one

$$\frac{1}{(-\sigma)^q}=\frac{1}{\Gamma q}\,e^{q\pi i}\int_0^\infty d\phi\,.\,\phi^{q-1}\,e^{i\sigma\phi} \qquad (\sigma \text{ negative})$$

(in which formulæ q must be positive and less than 1), we see that the real part in question is $=0$, or is

$$-\frac{\Gamma q\sin(q+1)\pi}{2(1-\sigma)^q},\ =\frac{\pi}{2\Gamma(1-q)}\,\frac{1}{(1-\sigma)^q},$$

according as $\sigma>1$ or $\sigma<1$.

155. If the point is interior, $\frac{a^2}{f^2}\ldots+\frac{c^2}{h^2}<1$, and consequently also $\sigma<1$, and the value, writing $(\Gamma\frac{1}{2})^2$ instead of π, is

$$=\frac{(\Gamma\frac{1}{2})^s}{\Gamma(\frac{1}{2}s+q)\Gamma(1-q)}(f\ldots h)\int_0^\infty dt\,.\,t^{-q-1}(t+f^2\ \ .\,t+h^2)^{-\frac{1}{2}}\left(1-\frac{a^2}{f^2+t}\cdots-\frac{c^2}{h^2+t}\right)^{-q}.$$

But if the point be exterior, $\frac{a^2}{f^2}\ldots+\frac{c^2}{h^2}>1$, and hence, writing θ for the positive root of the equation, $\sigma=1$; viz. θ is the positive root of the equation $\frac{a^2}{f^2+\theta}\ldots+\frac{c^2}{h^2+\theta}=1$, then $t=0$, σ is greater than 1, and continues so as t increases, until, for $t=\theta$, σ becomes $=1$, and for larger values of t we have $\sigma<1$, and the expression thus is

$$=\frac{(\Gamma\frac{1}{2})^s}{\Gamma(\frac{1}{2}s+q)\Gamma(1-q)}(f.\ .h)\int_\theta^\infty dt\,.\,t^{-q-1}(t+f^2\ldots\ t+h^2)^{-\frac{1}{2}}\left(1-\frac{a^2}{f^2-t}\cdots-\frac{c^2}{h^2+t}\right)^{-q};$$

viz. the two expressions in the cases of an interior point and an exterior point respectively give the value of the integral

$$\int\frac{dx.\ dz}{\{(a-x)^2\ .\ +(c-z)^2\}^{\frac{1}{2}s+q}}.$$

This is in fact the formula of Annex IV. No. 110, writing therein $e=0$ and $m=-q$.

156. BOOLE's researches are contained in two memoirs dated 1846, "On the Analysis of Discontinuous Functions," Trans. Royal Irish Academy, vol xxi (1848), pp. 124–139, and "On a certain Multiple Definite Integral," do. pp. 140–150 (the particular theorem about to be referred to is stated in the postscript of this memoir), and in the memoir "On the Comparison of Transcendents, with certain applications to the theory of Definite Integrals," Phil. Trans. vol. 147, for 1857, pp 745–803, the theorem being the third example, p 794. The method is similar to that of, and was in fact suggested by, LEJEUNE-DIRICHLET, the auxiliary theorem made use of in the memoir of 1857 for the representation of the discontinuity being

$$\frac{f(x)}{t^i}=\frac{1}{\pi\Gamma i}\int_{-\infty}^\infty\int_0^\infty\int_0^\infty da\,dv\,ds\cos\{(a-x-ts)v+\tfrac{1}{2}i\pi\}v^is^{i-1}f(a),$$

which is a deduction from FOURIER's theorem.

Changing the notation (and in particular writing s and $\frac{1}{2}s+q$ for his n and i) the method is here applied to the determination of the s-tuple integral

$$\mathrm{V}=\int dx\ldots dz\,\frac{\phi\left(\frac{x^2}{f^2}\cdots+\frac{z^2}{h^2}\right)}{\{(a-x)^2\ldots+(c-z)^2+e^2\}^{\frac{1}{2}s+q}}$$

(where ϕ is an arbitrary function) over the ellipsoid $\frac{x^2}{f^2}\ldots+\frac{z^2}{h^2}=1$.

157. The process is as follows: we have

$$\frac{\phi\left(\frac{x^2}{f^2}\cdots+\frac{z^2}{h^2}\right)}{\{(a-x)^2\ldots+(c-z)^2+e^2\}^{\frac{1}{2}s+q}}=\frac{1}{\pi\Gamma(\frac{1}{2}s+q)}\int_0^1\int_0^\infty\int_0^\infty du\,dv\,d\tau\,v^{\frac{1}{2}s+q}\tau^{\frac{1}{2}s+q-1}$$
$$\cos\left\{\left(u-\frac{x^2}{f^2}\cdots-\frac{z^2}{h^2}-\tau\left((a-x)^2\ldots+(c-z)^2+e^2\right)v\right)+\tfrac{1}{2}(\tfrac{1}{2}s+q)\pi\right\}\phi u;$$

viz. the right-hand side is here equal to the left-hand side or is $=0$, according as $\frac{x^2}{f^2}\ldots+\frac{z^2}{h^2}<1$ or >1 V is consequently obtained by multiplying the right-hand side by $dx\ldots dz$ and integrating from $-\infty$ to $+\infty$ for each variable.

Hence, changing the order of the integration,

$$\mathrm{V}=\frac{1}{\pi\Gamma(\frac{1}{2}s+q)}\int_0^1\int_0^\infty\int_0^\infty du\,dv\,d\tau\,v^{\frac{1}{2}s+q}\tau^{\frac{1}{2}s+q-1}\phi u\,.\,\Omega,$$

where

$$\Omega=\int dx\quad dz\cos\left\{\left(u-e^2\tau-\frac{x^2}{f^2}\,..-\frac{z^2}{h^2}+\tau\{(a-x)^2\,.+(c-z)^2\}\right)v+\tfrac{1}{2}(\tfrac{1}{2}s+q)\pi\right\}.$$

Now

$$\frac{x^2}{f^2}+\tau(a-x)^2=\frac{1+f^2\tau}{f^2}\xi^2+\frac{\tau a^2}{1+f^2\tau},\qquad \frac{z^2}{h^2}+\tau(c-z)^2=\frac{1+h^2\tau}{h^2}\zeta^2+\frac{\tau c^2}{1+h^2\tau},$$

if

$$\xi=x-\frac{f^2\tau a}{1+f^2\tau},\qquad ..\,\zeta=z-\frac{h^2\tau c}{1+h^2\tau}.$$

158 Substituting, and integrating with respect to $\xi\,.\,.\,\zeta$ between the limits $-\infty$, $+\infty$, we have

$$\Omega=\frac{(f\quad h)\pi^{\frac{1}{2}s}}{(1+f^2\tau\quad 1+h^2\tau)^{\frac{1}{2}}v^{\frac{1}{2}s}}\cos\left\{\left(u-e^2\tau-\frac{a^2\tau}{1+f^2\tau}\,.\,-\frac{c^2\tau}{1+h^2\tau}\right)v+\tfrac{1}{2}q\pi\right\},$$

or, what is the same thing, writing $\frac{1}{t}$ in place of τ, this is

$$\Omega=\frac{(f\quad h)\pi^{\frac{1}{2}s}t^{\frac{1}{2}}}{(f^2+t\,..h^2+t)^{\frac{1}{2}}v^{\frac{1}{2}s}}\cos\left\{\left(u-\frac{a^2}{f^2+t}\cdots-\frac{c^2}{h^2+t}-\frac{e^2}{t}\right)v+\tfrac{1}{2}q\pi\right\},$$

that is, writing

$$\sigma=\frac{a^2}{f^2+t}\cdots+\frac{c^2}{h^2+t}+\frac{e^2}{t},$$

we have

$$\mathrm{V}=\frac{\pi^{\frac{1}{2}s-1}(f\quad h)}{\Gamma(\frac{1}{2}s+q)}\int_0^1\int_0^\infty\int_0^\infty du\,dv\,dt\,\frac{t^{-q-1}v^q\cos\{(u-\sigma)v+\frac{1}{2}q\pi\}\phi u}{(t+f^2\,.\,t+h^2)^{\frac{1}{2}}};$$

or, writing $\pi^{\frac{1}{2}s-1}=\frac{1}{\pi}(\Gamma\frac{1}{2})^s$, this is

$$=\frac{(\Gamma\frac{1}{2})^s(f\,.h)}{\Gamma(\frac{1}{2}s+q)}\int_0^\infty dt\,.\,t^{-q-1}(t+f^2\ldots t+h^2)^{-\frac{1}{2}}\,\frac{1}{\pi}\int_0^1\int_0^\infty du\,dv\,.\,v^q\cos\{(u-\sigma)v+\tfrac{1}{2}q\pi\}\phi u$$

159. BOOLE writes

$$\frac{1}{\pi}\int_0^1\int_0^\infty du\,dv\,d^q\cos\{(u-\sigma)v+\tfrac{1}{2}q\pi\}\phi u=\left(-\frac{d}{d\sigma}\right)^q\phi(\sigma),$$

viz. starting from FOURIER's theorem,

$$\frac{1}{\pi}\int_0^1\int_0^\infty du\,dv\cos(u-\sigma)v\,.\,\phi u=\phi(\sigma)$$

(where $\phi(\sigma)$ is regarded as vanishing except when σ is between the limits 0, 1, and the limits of u are taken to be 1, 0 accordingly), then, according to an admissible theory of

general differentiation, we have the result in question He has in the formula $\frac{1}{s}$ instead of my t; and he proceeds, "Here σ increases continually with s As s varies from 0 to ∞, σ also varies from 0 to ∞. To any positive limits of σ will correspond positive limits of s; and these, as will hereafter appear [refers to his note B], will in certain cases replace the limits 0 and ∞ in the expression for V."

160. It seems better to deal with the result in the following manner, as in part shown p. 803 of BOOLE's memoir. Writing the integral in the form

$$V=\frac{(\Gamma\frac{1}{2})^s\,(f\ldots h)}{\pi\Gamma(\frac{1}{2}s+q)}\int_0^1\int_0^\infty du\,dt\;\;t^{-q-1}(t+f^2\ldots t+h^2)^{-\frac{1}{2}}\phi(u)\int_0^\infty dv\;\;v^q\cos\{(u-\sigma)v+\tfrac{1}{2}q\pi\},$$

effect the integration in regard to v, viz according as u is greater or less than σ, then

$$\int_0^\infty dv\;\;v^q\cos\{(u-\sigma)v+\tfrac{1}{2}q\pi\}=\frac{\Gamma(q+1)\sin(q+1)\pi}{(u-\sigma)^{q+1}},\text{ or } 0,$$

$$=\frac{\pi}{\Gamma(-q)\,(u-\sigma)^{q+1}}\quad,\text{ or } 0,$$

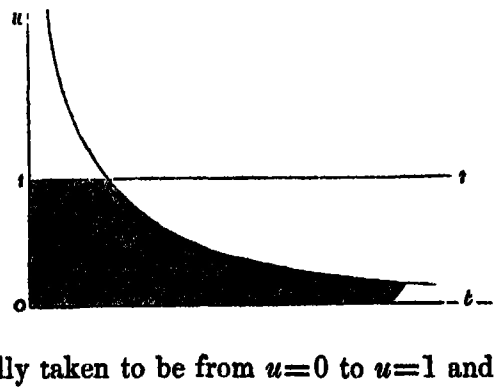
and consequently, writing for σ its value,

$$V=\frac{(\Gamma\frac{1}{2})^s(f\ldots h)}{\Gamma(-q)\Gamma(\frac{1}{2}s+q)}\int_0^1\int_0^\infty du\,dt\left\{t^{-q-1}(t+f^2\ldots t+h^2)^{-\frac{1}{2}}\left(u-\frac{a^2}{f^2+t}\ldots-\frac{c^2}{h^2+t}-\frac{e^2}{t}\right)^{-q-1}\phi u,\right.$$
$$\left.\text{or } 0 \text{ as above}\right\}$$

161. To further explain this, consider t as an x-coordinate and u as a y-coordinate, then tracing the curve

$$y=\frac{a^2}{f^2+x}\;\ldots+\frac{c^2}{h^2+x}+\frac{e^2}{x}$$

for positive values of x, this is a mere hyperbolic branch, as shown in the figure, viz $x=0$, $y=\infty$; and as x continually increases to ∞, y continually decreases to zero



The limits are originally taken to be from $u=0$ to $u=1$ and $t=0$ to $t=\infty$, viz. over the infinite strip bounded by the lines tO, O1, 11; but within these limits the function under the integral sign is to be replaced by zero whenever the values u, t are such that u is less than $\frac{a^2}{f^2+t}\ldots+\frac{c^2}{h^2+t}+\frac{e^2}{t}$, viz. when the values belong to a point in the shaded

portion of the strip, the integral is therefore to be extended only over the unshaded portion of the strip; viz. the value is

$$V=\frac{(\Gamma\tfrac{1}{2})^s(f\,..\,h)}{\Gamma(-q)\Gamma(\tfrac{1}{2}s+q)}\iint du\,dt\,.\,t^{-q-1}(t+f^2\ldots t+h^2)^{-\frac{1}{2}}\left(u-\frac{a^2}{f^2+t}\ldots-\frac{c^2}{h^2+t}-\frac{e^2}{t}\right)^{-q-1}\phi u,$$

the double integral being taken over the unshaded portion of the strip; or, what is the same thing, the integral in regard to u is to be taken from $u=\frac{a^2}{f^2+t}\ldots+\frac{c^2}{h^2+t}+\frac{e^2}{t}$ (say from $u=\sigma$) to $u=1$, and then the integral in regard to t is to be taken from $t=\theta$ to $t=\infty$, where, as before, θ is the positive root of the equation $\sigma=1$, that is of $\frac{a^2}{f^2+\theta}\ldots+\frac{c^2}{h^2+\theta}+\frac{e^2}{\theta}=1.$

162. Write $u=\sigma+(1-\sigma)x$, and therefore $u-\sigma=(1-\sigma)x$, $1-u=(1-\sigma)(1-x)$ and $du=(1-\sigma)dx$, then the limits $(1, 0)$ of x correspond to the limits $(1, \sigma)$ of u, and the formula becomes

$$V=\frac{(\Gamma\tfrac{1}{2})^s.(f\,.\,h)}{\Gamma(-q)\Gamma(\tfrac{1}{2}s+q)}\int_\theta^\infty dt\,.\,t^{-q-1}(t+f^2\ldots t+h^2)^{-\frac{1}{2}}(1-\sigma)^{-q-1}\int_0^1 dx\,.\,x^{-q-1}.\,\phi\{\sigma+(1-\sigma)x\},$$

where σ is retained in place of its value $\frac{a^2}{f^2+t}\ldots+\frac{c^2}{h^2+t}+\frac{e^2}{t}$. This is in fact a form (deduced from BOOLE's result in the memoir of 1846) given by me, Cambridge and Dublin Mathematical Journal, vol ii. (1847), p. 219.

If in particular $\phi u=(1-u)^{q+m}$, then $\phi\{\sigma+(1-\sigma)x\}=(1-\sigma)^{q+m}(1-x)^{q+m}$, and thence

$$\int_0^1 x^{-q-1}\{\phi\sigma+(1-\sigma)x\}dx=(1-\sigma)^m\int_0^1 x^{-q-1}(1-x)^{q+m}dx,$$
$$=\frac{\Gamma(-q)\Gamma(1+q+m)}{\Gamma(1+m)}(1-\sigma)^m;$$

and thence restoring for σ its value, we have

$$V=\frac{(\Gamma\tfrac{1}{2})^s\Gamma(1+q+m)}{\Gamma(\tfrac{1}{2}s+q)\Gamma(1+m)}(f\ldots h)\int_\theta^\infty dt\,.\,t^{-q-1}(t+f^2\ldots t+h^2)^{-\frac{1}{2}}\left(1-\frac{a^2}{f^2+t}\ldots-\frac{c^2}{h^2+t}-\frac{e^2}{t}\right)^m$$

as the value of the integral

$$\int\frac{\left(1-\frac{x^2}{f^2}\ldots-\frac{z^2}{h^2}\right)^{q+m}dx\ldots dz}{\{(a-x)^2\ldots+(c-z)^2+e^2\}^{\frac{1}{2}s+q}}$$

over the ellipsoid $\frac{x^2}{f^2}\ldots+\frac{z^2}{h^2}=1$. This is in fact the theorem of Annex IV. No. 110 in its general form; but the proof assumes that q is positive.

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LONDON

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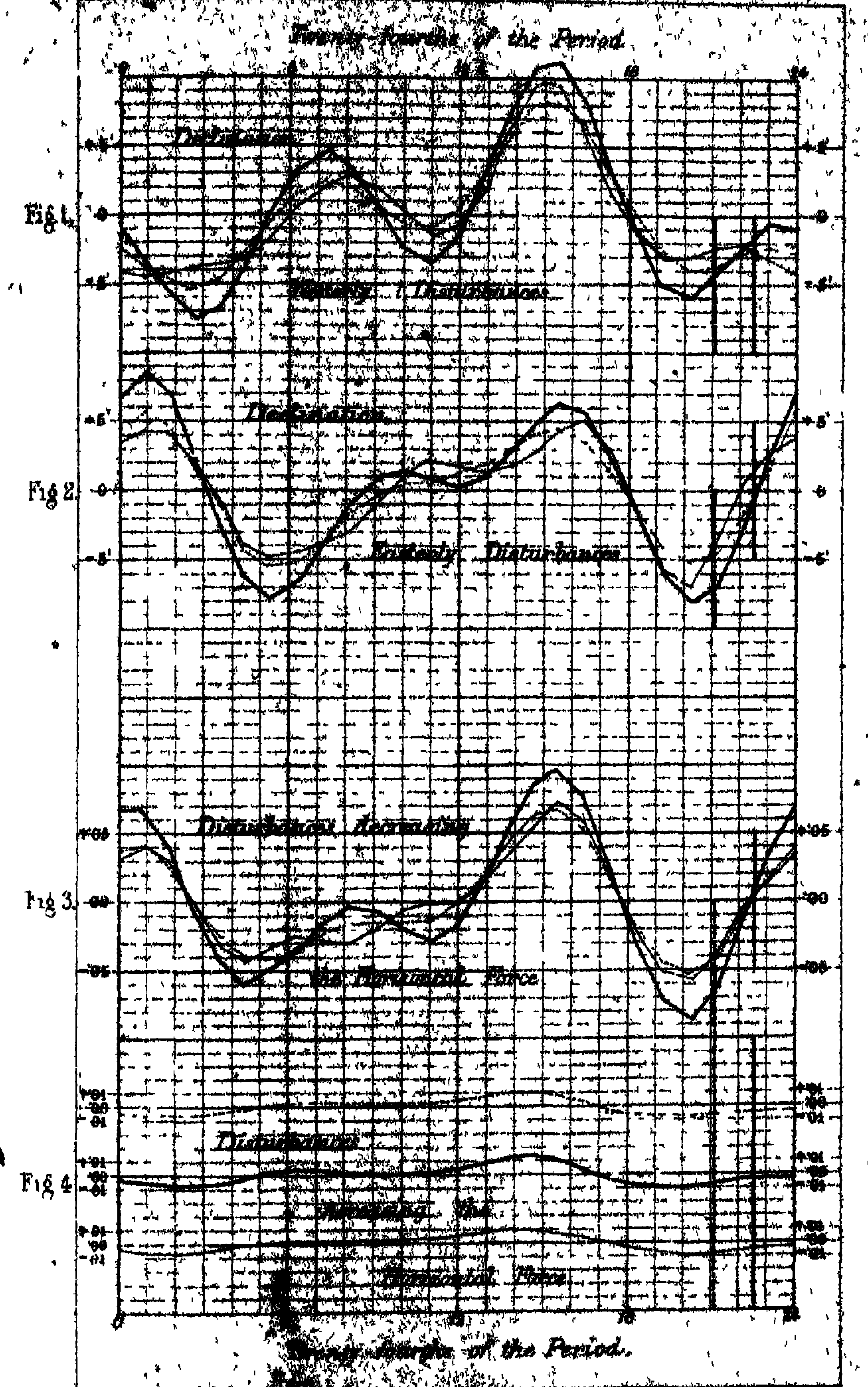
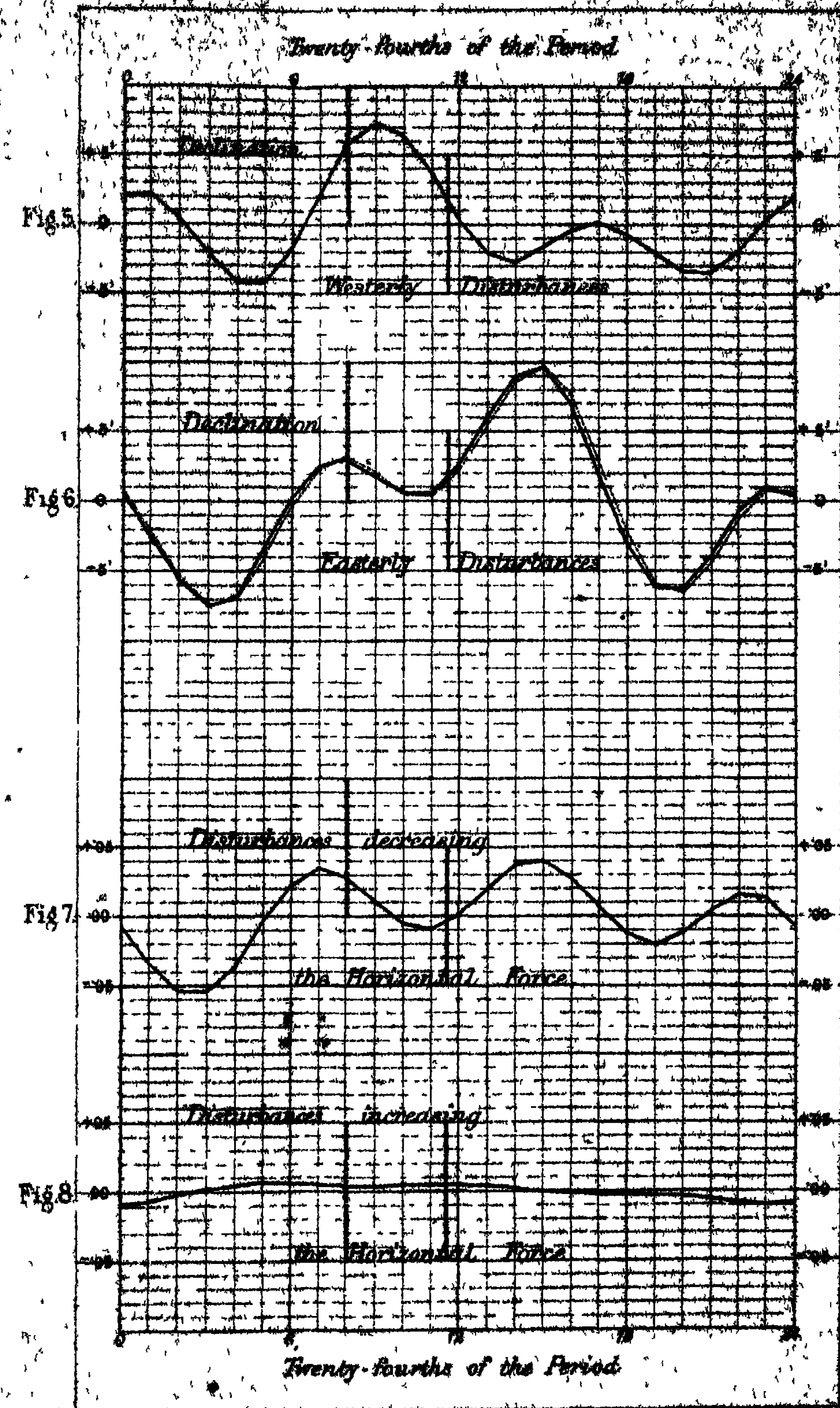
Disturbance Variations of Magnetic Declination and Horizontal Force in the [illegible] Periods of Mercury, Venus and the Earth

MERCURY



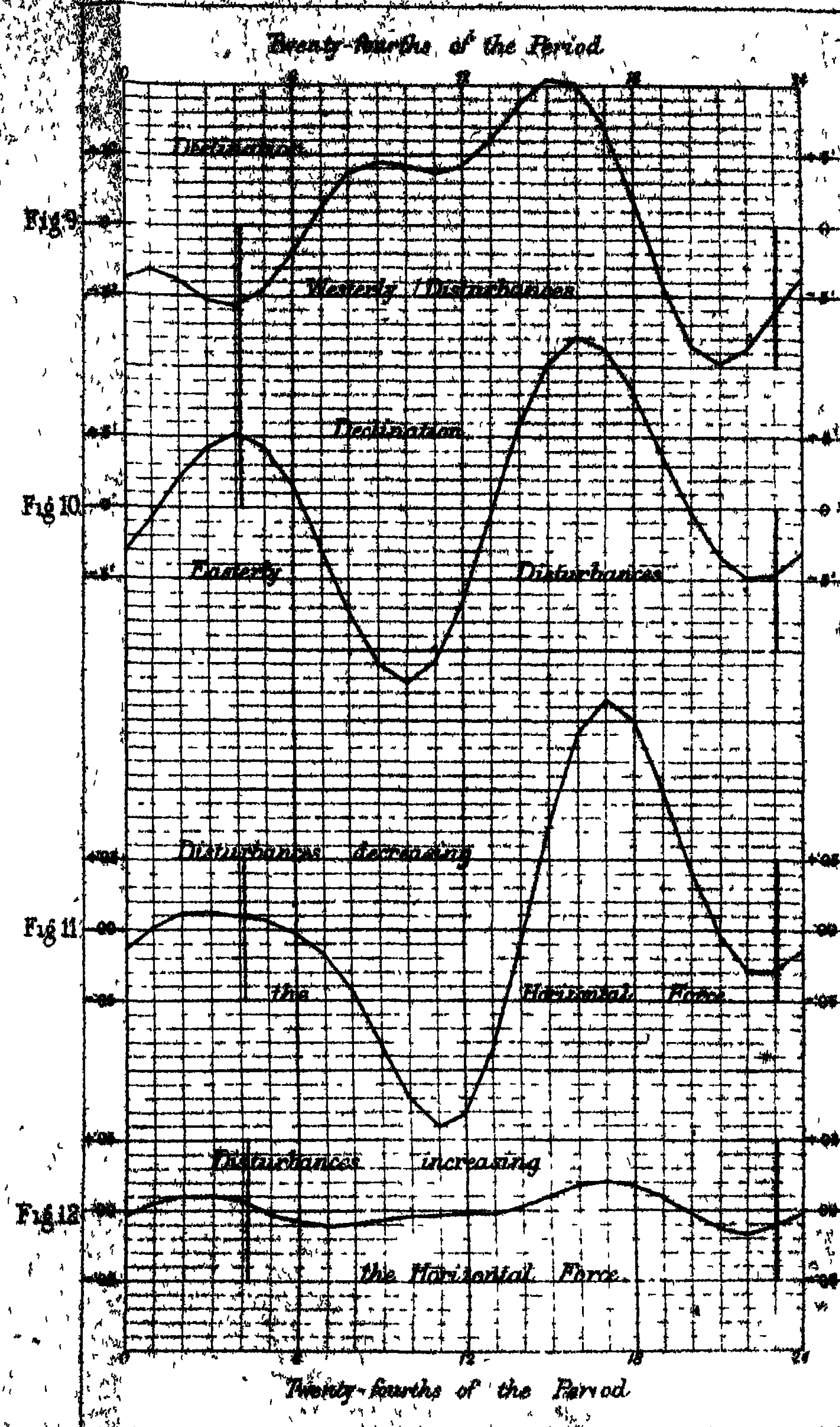
The vertical dotted and thick lines mark the times of the ascending Node and Perihelion respectively.

VENUS



The vertical dotted and thick lines mark the times of the ascending Node and Perihelion respectively.

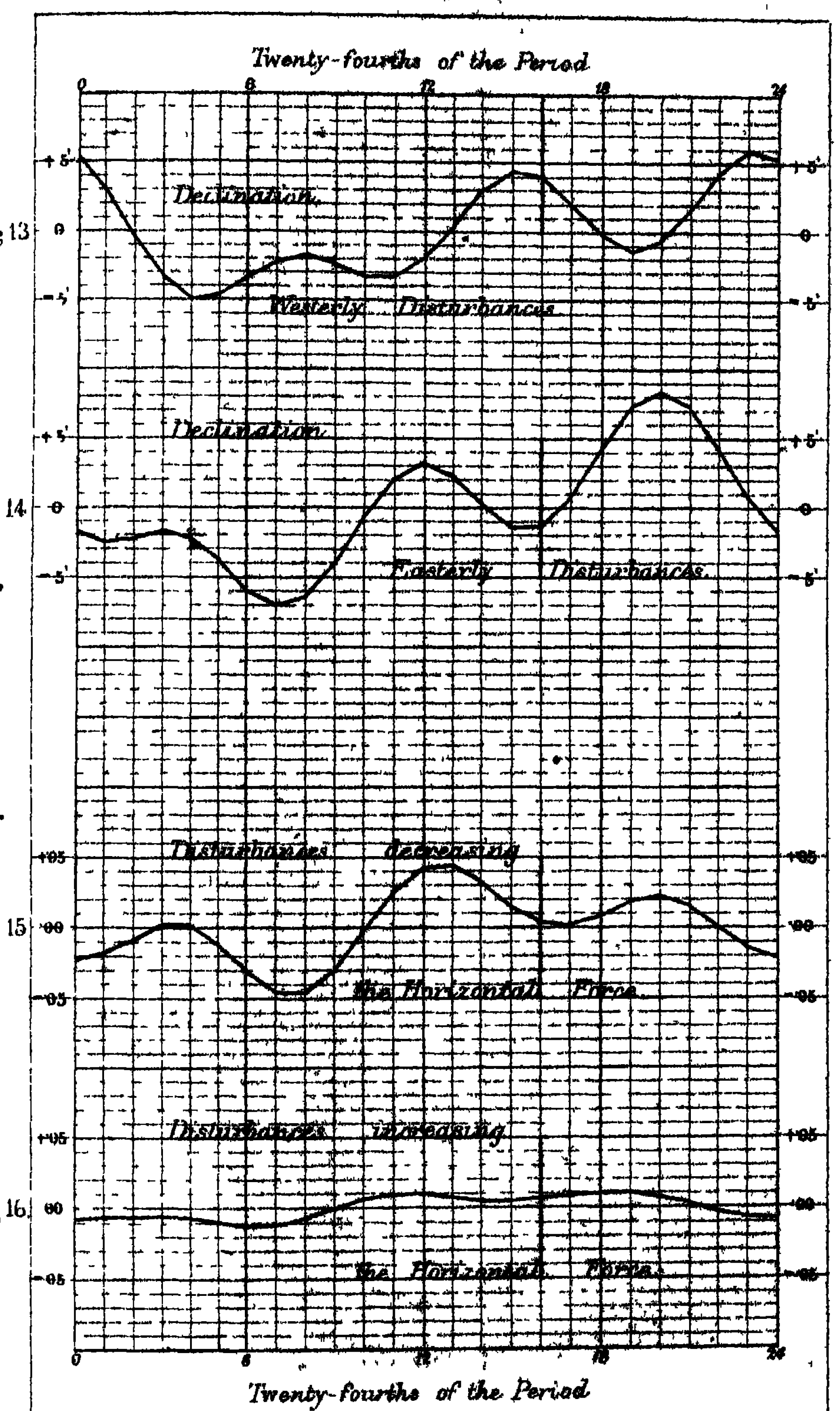
THE EARTH



The vertical dotted lines mark the time of the Vernal Equinox, and the Vertical thick lines the time of Perihelion

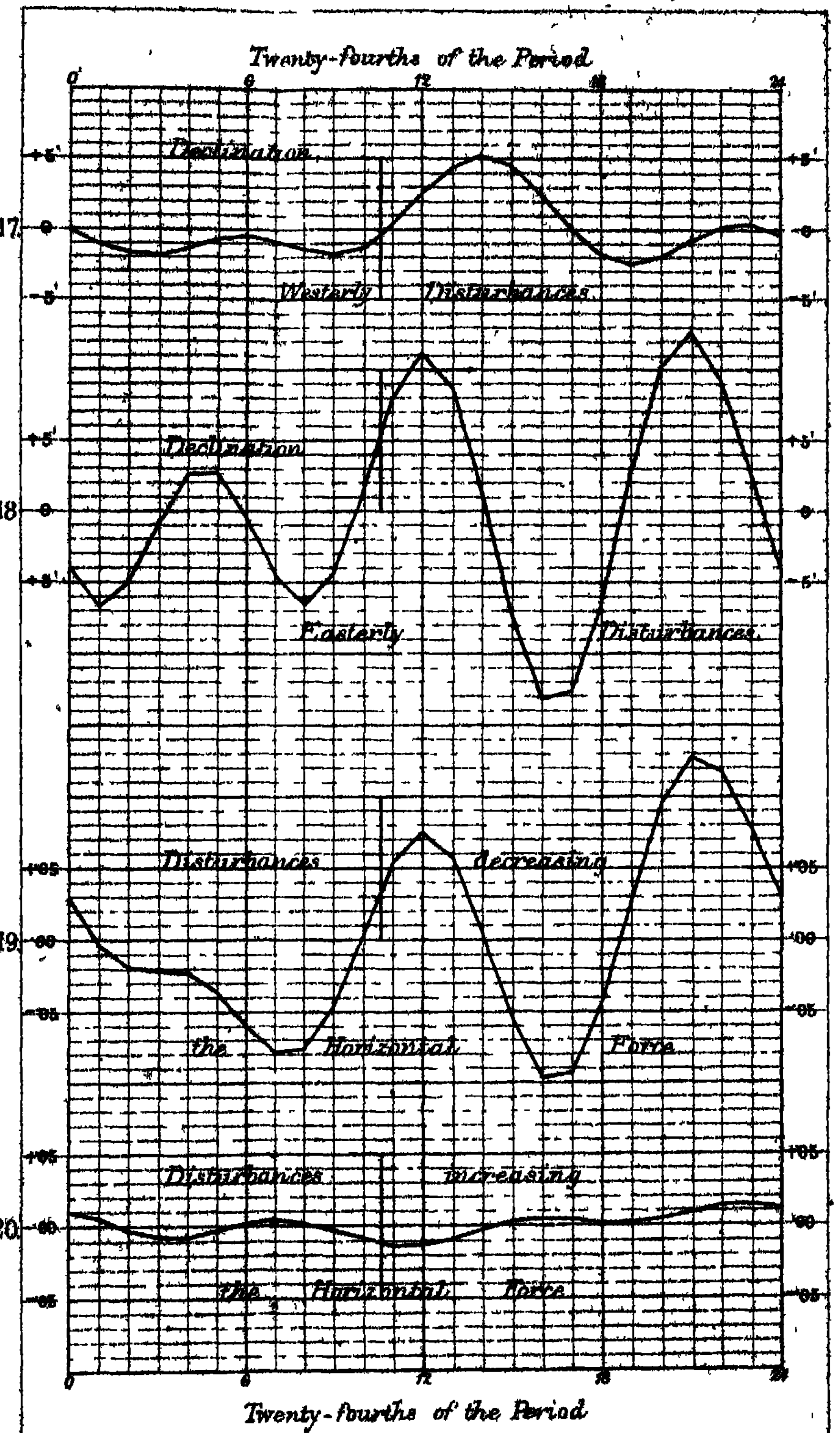
Disturbance Variations of Magnetic Declination and Horizontal Force in the Synodic Periods of Mercury, Venus and Jupiter

MERCURY.



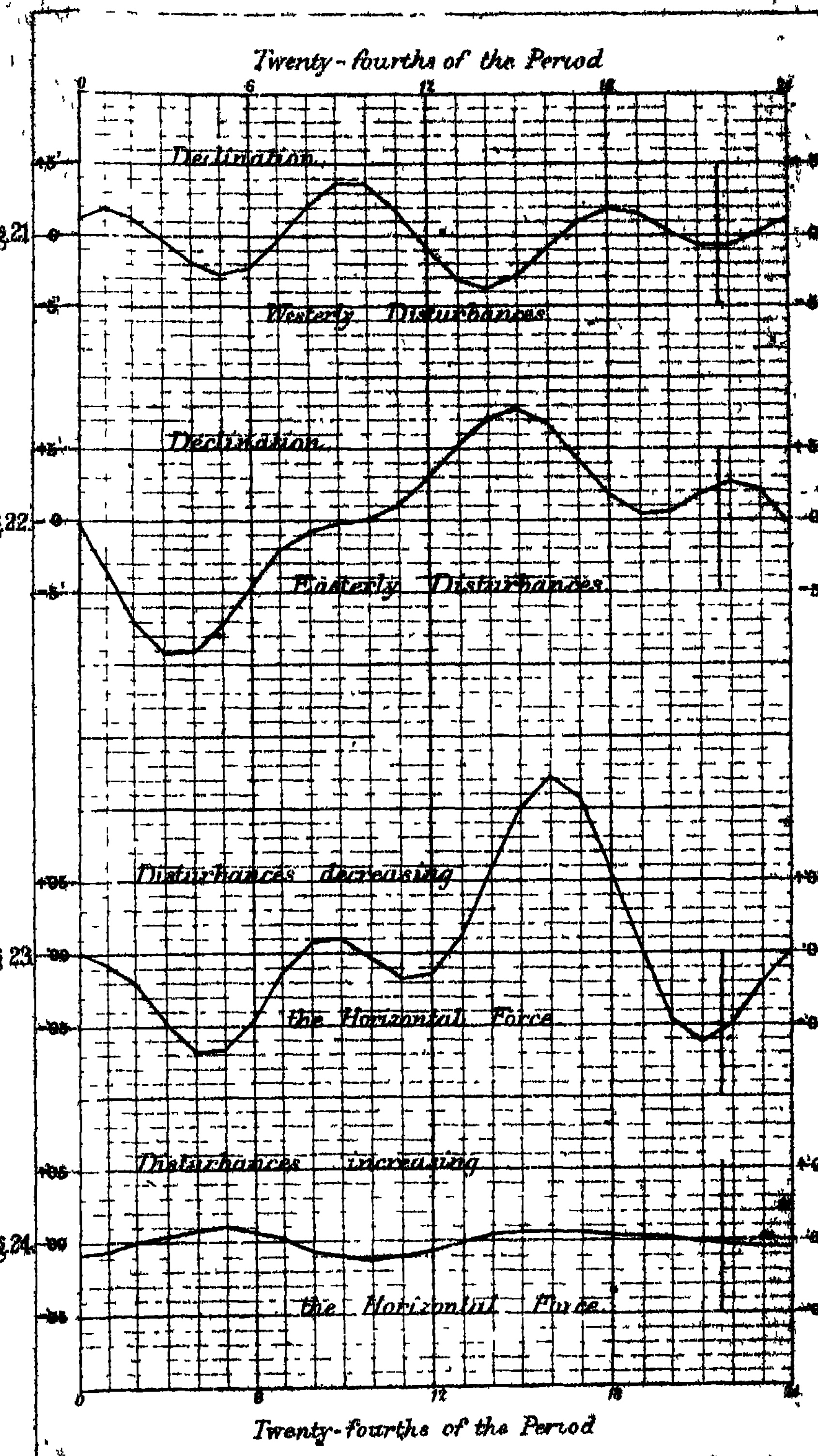
The vertical thick lines mark the time of inferior Conjunction.

VENUS



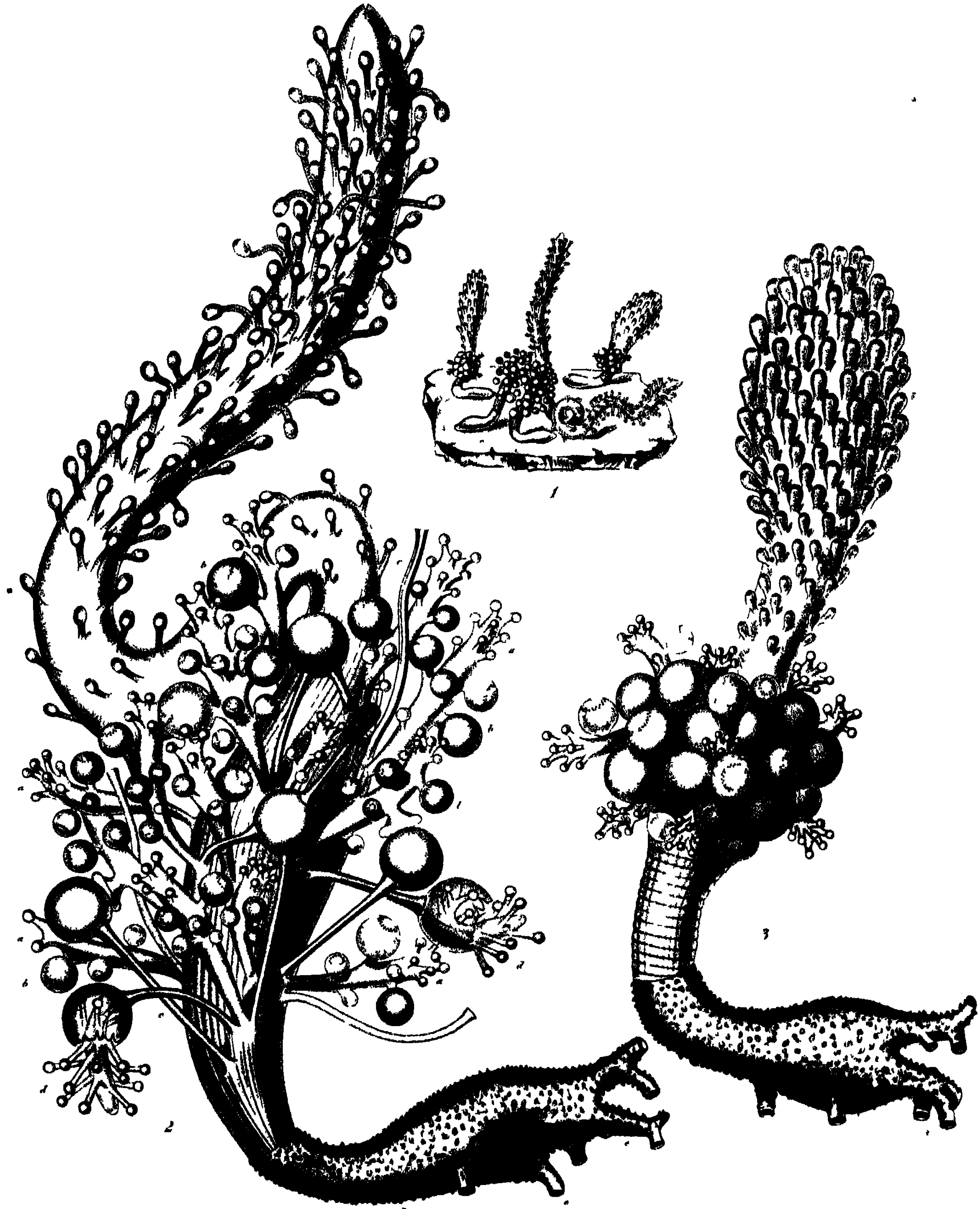
The vertical thick lines mark the time of inferior Conjunction.

JUPITER



The vertical thick lines mark the time of Opposition.

W H Wesley lith.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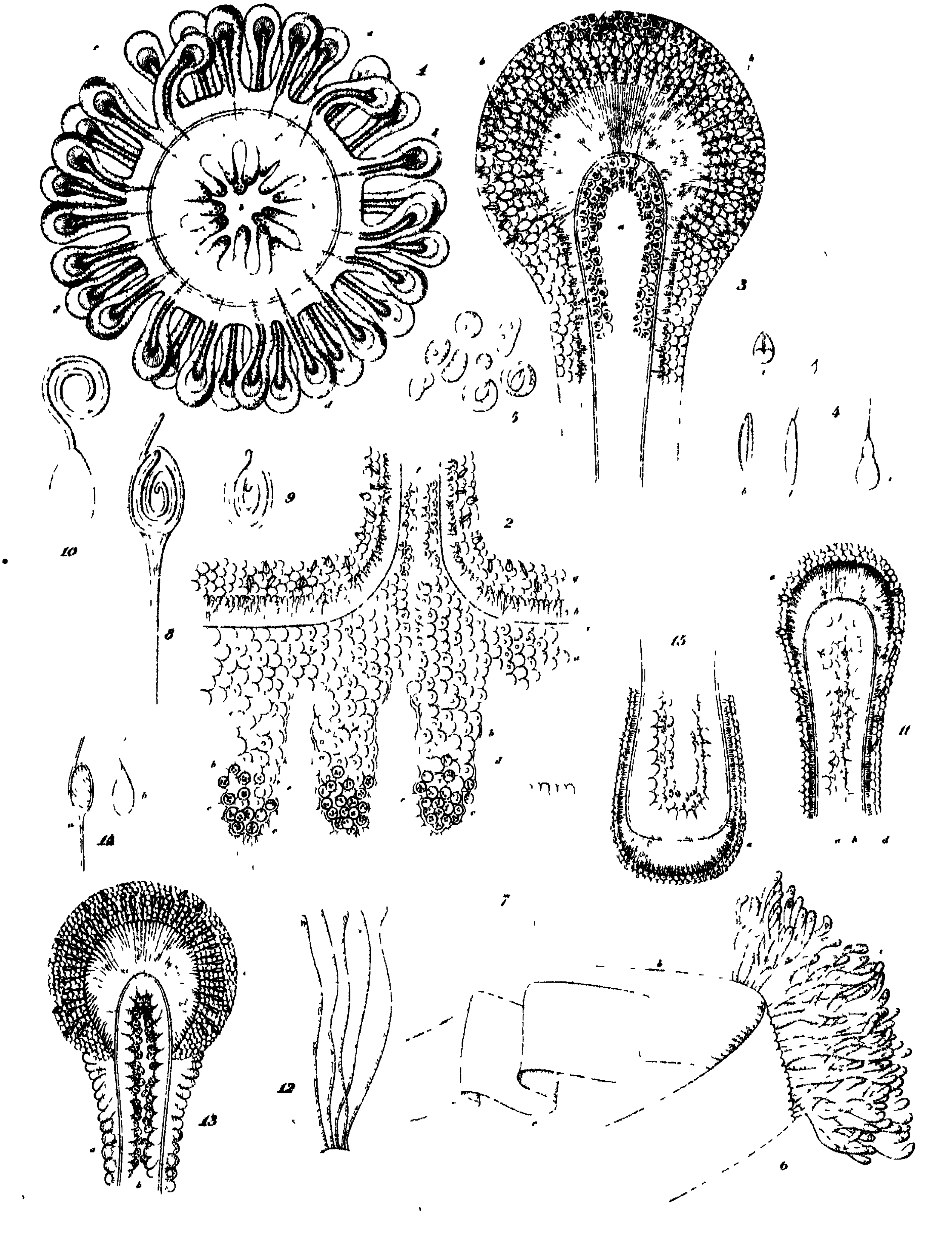
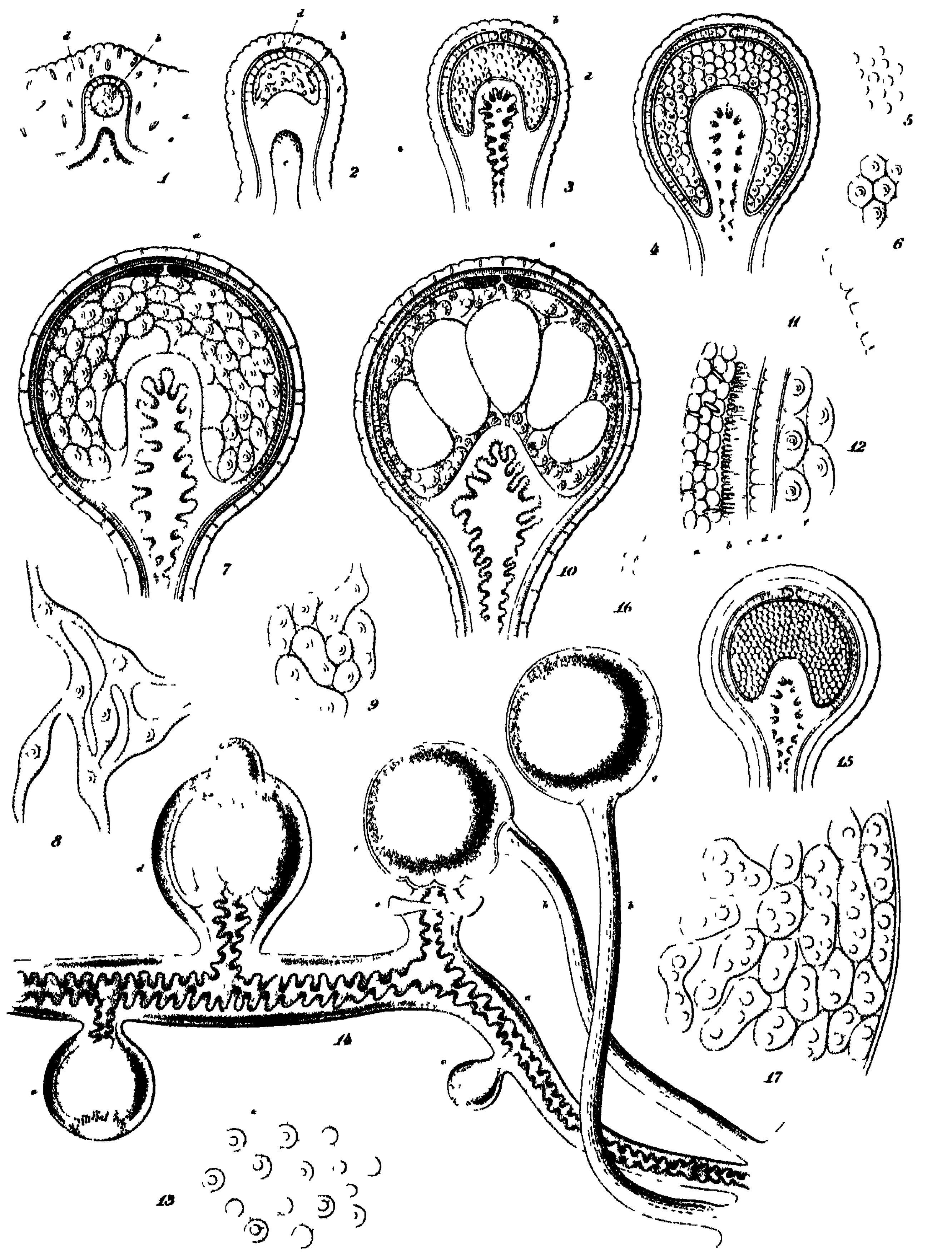
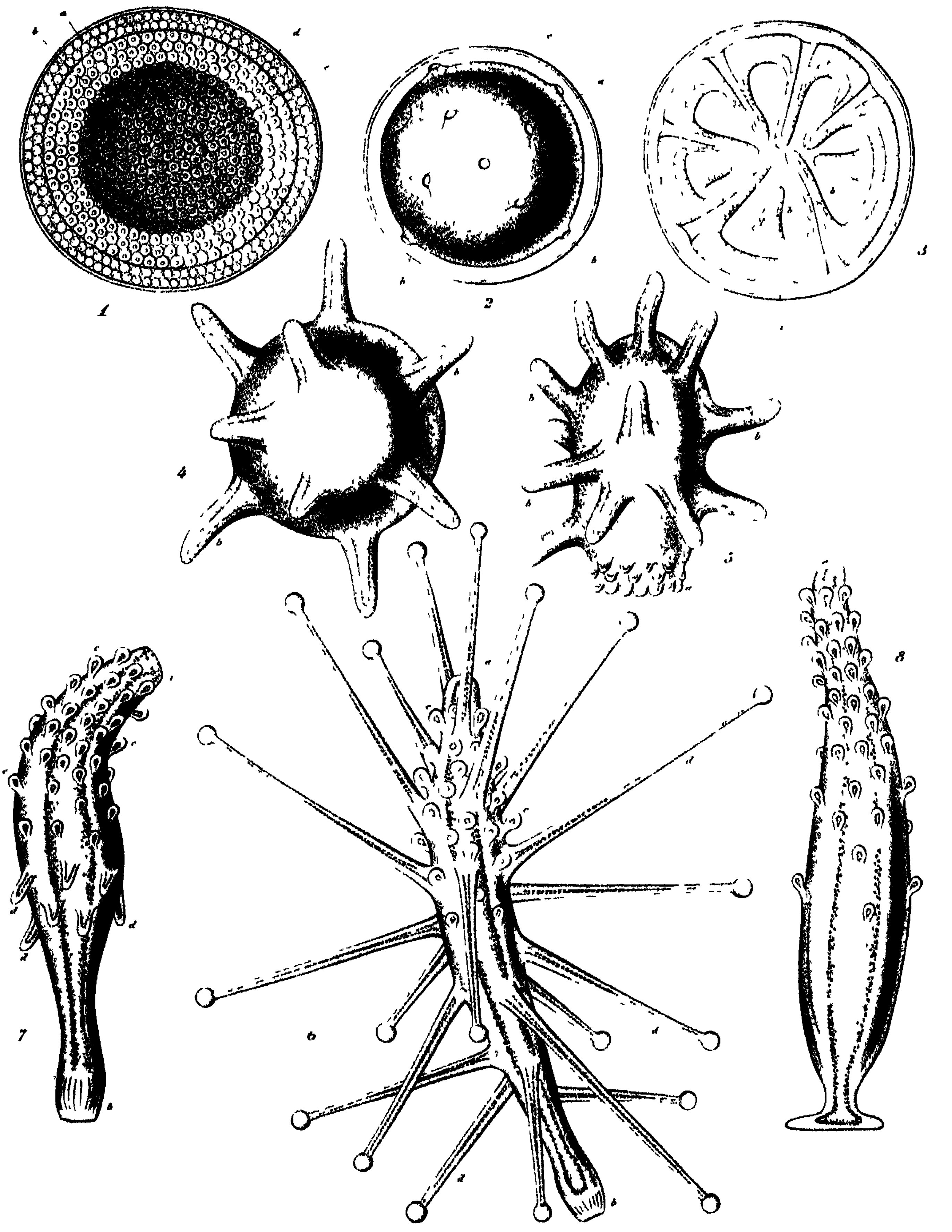
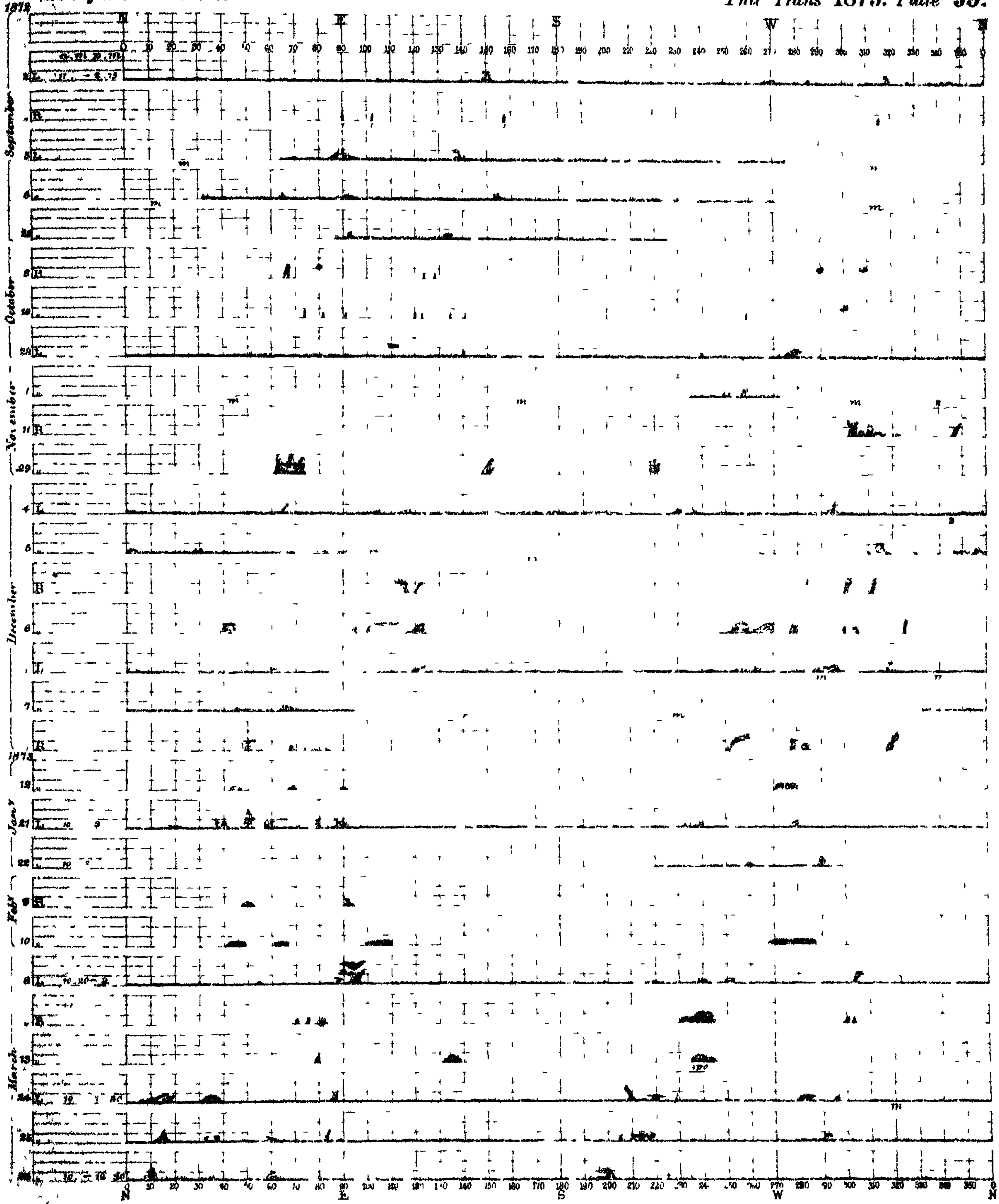


G. J. Allman ad Nat. del. Lonsdale ... Copenhagen







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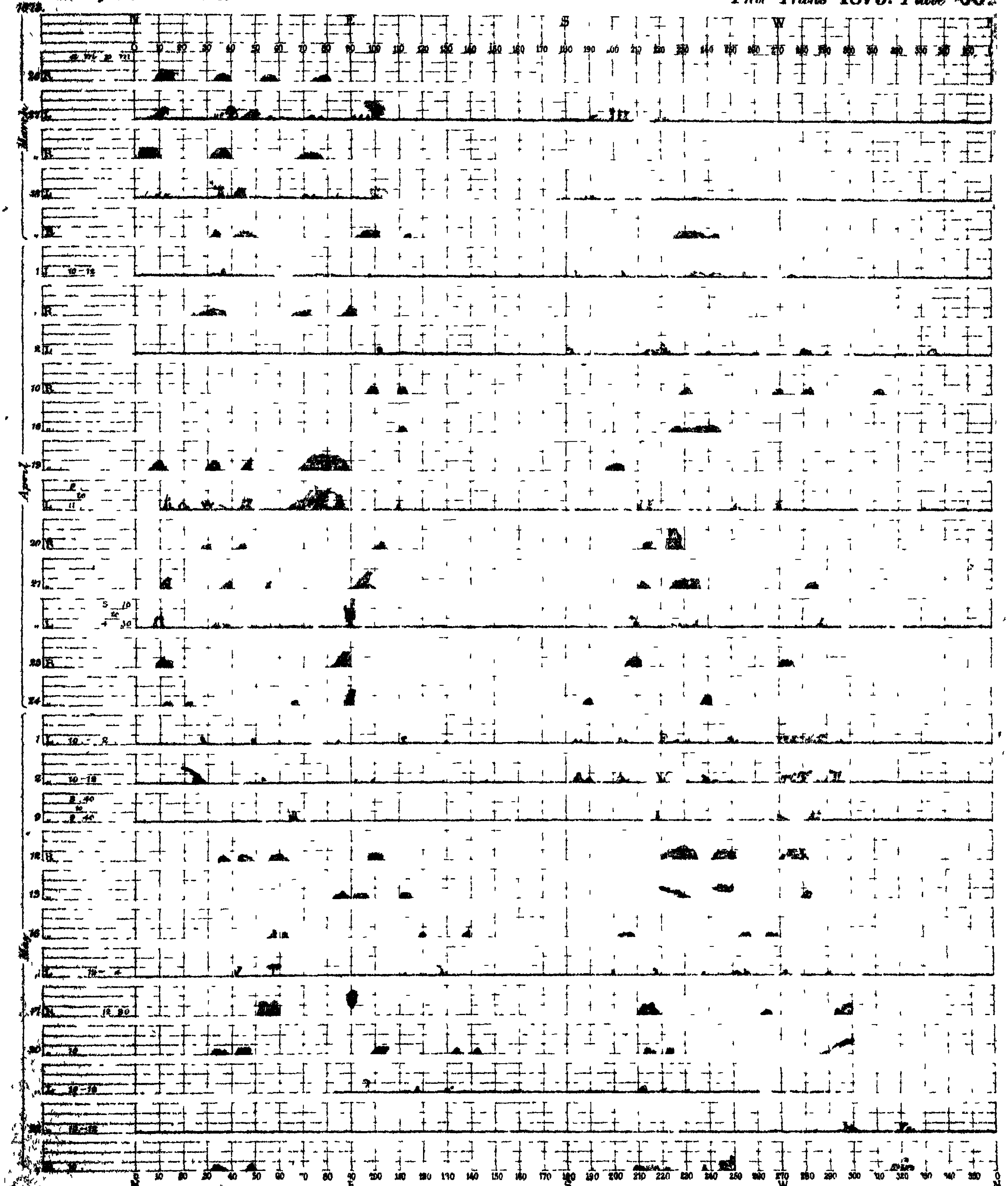

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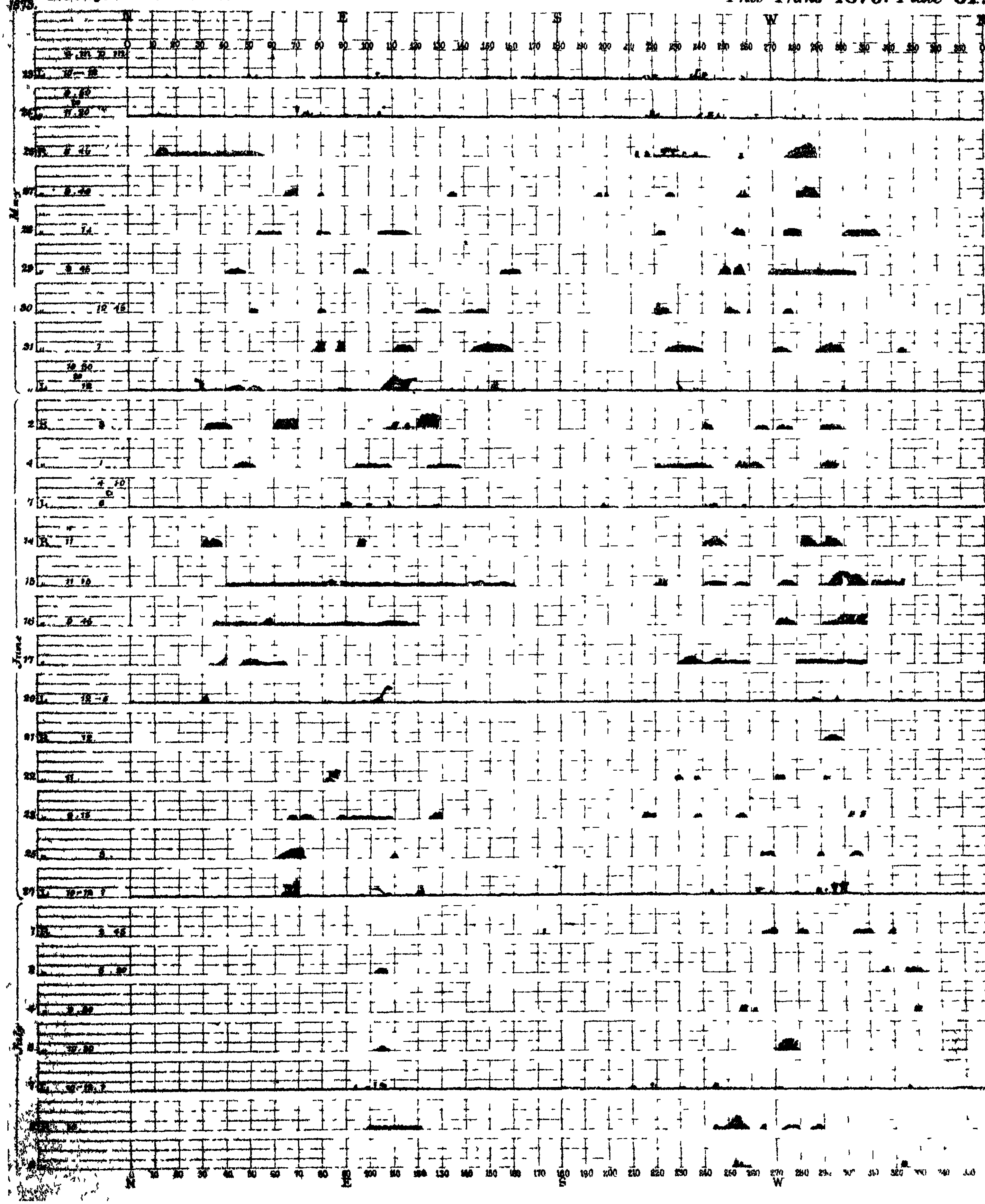

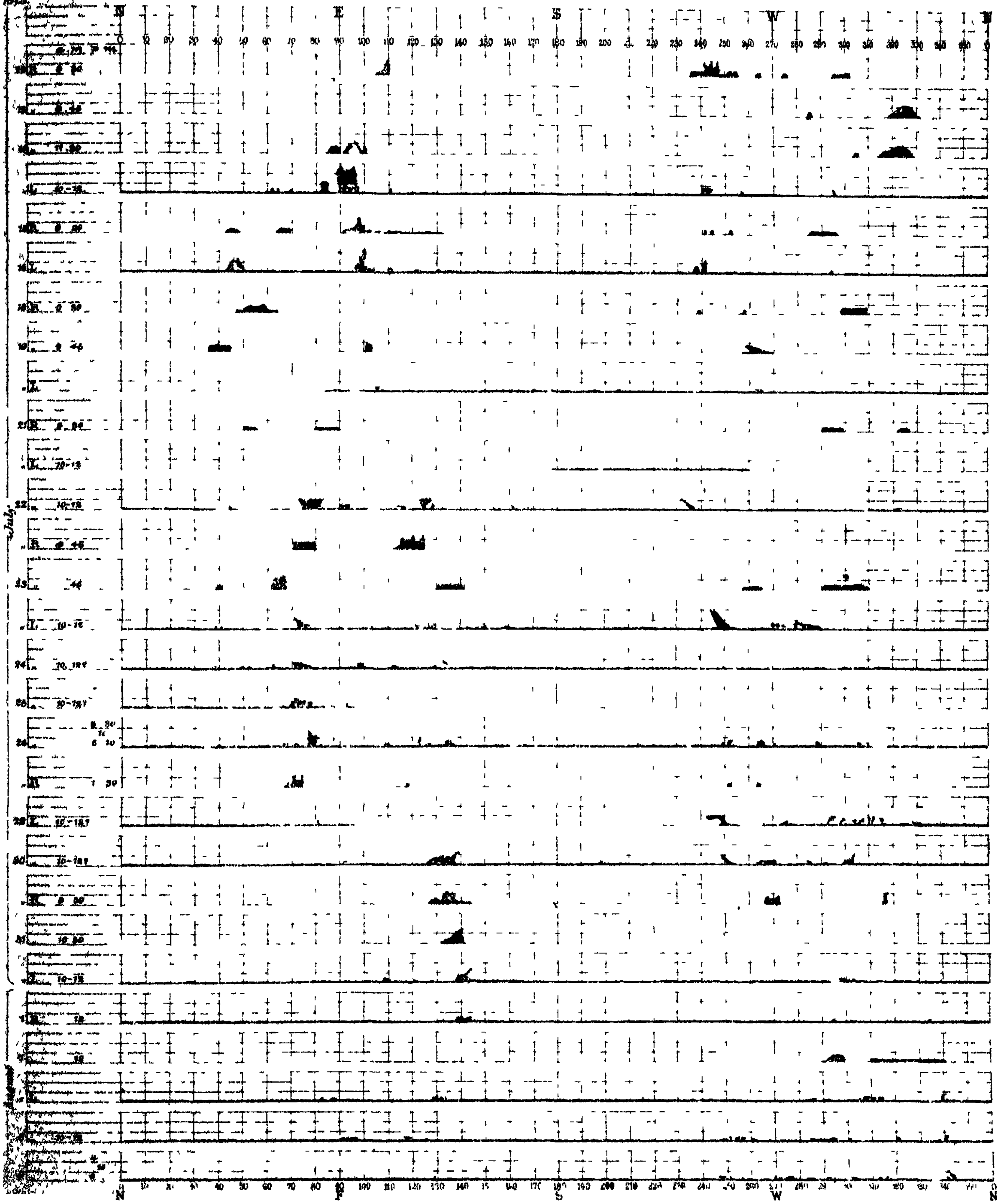

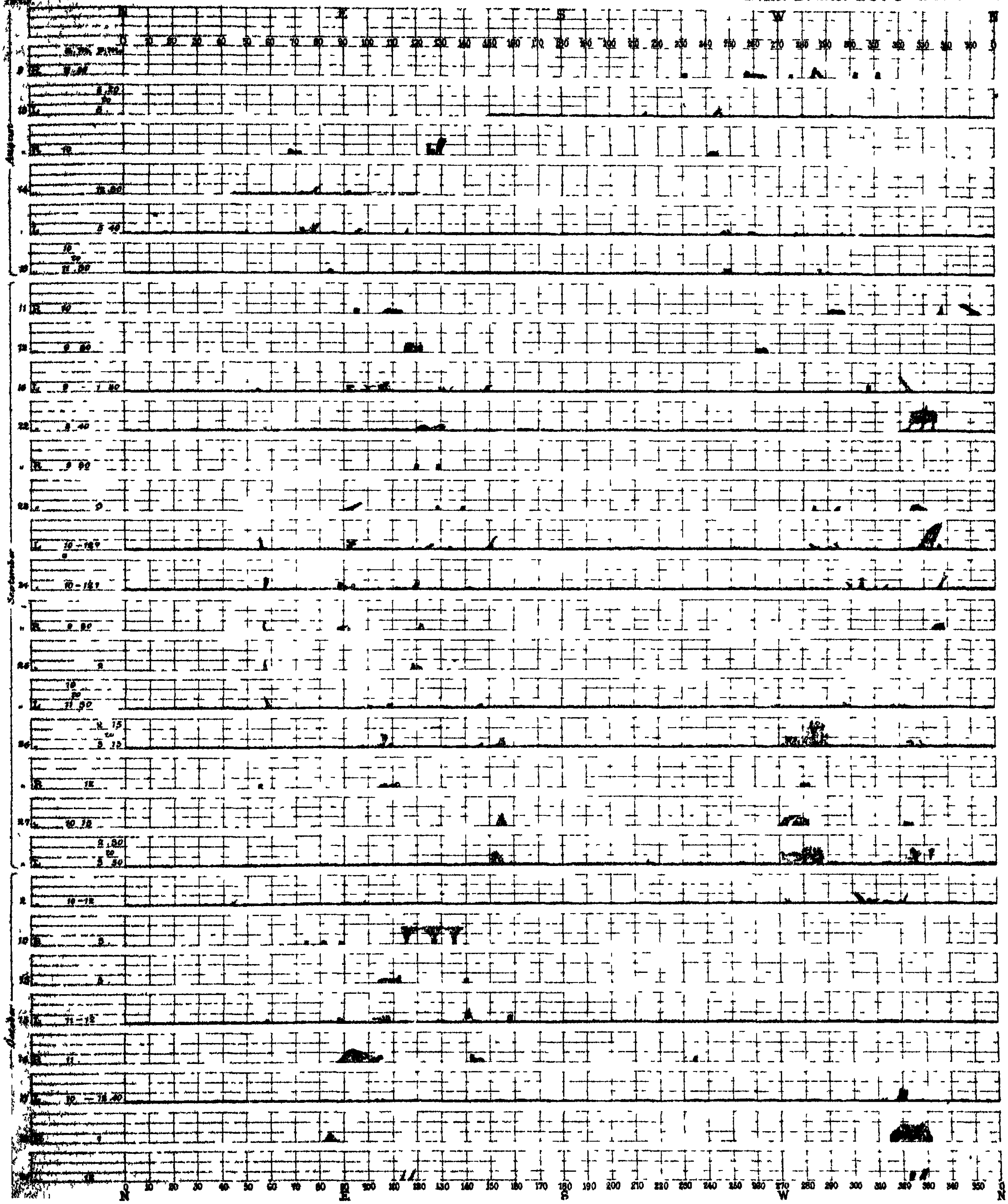
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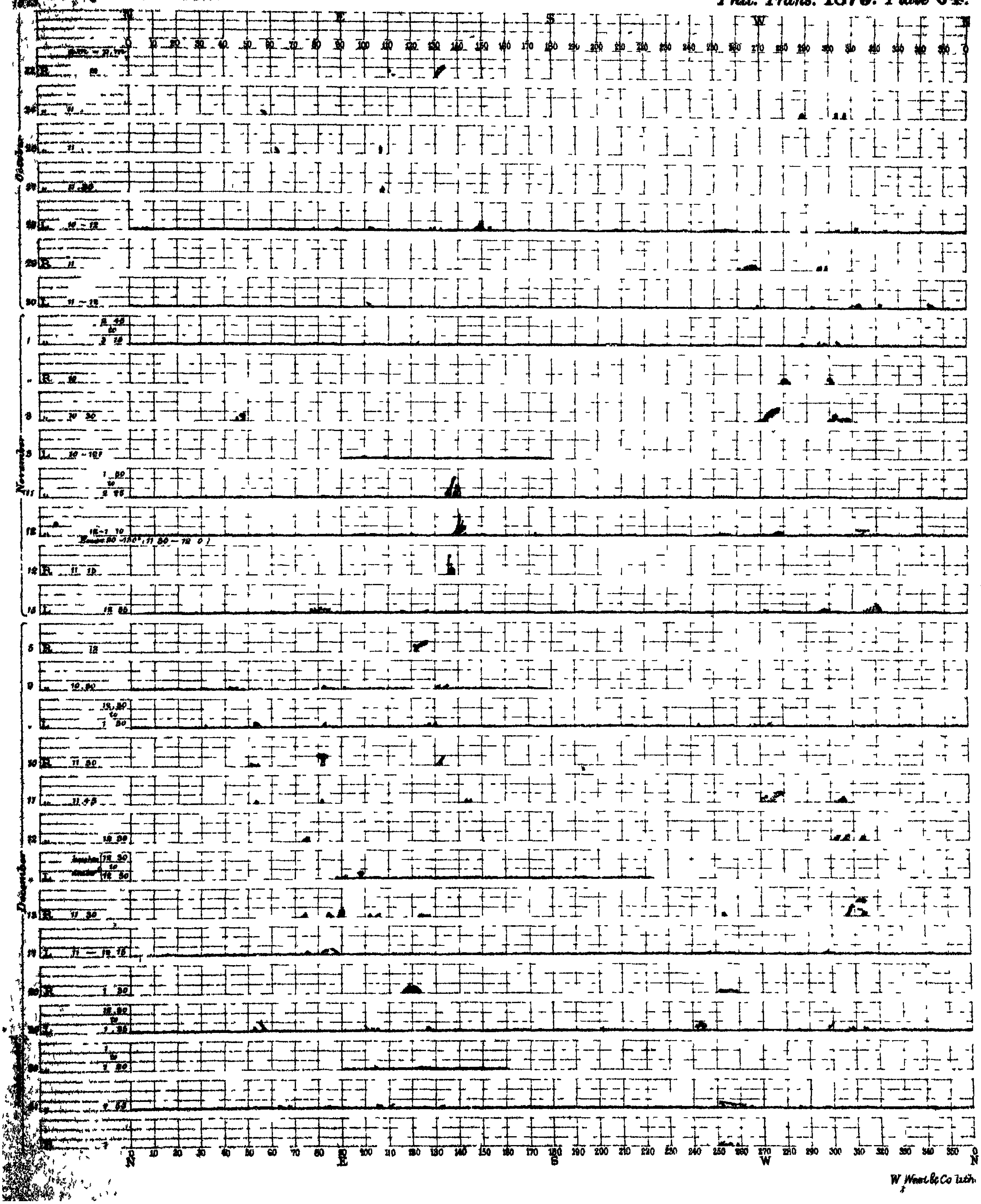

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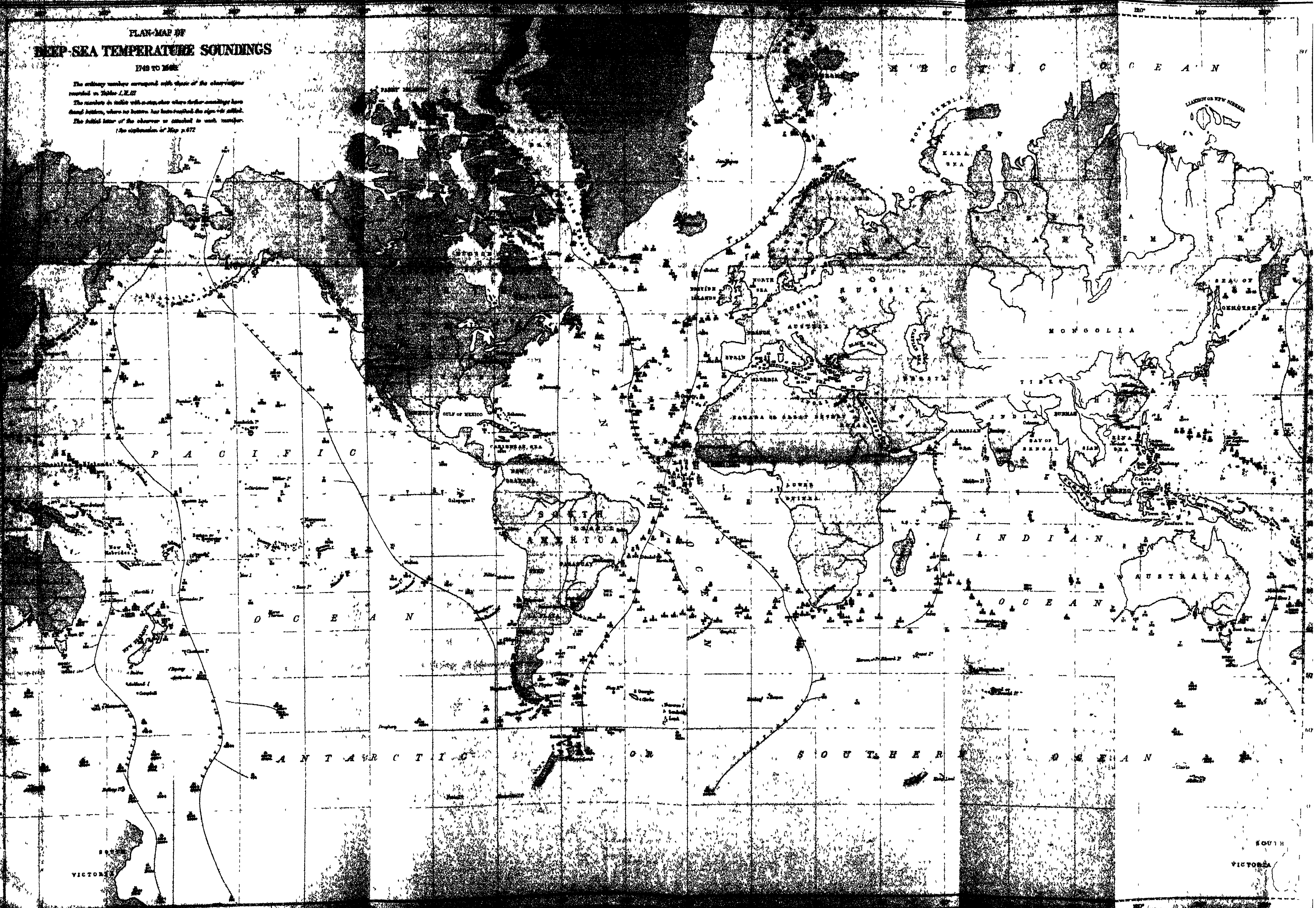
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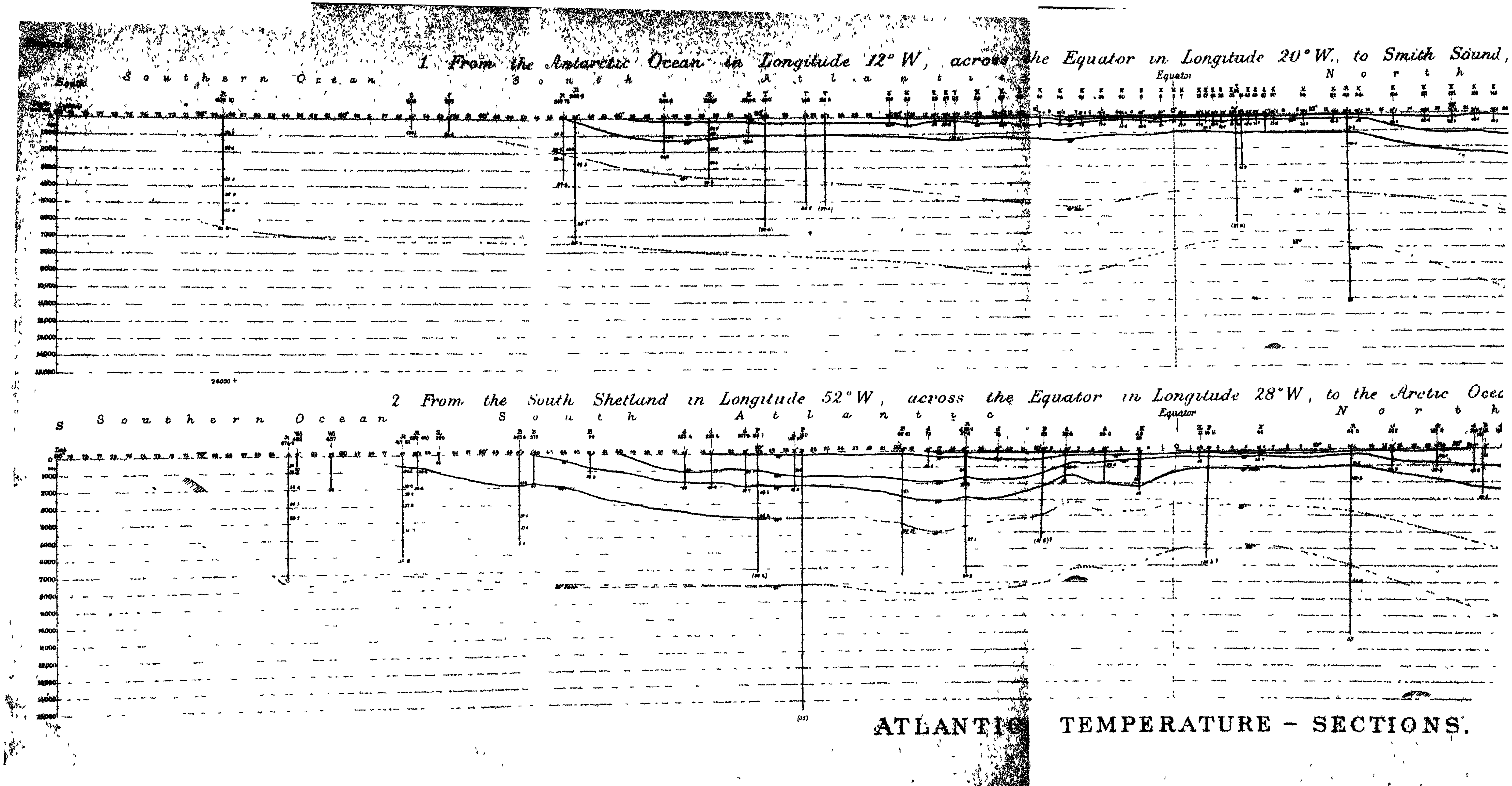
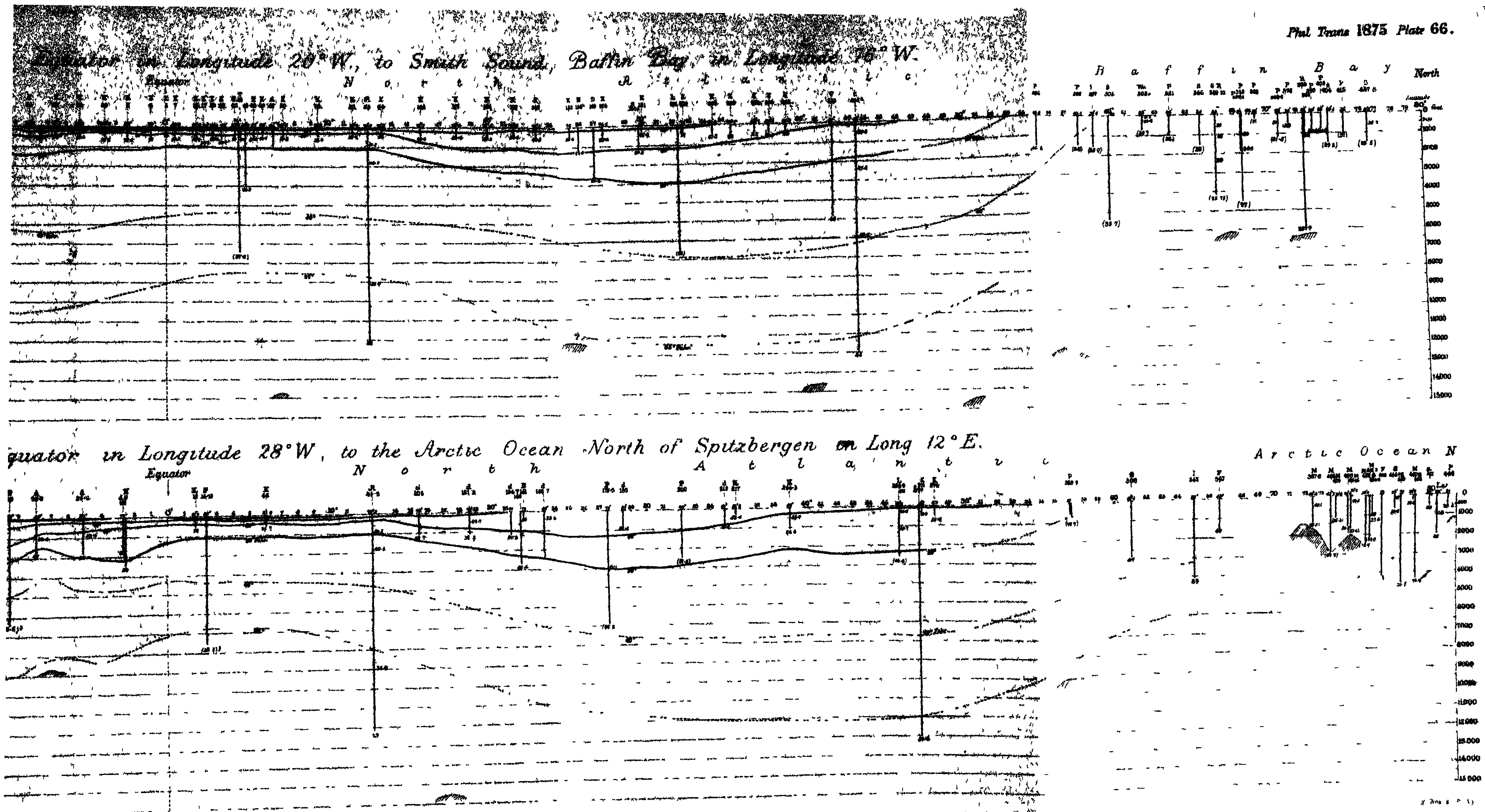
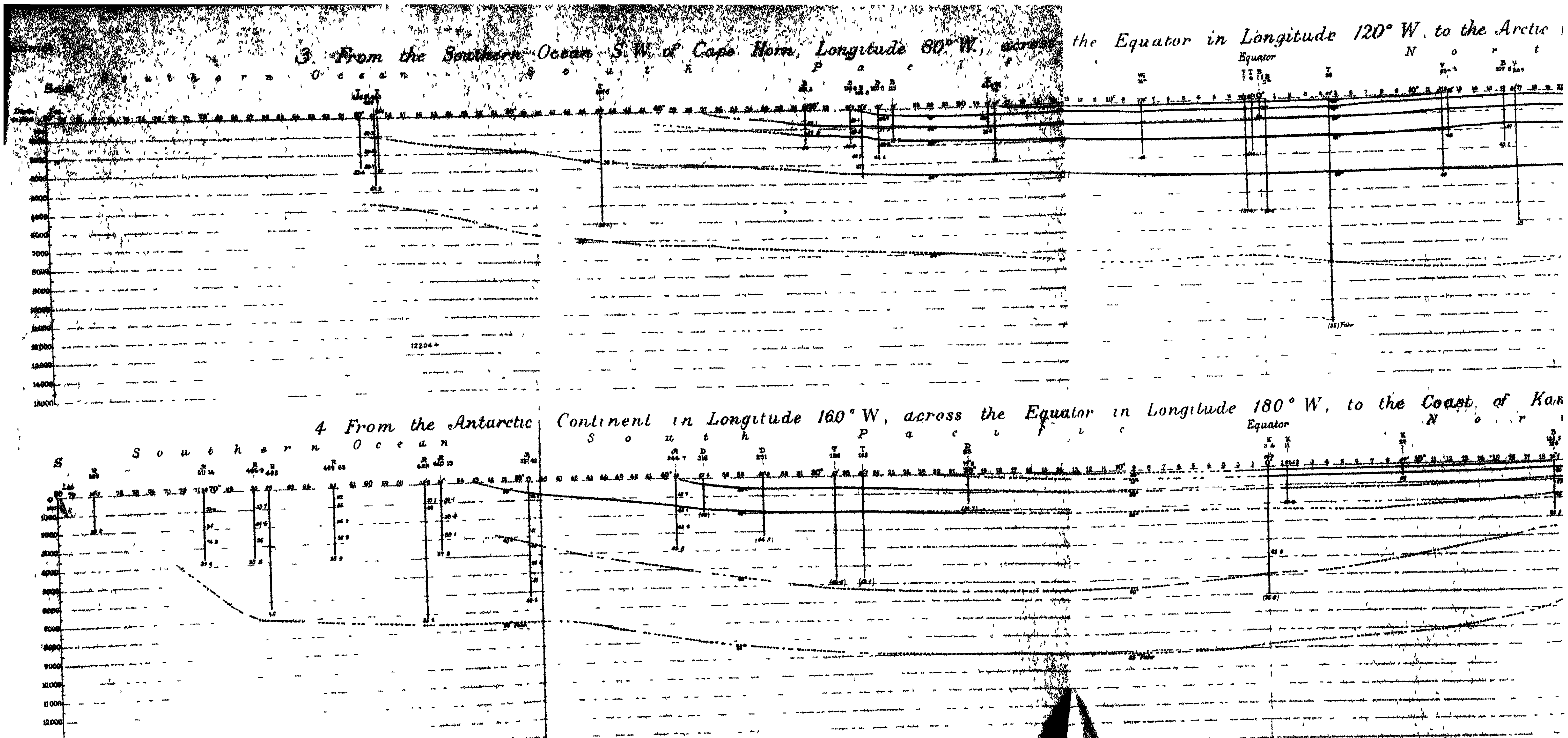
PLAN-MAP OF
DEEP-SEA TEMPERATURE SOUNDINGS
The ordinary numbers correspond with those of the observations recorded in Tables I, II, III
ARCTIC OCEAN
NOVA ZEMBLA
KARA SEA
NORTH SEA
BRITISH ISLANDS
RUSSIA
AUSTRIA
SPAIN
BLACK SEA
CASPIAN SEA
ALGERIA
SAHARA OR GREAT DESERT
ARABIA
MONGOLIA
TIBET
INDIA
BURMAH
BAY OF BENGAL
SEA OF OKHOTSK
GULF OF MEXICO
CARIBBEAN SEA
SOUTH AMERICA
ATLANTIC OCEAN
PACIFIC OCEAN
INDIAN OCEAN
AUSTRALIA
ANTARCTIC OR SOUTHERN OCEAN
SOUTH VICTORIA



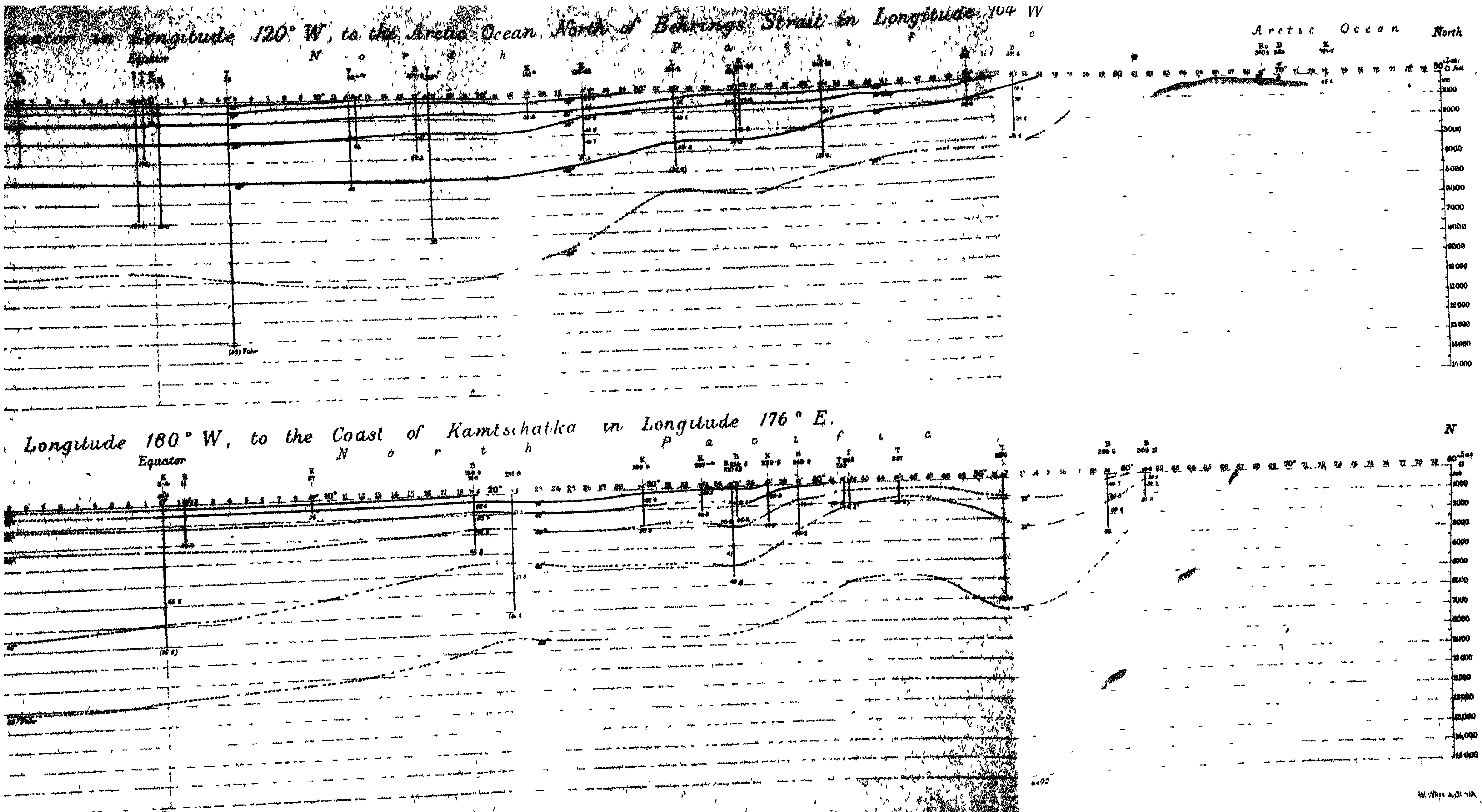
ATLANTIC TEMPERATURE - SECTIONS.



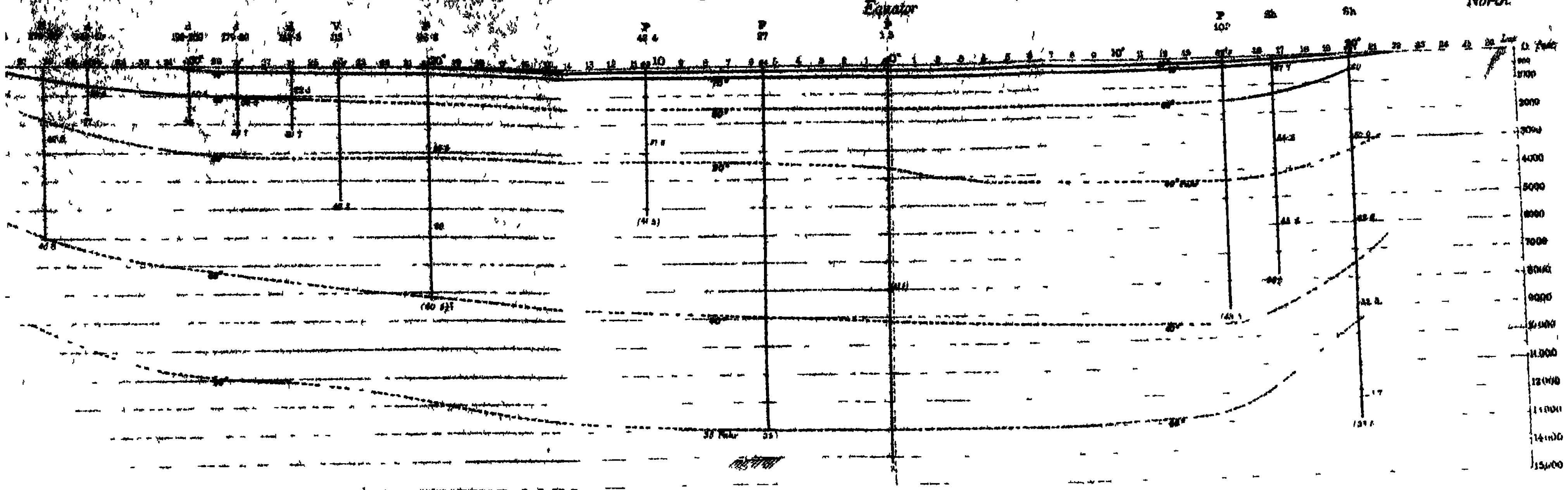
TEMPERATURE – SECTIONS.

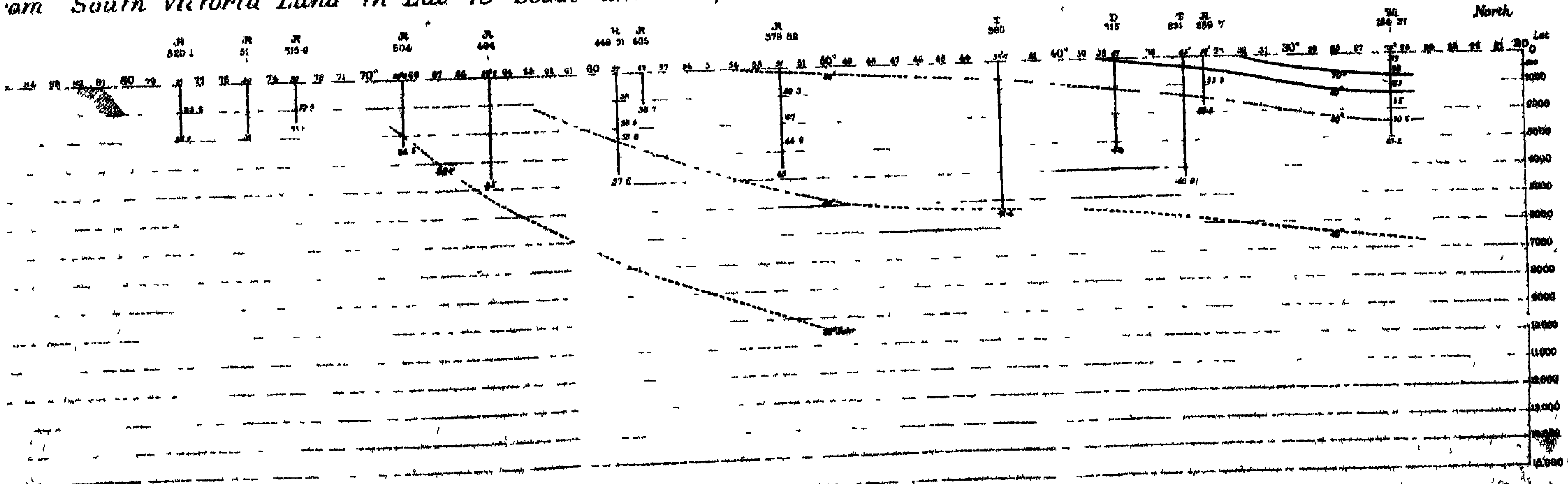


PAC...I... TEMPERATURE – SECTIONS.



TEMPERATURE – SECTIONS.





TEMPERATURE – SECTIONS IN THE INDIAN AND SOUTHERN OCEANS.

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